ISSN 0971-0388

Aligarh Journal of Statistics
Vol. 44(2024), 51-72

Odd Inverse Chen G-Family of Distributions with Applications

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[Received on May, 2023. Accepted on January, 2024]

ABSTRACT

This article is dedicated to the study of the new class of distributions and one of its particular members. Based on the odd ratio of baseline distribution, we have developed the odd inverse Chen-G family of distributions. General properties of the suggested family of distributions are provided. Using Weibull distribution as a baseline distribution, we have introduced a member of the suggested family having reverse-j or increasing or inverted bathtub shaped hazard function. Some statistical properties of this suggested distribution are explored. The associated parameters of the new distribution are estimated through MLE method. To assess the estimation procedure, we have conducted a Monte Carlo simulation and found that even for small samples, biases and mean square errors decrease as the sample size increases. Two real datasets are considered for the application of the suggested distribution. Using some criteria for model selection and goodness of fit test statistics, we empirically proved that the proposed model performs better than some existing models under study.

1. Introduction

The modelling of datasets using statistical distributions is a common practice in investigating real-world phenomena. Classical continuous parametric distributions such as Weibull, gamma, beta, log-normal, and exponential have been extensively used for this purpose. However, these distributions may not always produce a reasonable fit when dealing with complex datasets. Researchers have been incessantly developing new models that generalize the existing ones to overcome this limitation. These latest developments often employ techniques such as exponentiation and the T-X approach to generate more flexible distributions. Recently, several families of distributions with odd ratio of the distribution function of base distribution have been proposed in the literature; some of them are the extended odd Weibull-G family Alizadeh *et al*., (2018), the Kumaraswamy-Odd Rayleigh-G (Falgore and Doguwa, 2020), odd generalized half logistic Weibull-G (Chipepa *et al*., 2020), transmuted odd log-logistic-G (Alizadeh *et al*., 2020), odd power generalized Weibull-G (Moakofi *et al*., 2021), exponentiated odd Weibull-Topp-Leone-G (Chamunorwa *et al*., 2021), Topp-Leone Odd Burr III-G (Moakofi *et al*., 2022), and odd Lomax generalized exponential distributions (Sapkota and Kumar, 2022). This article aims to introduce and discuss a new family of distributions and its applications in modelling complex datasets. In the field of reliability analysis, the Chen distribution has gained attention as a novel lifetime distribution with an increasing or a bathtub-shaped failure rate function. Chen (2000) introduced this distribution with non-negative parameters α and β and, which exhibits a unique behaviour in modelling reliability systems. The Chen distribution has a flexible

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hazard function that can describe a wide variety of failure rates, making it a useful tool in modelling complex real-world problems. The PDF and CDF of Chen distribution are

$$k(t;\alpha,\beta) = \alpha\beta t^{\beta-1} e^{t^{\beta}} e^{\alpha(1-e^{t^{\beta}})}; t>0 \quad \text{and} \tag{1.1}$$

$$K(t;\alpha,\beta) = 1 - e^{\alpha(1-e^{t^{\beta}})}; t>0,$$

respectively. Srivastava and Srivastava (2014) defined a new model by inverting the Chen distribution with increasing or bathtub shaped hazard rate function called the inverse Chen distribution with non-negative parameters α and β. The CDF and PDF of the inverse Chen distribution are

$$K(t;\alpha,\beta) = e^{\alpha(1-e^{t^{-\beta}})}; t>0 \quad \text{and} \tag{1.2}$$

$$k(t;\alpha,\beta) = \alpha\beta t^{-(\beta+1)} e^{t^{-\beta}} e^{\alpha(1-e^{t^{-\beta}})}; t>0 \ ,$$

respectively. Similarly (Day *et al*., 2017), who introduced an exponentiated Chen distribution with an additional shape parameter to improve flexibility over the Chen distribution, El-Morshedy *et al.* (2020) utilized the Chen distribution to develop the odd Chen generator using the T-X approach proposed by (Alzaatreh *et al*., 2013) having two parameters. Anzagra *et al*. (2022) have also employed the T-X approach with W(x) as the odds ratio of the baseline CDF G(t) to propose a new family of two-parameter Chen-G distributions. This family has introduced three members, namely the Odd Chen Lomax, Odd Chen Burr-III, and Odd Chen Weibull, all exhibiting bathtub-shaped or increasing failure rate functions. The CDF and PDF of the Odd Chen G-family of distributions are

$$K(t;\alpha,\beta,\xi) = \int_{0}^{\frac{G(t;\xi)}{1-G(t;\xi)}} f(u)du = 1 - \exp\left[\alpha\left\{1 - e^{\left(\frac{G(t;\xi)}{1-G(t;\xi)}\right)^{\beta}}\right\}\right]; \alpha>0, \beta>0, t>0 \tag{1.3}$$

$$k(t;\alpha,\beta,\xi) = \alpha\beta\left(\frac{G(t;\xi)}{1-G(t;\xi)}\right)^{\beta-1} e^{\left(\frac{G(t;\xi)}{1-G(t;\xi)}\right)^{\beta}} \exp\left[\alpha\left\{1 - e^{\left(\frac{G(t;\xi)}{1-G(t;\xi)}\right)^{\beta}}\right\}\right] \frac{g(t;\xi)}{\left[1-G(t;\xi)\right]^2}; t>0.$$

Another odd Chen distribution’s family was introduced by (Eliwa *et al*., 2020) by using exponentiated approach by adding one extra parameter to the odd Chen distribution’s family. One member of this family of distributions is studied using two data sets under classical approaches.

The main aim of this article is to introduce a novel family of distributions known as the Odd inverse Chen-G family of distributions (OICh-G FD), which is capable of generating more robust compound probability distributions for modeling real-life datasets. The proposed family involves incorporating two additional parameters to the baseline distribution, providing an opportunity to capture the variability in the dataset such as skewness (left-right), symmetry, kurtosis (leptokurtosis-mesokurtic-platykurtosis), and different shapes of failure rates (increasing, bathtub, J, inverse J, decreasing, and unimodal-bathtub). One member of the OICh-G FD that is particularly noteworthy is investigated using the Weibull as the base distribution, which is widely used in reliability theory and life-testing (Marshal and Olkin, 2007).

2. Methodology

Suppose that T be the lifetime of a system or a component follows the inverse Chen distribution, as defined in Equation (1.2). If we let X represent the odds ratio, then the risk that the system or component will not be operational at time x can be expressed as $G(x;\xi)\,/\left[1-G(x;\xi)\right]$. Here, $G(x;\xi)$ be the baseline CDF that has a parameter vector of $\xi_{(1\times p)}$.

$$F(x) = \mathrm{P}(\mathrm{X} \le x) = \Psi\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right),$$

where the odd ratio $G(x;\xi)\,/\left[1-G(x;\xi)\right]$ holds the conditions provided below:

1. $G(x;\xi)\,/\left[1-G(x;\xi)\right] \in [c,d]$ for $0<c<d<\infty$
2. $G(x;\xi)\,/\left[1-G(x;\xi)\right]$ is monotonically non-decreasing and differentiable.
3. $G(x;\xi)\,/\left[1-G(x;\xi)\right] \to c$ as $x \to 0$ and $G(x;\xi)\,/\left[1-G(x;\xi)\right] \to d$ as $x \to \infty$.

The T-X approach is utilized in proposing a new family of distributions known as the Odd Inverse Chen-G (OICh-G) family of distributions in this study. Let ξ be a vector of parameters of baseline CDF $G(x;\xi)$ and then CDF $F(x)$ of the OICh-G FD can be obtained as

$$F(x;\alpha,\beta,\xi) = \int_0^{\frac{G(x;\xi)}{1-G(x;\xi)}} f(t)dt = \exp\left[\alpha\left\{1-e^{\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{-\beta}}\right\}\right]; \alpha>0, \beta>0, x\in \text{¡} \qquad (2.1)$$

where α and β are extra shape parameters. Differentiating the Equation (2.1), the PDF $f(x)$ of the OICh-G FD is obtained as

$$f(x;\alpha,\beta,\xi) = \alpha\beta g(x;\xi)\frac{\left(1-G(x;\xi)\right)^{\beta-1}}{\left(G(x;\xi)\right)^{(\beta+1)}} e^{\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{-\beta}} \exp\left[\alpha\left\{1-e^{\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{-\beta}}\right\}\right] \qquad (2.2)$$

The Reliability function of OICh-G FD is given as

$$R(x) = 1-\exp\left[\alpha\left\{1-e^{\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{-\beta}}\right\}\right]. \qquad (2.3)$$

The Hazard function of OICh-G FD is given as

$$H(x;\alpha,\beta,\xi) = \alpha\beta g(x;\xi)\left(G(x;\xi)\right)^{-(\beta+1)}\left(1-G(x;\xi)\right)^{\beta-1}\exp\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{-\beta}$$
$$\exp\left[\alpha\left\{1-\exp\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{-\beta}\right\}\right]\left[1-\exp\left\{\alpha\left\{1-\exp\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{-\beta}\right\}\right\}\right]^{-1} \qquad (2.4)$$

2.3. The quantile function (QF)

The quantile function is useful in statistical analysis and modelling, as it provides a way to estimate percentiles and other summary statistics of a probability distribution. Suppose $Q(p)$ is the smallest value of X for which the probability that X is less than or equal to that value is at least p. The quantile function $Q(p;\xi)$ of CDF $F(x;\xi)$ of OICh-G FD can be obtained as

$$Q(p;\xi)=G^{-1}\left[\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln p\right)\right\}^{\frac{1}{\beta}}\right\}^{-1}\right];\ p\in(0,1). \tag{2.5}$$

The random deviate function of OICh-G FD can be generated by

$$x=G^{-1}\left[\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln u\right)\right\}^{\frac{1}{\beta}}\right\}^{-1}\right];\mathrm{u}\in(0,1). \tag{2.6}$$

Using Equation (2.5), we can calculate the median, upper and lower quartile, quartile deviation (QD), coefficient of QD, skewness and kurtosis presented in Table 1.

Table 1: Various measures based on quantiles.

Median	$G^{-1}\left[\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln(0.5)\right)\right\}^{\frac{1}{\beta}}\right\}^{-1}\right]$
Lower Quartile	$G^{-1}\left[\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln(0.25)\right)\right\}^{\frac{1}{\beta}}\right\}^{-1}\right]$
Upper Quartile	$G^{-1}\left[\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln(0.75)\right)\right\}^{\frac{1}{\beta}}\right\}^{-1}\right]$
QD	$\frac{1}{2}\left[G^{-1}\left[\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln(0.75)\right)\right\}^{\frac{1}{\beta}}\right\}^{-1}\right]-G^{-1}\left[\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln(0.25)\right)\right\}^{\frac{1}{\beta}}\right\}^{-1}\right]\right]$
Coeffcient of QD	$\dfrac{\left[G^{-1}\left[\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln(0.75)\right)\right\}^{\frac{1}{\beta}}\right\}^{-1}\right]-G^{-1}\left[\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln(0.25)\right)\right\}^{\frac{1}{\beta}}\right\}^{-1}\right]\right]}{\left[G^{-1}\left[\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln(0.75)\right)\right\}^{\frac{1}{\beta}}\right\}^{-1}\right]+G^{-1}\left[\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln(0.25)\right)\right\}^{\frac{1}{\beta}}\right\}^{-1}\right]\right]}$
Skewness (Kenney and Keeping 1962)	$\dfrac{Q\left(\frac{3}{4};\xi\right)+Q\left(\frac{1}{4};\xi\right)-2Q\left(\frac{1}{2};\xi\right)}{Q\left(\frac{3}{4};\xi\right)-Q\left(\frac{1}{4};\xi\right)}$
Kurtosis (Moors, 1988)	$\dfrac{Q\left(\frac{3}{8};\xi\right)+Q\left(\frac{7}{8};\xi\right)-Q\left(\frac{5}{8};\xi\right)-Q\left(\frac{1}{8};\xi\right)}{Q\left(\frac{3}{4};\xi\right)-Q\left(\frac{1}{4};\xi\right)}$

3. Statistical Properties

3.1 Linear form of OICh-G FD

One can derive useful linear expansions using the exponentiated approach. Specifically, the exponentiated-G with power parameter z > 0 has a CDF

$$G_Z(x;\phi)=\left[G(x;\phi)\right]^Z \text{, where } x\in\Re \text{ and } \phi \text{ is a parameter space.} \tag{3.1}$$

The corresponding PDF is

$$g_Z(x;\phi)=zg(x;\phi)\left[G(x;\phi)\right]^{(z-1)},\ x\in\Re. \tag{3.2}$$

These notations will be used in the following discussion. Exponentiated distributions have well-known properties for a wide range of baseline CDF $G(x;\phi)$ for more information (see Nadarajah and Gupta, 2007; Lemonte *et al*., 2013). Now, using the following series, we can write the PDF of OICh-G FD into a linear form as

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\ ;-\infty<x<\infty$$

$$(1-x)^m=\sum_{n=0}^{\infty}(-1)^n\binom{m}{n}x^n.$$

Appling exponential series expansion, the OICh-G PDF defined in Equation (2.2) becomes

$$f(x;\alpha,\beta,\xi)=\alpha\beta e^{\alpha}g(x;\xi)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\frac{(i+1)^j}{j!}\left(1-G(x;\xi)\right)^{\beta(j+1)-1}\left\{G(x;\xi)\right\}^{-(\beta(j+1)+1)}. \tag{3.3}$$

Further expanding Equation (3.3) we get

$$f(x;\alpha,\beta,\xi)=\alpha\beta e^{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}\alpha^i}{i!}\frac{(i+1)^j}{j!}\binom{\beta(j+1)-1}{k}g(x;\xi)\left(G(x;\xi)\right)^{-(\beta(j+1)+1)+k}. \tag{3.4}$$

Now Equation (3.4) can also be written in the simplest form

$$f(x;\alpha,\beta,\xi)=\alpha\beta e^{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}v_{ijk}\,g(x;\xi)\left(G(x;\xi)\right)^{-(\beta(j+1)+1)+k}\ ;\alpha,\beta>0. \tag{3.5}$$

where, $$v_{ijk}=\frac{(-1)^{i+k}\alpha^i}{i!}\frac{(i+1)^j}{j!}\binom{\beta(j+1)-1}{k}.$$

3.2 Moments

The rth order non-central moment (μ_r') for the OICh-G FD is

$$\begin{aligned}\mu_r' &= E(X^r)=\int_{-\infty}^{\infty}x^r f(x)dx \\ &=\alpha\beta e^{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}v_{ijk}\int_{-\infty}^{\infty}x^r g(x;\xi)\left(G(x;\xi)\right)^{-(\beta(j+1)+1)+k}dx\end{aligned} \tag{3.6}$$

Let $G(x;\xi)=w$ then $g(x;\xi)dx=dw;\quad 0\le w\le 1$

$$\mu_r'=\alpha\beta e^{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}v_{ijk}\int_0^1 w^{-(\beta(j+1)+1)+k}Q_G^r(w)dw;\quad 0<w<1.$$

where $Q_G(w)$ is the baseline distribution's quantile function.

3.3 Moment Generating Function (MGF)

The MGF ($M_X(t)$) for the OICh-G FD is

$$\begin{aligned} M_X(t) &= \sum_{l=0}^{\infty} \frac{t^l}{l!} \int_{-\infty}^{\infty} x^l f(x) dx \\ &= \alpha\beta e^{\alpha} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^l}{l!} v_{ijk} \int_{-\infty}^{\infty} x^l \left(G(x;\xi)\right)^{-(\beta(j+1)+1)+k} g(x;\xi)\, dx \end{aligned} \tag{3.7}$$

In terms of quantile function MGF can be written as

$$M_X(t) = \alpha\beta e^{\alpha} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^l}{l!} v_{ijk} \int_{0}^{1} w^{-(\beta(j+1)+1)+k} Q_G^l(w) dw;\ 0 < w < 1,$$

where $G(x;\xi) = w$ and $Q_G(w)$ is the baseline distribution's quantile function.

3.5 Entropy

Entropy is a tool used to calculate a random variable's variation or unpredictability level. This concept has a wide variety of applications in diverse fields, such as econometrics, probability theory, engineering, and science in general. Entropy describes how much uncertainty or disorder there is in a particular system or situation. It is an important tool that helps us understand complex systems' behaviour and make predictions about their future states. Whether we are trying to analyze financial markets, design efficient machines, or model natural phenomena, entropy is an elemental concept that plays a critical role in numerous areas of research and development.

i) Renyi's Entropy

It measures the randomness or uncertainty of a probability distribution and has applications in various fields like engineering, econometrics, and financial mathematics. One of the pioneers in introducing the concept of entropy was given by (Renyi, 1961) to quantify the variability of uncertainty. These measures can be used to evaluate the degree of randomness or predictability in a given system and calculated as

$$R_\rho(X) = \frac{1}{1-\rho} \log \int_{-\infty}^{\infty} \{ f(x) \}^{\rho} dx \quad ;\ \rho > 0 \text{ and } \rho \neq 1.$$

Applying Taylor's series expansion $[f(x)]^\rho$ can be presented in the form

$$\begin{aligned} [f(x;\alpha,\beta,\xi)]^\rho = (\alpha\beta)^\rho e^{\alpha\rho} \left(g(x;\xi)\right)^\rho \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i (\alpha\rho)^i}{i!} \frac{(i+\rho)^j}{j!} \\ \left(1-G(x;\xi)\right)^{\beta(j+\rho)-\rho} \{G(x;\xi)\}^{-(\beta(j+\rho)+\rho)}. \end{aligned} \tag{3.8}$$

Further expanding Equation (3.8) using generalized binomial series expansion. The expression for $[f(x;\alpha,\beta,\xi)]^\rho$ becomes

$$\begin{aligned} [f(x;\alpha,\beta,\xi)]^\rho = (\alpha\beta)^\rho e^{\alpha\rho} \left(g(x;\xi)\right)^\rho \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{\beta(j+\rho)-\rho}{k} \frac{(-1)^{i+k} (\alpha\rho)^i}{i!} \frac{(i+\rho)^j}{j!} \\ \left(G(x;\xi)\right)^{-(\beta(j+\rho)+\rho)+k}. \end{aligned} \tag{3.9}$$

Now Equation (3.9) can also written in the simplest form

$$\left[f(x;\alpha,\beta,\xi)\right]^{\rho}=\left(\alpha\beta\right)^{\rho}e^{\alpha\rho}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Psi_{ijk}\left(g(x;\xi)\right)^{\rho}\left(G(x;\xi)\right)^{-(\beta(j+\rho)+\rho)+k}, \tag{3.10}$$

where, $$\Psi_{ijk}=\frac{(-1)^{i+k}(\alpha\rho)^{i}}{i!}\binom{\beta(j+\rho)-\rho}{k}\frac{(i+\rho)^{j}}{j!}.$$

Putting $\left[f(x;\alpha,\beta,\xi)\right]^{\rho}$ into the equation for $R_{\rho}(X)$, Renyi's Entropy for OICh-G FD becomes

$$R_{\rho}(X)=\frac{1}{1-\rho}\log\left[\left(\alpha\beta\right)^{\rho}e^{\alpha\rho}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Psi_{ijk}\int_{-\infty}^{\infty}\left(g(x;\xi)\right)^{\rho}\left(G(x;\xi)\right)^{-(\beta(j+\rho)+\rho)+k}dx\right] \tag{3.11}$$

ii) q-Entropy

The q-entropy is given by

$$H(\rho)=\frac{1}{1-\rho}\log\left[1-\int_{-\infty}^{\infty}\left\{f(x)\right\}^{\rho}dx\right];\ \rho>0 \text{ and } \rho\neq 1.$$

Putting $\left[f(x;\alpha,\beta,\xi)\right]^{\rho}$ from Equation (3.10) into the expression for $H(\rho)$, the q-Entropy for OICh-G FD becomes

$$H(\rho)=\frac{1}{1-\rho}\log\left[1-\left(\alpha\beta\right)^{\rho}e^{\alpha\rho}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Psi_{ijk}\int_{-\infty}^{\infty}\left(g(x;\xi)\right)^{\rho}\left(G(x;\xi)\right)^{-(\beta(j+\rho)+\rho)+k}dx\right];\ \rho>0 \text{ and } \rho\neq 1.$$

iii) Shannon's Entropy

It is a special case of the Rényi's entropy when $\rho\uparrow 1$.Shannon entropies can be defined as $\eta_X=E(-\log f(x))$. For the OICh G-FD one can compute using

$$\eta_X=E\left[-\log\left\{\alpha\beta e^{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}v_{ijk}\,g(x;\xi)\left(G(x;\xi)\right)^{-(\beta(j+1)+1)+k}\right\}\right].$$

4. Estimation Method

4.1. Maximum Likelihood Estimation (MLE)

The parameters of the OICh-G family are estimated in this sub-section using MLE. Let $u=(\alpha,\beta,\xi)^{T}$ be $(p\times 1)$ vector of parameter, the log density and total log-likelihood function respectively is given by

$$l(\alpha,\xi\,|\,x)=\log(\alpha\beta)-(\beta+1)\log\left(\frac{G(x;\xi)}{1-G(x,\xi)}\right)+\left[\frac{G(x;\xi)}{1-G(x,\xi)}\right]^{-\beta}+\alpha\left[1-e^{\left(\frac{G(x;\xi)}{1-G(x,\xi)}\right)^{-\beta}}\right]+\log g(x;\xi)-2\log\left(1-G(x;\xi)\right)$$

and

$$l(\alpha,\beta,\xi\,|\,\underline{x})=n\log\alpha+n\log\beta-(\beta+1)\sum_{i=1}^{n}\log\left(\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right)+\sum_{i=1}^{n}\left[\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right]^{-\beta}+\alpha\sum_{i=1}^{n}\left[1-e^{\left(\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right)^{-\beta}}\right]$$
$$+\sum_{i=1}^{n}\log g(x_i;\xi)-2\sum_{i=1}^{n}\log\left(1-G(x_i;\xi)\right). \tag{4.1}$$

Differentiating Equation (4.1) gives the score function's components $V(u)=\left(\frac{\partial l}{\partial\alpha},\frac{\partial l}{\partial\beta},\frac{\partial l}{\partial\xi}\right)^T$ as follows

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\left[1-e^{\left(\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right)^{-\beta}}\right] \tag{4.2}$$

$$\begin{aligned}\frac{\partial l}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left(\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right)+\sum_{i=1}^{n}\left[\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right]^{-\beta}\log\left[\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right]\\+\alpha\sum_{i=1}^{n}e^{\left(\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right)^{-\beta}}\left[\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right]^{-\beta}\log\left[\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right]\end{aligned} \tag{4.3}$$

$$\begin{aligned}\frac{\partial l}{\partial\xi}=(\beta+1)\sum_{i=1}^{n}\left(\frac{1-G(x_i;\xi)}{G(x_i,\xi)}\right)\frac{G_k^{'}(x_i;\xi)}{\left[1-G(x_i,\xi)\right]^2}+\beta\sum_{i=1}^{n}\left[\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right]^{-(\beta+1)}\frac{G_k^{'}(x_i;\xi)}{\left[1-G(x_i,\xi)\right]^2}\\-\alpha\beta\sum_{i=1}^{n}e^{\left(\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right)^{-\beta}}\left[\frac{G(x_i;\xi)}{1-G(x_i,\xi)}\right]^{-(\beta+1)}\frac{G_k^{'}(x_i;\xi)}{\left[1-G(x_i,\xi)\right]^2}+\sum_{i=1}^{n}\frac{g_k^{'}(x_i;\xi)}{g(x_i;\xi)}+2\sum_{i=1}^{n}\frac{g_k^{'}(x_i;\xi)}{\left[1-G(x_i;\xi)\right]}\end{aligned} \tag{4.4}$$

where $g_k^{'}(x_i,\xi)=\frac{dg(x_i,\xi)}{d\xi}$, $g_k^{''}(x_i,\xi)=\frac{d^2g(x_i,\xi)}{d^2\xi}$, $G_k^{'}(x_i,\xi)=\frac{dG(x_i,\xi)}{d\xi}$

and $G_k^{''}(x_i,\xi)=\frac{d^2G(x_i,\xi)}{d^2\xi}$.

To determine the parameter estimators, Equations (4.2), (4.3), and (4.4) are set to zero, and numerical methods like the Newton-Raphson algorithms are used iteratively to solve them. The observed information matrix $J(v)$ is essential for calculating the parameter's confidence intervals. It can be computed as $J(v)=\frac{\partial^2 l}{\partial i\partial j}$ for $(i,j=\alpha,\beta,\xi)$ whose elements can be evaluated numerically.

4.2 Method of Least Square Estimation (LSE)

Another estimation method was introduced by (Swain et al., 1988), named the ordinary LSEs and weighted LSEs to estimate the parameters of the OICh-G FD. Let $x_{(1)},\dots,x_{(n)}$ be order statistics of size n form $F(x;\alpha,\beta,\xi)$. The LSE for the OICh-G FD can be obtained by minimizing

$$K(X;\alpha,\beta,\xi)=\sum_{i=1}^{n}\left[F(x_{(i)};\alpha,\beta,\xi)-\frac{i}{n+1}\right]^2 \tag{4.5}$$

w. r. t. α,β and ξ. The least square estimates for the OICh-G FD is also become

$$K(X;\alpha,\beta,\xi)=\sum_{i=1}^{n}\left[\exp\left\{\alpha\left\{1-\exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)^{-\beta}\right\}\right\}-\frac{i}{n+1}\right]^2 \tag{4.6}$$

Differentiating Equation (4.6) w. r. t. α, β and ξ we get

$$\frac{dK}{d\alpha} = -2\alpha \sum_{i=1}^{n} \left[\exp\left\{ \alpha \left\{ 1 - \exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)^{-\beta} \right\} \right\} - \frac{i}{n+1} \right] \exp\left\{ \alpha \left\{ 1 - \exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)^{-\beta} \right\} \right\} \exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)^{-\beta}.$$

$$\frac{dK}{d\beta} = 2\alpha \sum_{i=1}^{n} \left[\exp\left\{ \alpha \left\{ 1 - \exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)^{-\beta} \right\} \right\} - \frac{i}{n+1} \right] \exp\left\{ \alpha \left\{ 1 - \exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)^{-\beta} \right\} \right\} \exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)^{-\beta} \left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)^{-\beta} \log\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)$$

$$\frac{dK}{d\xi} = 2\alpha\beta \sum_{i=1}^{n} \left[\exp\left\{ \alpha \left\{ 1 - \exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)^{-\beta} \right\} \right\} - \frac{i}{n+1} \right] \exp\left\{ \alpha \left\{ 1 - \exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)^{-\beta} \right\} \right\} \exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)^{-\beta} \left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)^{-(\beta+1)} \frac{G_k^{'}\left(x_{(i)};\xi\right)}{\left[1-G\left(x_{(i)},\xi\right)\right]^2}$$

where $G_k^{'}(x_{(i)},\xi) = \dfrac{dG(x_{(i)},\xi)}{d\xi}$. By solving $\dfrac{dK}{d\alpha} = 0, \dfrac{dK}{d\beta} = 0$ and $\dfrac{dK}{d\xi} = 0$ simultaneously will get the LSEs.

4.3 Cramer-von Mises Estimator (CVME)

CVMEs are derived by measuring the discrepancy between the empirical CDF (based on the observed data) and the estimated CDF (based on the assumed theoretical distribution). By minimizing this difference, the CVMEs provide an estimate of the parameters that best fit the observed data. In the context of estimating the parameters of the CDF for the OICh-G FD, the CVMEs can be used to obtain estimates that are as accurate as possible while minimizing.

$$C(X;\alpha,\beta,\xi) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{(i)};\alpha,\beta,\xi) - \frac{2i-1}{2n} \right]^2,$$

with respect to α, β and ξ. The CVMEs for the OICh-G FD is also become

$$C(X;\alpha,\beta,\xi) = \frac{1}{12n} + \sum_{i=1}^{n} \left[\exp\left\{ \alpha \left\{ 1 - \exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)} \right)^{-\beta} \right\} \right\} - \frac{2i-1}{2n} \right]^2 \qquad (4.7)$$

Differentiating Equation (4.7) with respect to α, β and ξ we get

$$\frac{dC}{d\alpha}=-2\alpha\sum_{i=1}^{n}\left[\exp\left\{\alpha\left\{1-\exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)^{-\beta}\right\}\right\}-\frac{2i-1}{2n}\right]\exp\left\{\alpha\left\{1-\exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)^{-\beta}\right\}\right\}$$

$$\exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)^{-\beta}$$

$$\frac{dC}{d\beta}=2\alpha\sum_{i=1}^{n}\left[\exp\left\{\alpha\left\{1-\exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)^{-\beta}\right\}\right\}-\frac{2i-1}{2n}\right]\exp\left\{\alpha\left\{1-\exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)^{-\beta}\right\}\right\}$$

$$\exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)^{-\beta}\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)^{-\beta}\log\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)$$

$$\frac{dC}{d\xi}=2\alpha\beta\sum_{i=1}^{n}\left[\exp\left\{\alpha\left\{1-\exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)^{-\beta}\right\}\right\}-\frac{2i-1}{2n}\right]\exp\left\{\alpha\left\{1-\exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)^{-\beta}\right\}\right\}$$

$$\exp\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)^{-\beta}\left(\frac{G(x_{(i)};\xi)}{1-G(x_{(i)};\xi)}\right)^{-(\beta+1)}\frac{G_k^{'}\left(x_{(i)};\xi\right)}{\left[1-G\left(x_{(i)},\xi\right)\right]^2}$$

where $G_k^{'}(x_i,\xi)=\frac{dG(x_i,\xi)}{d\xi}$. By solving $\frac{dC}{d\alpha}=0,\frac{dC}{d\beta}=0$ and $\frac{dC}{d\xi}=0$ simultaneously will get the CVMEs.

5. Special Distribution

Generalization of several distributions can be made using the OICh-G FD. The special distributions odd inverse Chen Weibull (OIChW) is defined and some properties are presented in this sub-section.

5.1 Odd inverse Chen Weibull (OIChW) Distribution

The Weibull distribution is considered as base distribution whose CDF and PDF are $G\left(x;\xi\right)=1-e^{-\lambda x^{\theta}}$ and $g\left(x;\xi\right)=\lambda\theta x^{\theta-1}e^{-\lambda x^{\theta}}\;\;;\lambda>0,\theta>0,x>0$ respectively. Now, the CDF and PDF of OIChW distribution are given by

$$F(x;\alpha,\beta,\lambda,\theta)=\exp\left[\alpha\left(1-\exp\left\{\left(e^{\lambda x^{\theta}}-1\right)^{-\beta}\right\}\right)\right];\alpha>0,\beta>0,\lambda>0,\theta>,x>0 \qquad (5.1)$$

and $f(x;\alpha,\beta,\lambda,\theta)=\alpha\beta\lambda\theta x^{\theta-1}e^{\lambda x^{\theta}}\left(e^{\lambda x^{\theta}}-1\right)^{-(\beta+1)}\exp\left(e^{\lambda x^{\theta}}-1\right)^{-\beta}\exp\left[\alpha\left(1-\exp\left(e^{\lambda x^{\theta}}-1\right)^{-\beta}\right)\right]$. (5.2)

Using Equation (3.5) in Equation (5.2) becomes

$$f(x;\alpha,\beta,\lambda,\theta)=\alpha\beta\lambda\theta e^{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}v_{ijk}x^{\theta-1}e^{-\lambda x^{\theta}}\left(1-e^{-\lambda x^{\theta}}\right)^{-(\beta(j+1)+1)+k}$$

$$=\alpha\beta\lambda\theta e^{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}v_{ijkl}x^{\theta-1}e^{-\lambda(l+1)x^{\theta}}$$

$$=\alpha\beta\theta e^{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}v_{ijkl}k(\omega,\theta),$$

where $k(\omega,\theta)$ is the PDF of Weibull distribution with parameters $\omega=\lambda(l+1)$ and θ and

$$v_{ijkl}=\frac{(-1)^{i+k}\alpha^{i}}{i!}\frac{(i+1)^{j}}{j!(l+1)}\binom{\beta(j+1)-1}{k}\binom{((\beta(j+1)+1)-k)+l-1}{l}.$$

The reliability and hazard function, respectively is given by

$$R(x)=1-\exp\left[\alpha\left(1-\exp\left(e^{\lambda x^{\theta}}-1\right)^{-\beta}\right)\right];\alpha>0,\beta>0,\lambda>0,\theta>,x>0$$

and

$$H(x)=\frac{\alpha\beta\lambda\theta x^{\theta-1}e^{\lambda x^{\theta}}\left(e^{\lambda x^{\theta}}-1\right)^{-(\beta+1)}\exp\left(e^{\lambda x^{\theta}}-1\right)^{-\beta}\exp\left[\alpha\left(1-\exp\left(e^{\lambda x^{\theta}}-1\right)^{-\beta}\right)\right]}{\left[1-\exp\left\{\alpha\left(1-\exp\left(e^{\lambda x^{\theta}}-1\right)^{-\beta}\right)\right\}\right]}.$$

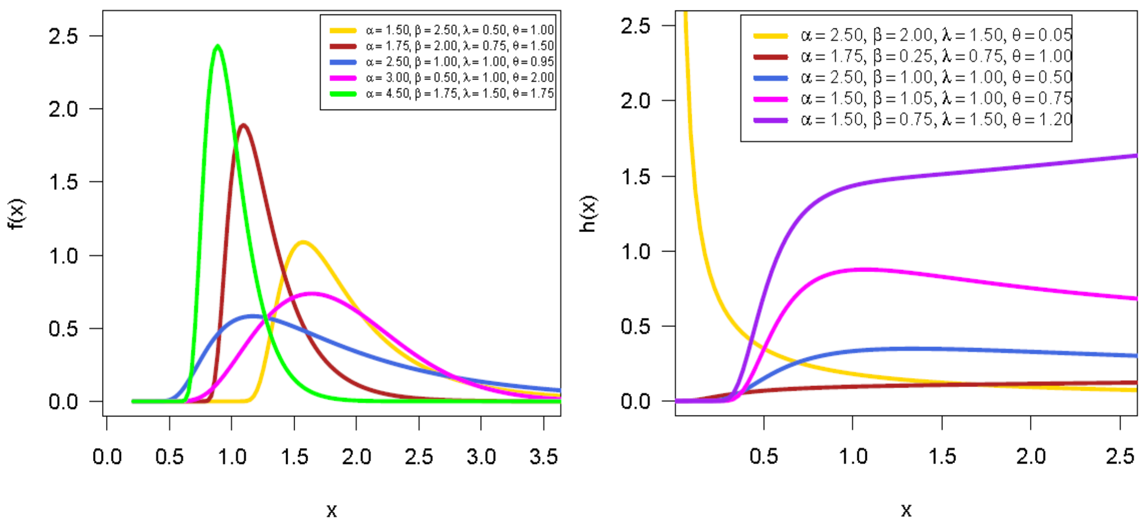

Figure 1: Shapes of PDF (left) and HRF (right) of OIChW distribution.

In Figure 1 (left), we can observe that the density plot of the OIChW distribution has different shapes, including decreasing, increasing, and right-skewed. On the other hand, Figure 1 (right) shows the HRF, which displays a varying pattern of failure rates for some specific values. The HRF can exhibit an upside-down bathtub shape, decreasing failure or increasing hazard rates. The QF and random deviate generation for the OIChW distribution, respectively is, given by

$$Q(p)=\left[\frac{1}{\lambda}\ln\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln p\right)\right\}^{-\frac{1}{\beta}}\right\}\right]^{\frac{1}{\theta}};0<p<1 \tag{5.3}$$

and $$x=\left[\frac{1}{\lambda}\ln\left\{1+\left\{\ln\left(1-\frac{1}{\alpha}\ln u\right)\right\}^{-\frac{1}{\beta}}\right\}\right]^{\frac{1}{\theta}};0<u<1$$

where u follows the $U(0,1)$ distribution.

5.2 Moments

The rth order moment about origin ($\mu_r^{'}$) for the OIChW distribution is

$$\mu_r^{'}=\int_0^{\infty}x^r f(x)dx$$

$$=\alpha\beta\lambda\theta e^{\alpha}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}v_{ijkl}\int_0^{\infty}x^{r+\theta-1}e^{-\lambda(l+1)x^{\theta}}dx$$

$$=\alpha\beta\lambda e^{\alpha}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}v_{ijkl}\frac{\Gamma\left(\frac{r}{\theta}+1\right)}{\left\{\lambda(l+1)\right\}^{\frac{r}{\theta}+1}}\ . \tag{5.4}$$

where $v_{ijkl}=\frac{(-1)^{i+k}\alpha^j}{i!}\frac{(i+1)^j}{j!}\binom{\beta(j+1)-1}{k}\binom{\left(\left(\beta(j+1)+1\right)-k\right)+l-1}{l}$.

5.3 Moment Generating Function

The MGF ($M_X(t)$) for the OIChW distribution is

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_{-\infty}^{\infty}x^r f(x)dx$$

$$=\alpha\beta\lambda\theta e^{\alpha}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}v_{ijkl}\int_0^{\infty}x^{r+\theta-1}e^{-\lambda(l+1)x^{\theta}}dx \tag{5.5}$$

$$=\alpha\beta\lambda e^{\alpha}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}v_{ijkl}\frac{\Gamma\left(\frac{r}{\theta}+1\right)}{\left(\lambda(l+1)\right)^{\frac{r}{\theta}+1}}.$$

5.4 Estimation

The parameters of the OIChW distribution are computed using MLE method. To find MLEs, we have to maximize the Equation (5.6). The log density and total log-likelihood function, respectively, are given by

$$l(\alpha,\beta,\lambda,\theta\,|\,x)=\log\left(\alpha\beta\lambda\right)+\log\theta+(\theta-1)\log x+\lambda x^{\theta}-(\beta+1)\log(e^{\lambda x^{\theta}}-1)$$
$$+(e^{\lambda x^{\theta}}-1)^{-\beta}+\alpha(1-e^{(e^{\lambda x^{\theta}}-1)^{-\beta}})$$

and

$$l(\alpha,\beta,\lambda,\theta\,|\,\underline{x}) = n\log\alpha + n\log\beta + n\log\theta + n\log\lambda + \lambda\sum_{i=1}^{n} x_i - (\beta+1)\sum_{i=1}^{n}\log(e^{\lambda x_i}-1) + \sum_{i=1}^{n}(e^{\lambda x_i}-1)^{-\beta} + \alpha\sum_{i=1}^{n}(1-e^{(e^{\lambda x_i}-1)^{-\beta}}). \tag{5.6}$$

To obtain the MLEs, we have to differentiate the Equation (5.6) w. r. t. model parameters. However, these equations are non-linear, so we used a **maxLik** R package to solve them numerically.

5.5 Simulation

We used the **maxLik** R package created by (Henningsen and Toomet, 2011) to generate samples from the QF specified in Equation (5.3) for different parameter combinations of the OIChW distribution. The MLEs were then computed for each sample using the maxLik() function along with the BFGS algorithm, allowing us to investigate parameter estimation issues, as well as to estimate the bias size and direction (i.e., overestimation or underestimation) of the MLEs. We employed sample sizes ranging from 100 to 300 in increments of 50 in our simulation, which we repeated 1000 times to obtain estimates of the bias, and mean square error (MSE). We presented the results in Tables 2, 3, and 4, which report the MLEs bias and MSEs for each parameter and a 95% confidence interval (CI) with a lower bound (LB) and upper bound (UB) for each estimated parameter. Our findings showed that the bias and MSE decreased as the sample size increased for three different parameter combinations, indicating that the MLE method is asymptotically efficient, consistent, and follows the invariance property.

Table 2: Biases, MSEs and CI for MLEs based on 1000 simulations for initials α=0.25, β=1.75, λ=0.5 and θ=0.5.

	Parameter	**Sample size (n)**				
		100	150	200	250	300
Bias	α	0.2117	0.0931	0.0793	0.1103	0.0529
	β	0.6508	0.6330	0.6369	0.8229	0.3303
	λ	-0.2387	-0.2158	-0.2094	-0.1749	-0.1729
	θ	1.4140	0.9978	0.8293	0.6587	0.559
MSE	α	2.1769	0.5654	0.3349	0.4090	0.2554
	β	20.312	16.4054	17.7771	18.6247	10.6431
	λ	0.1397	0.1237	0.1179	0.1040	0.0953
	θ	4.9501	2.7872	1.9411	1.4737	1.0188
CI	LB(α)	0.0085	0.0134	0.0152	0.0159	0.0173
	UB(α)	3.4537	2.2255	2.1076	2.1925	1.4935
	LB(β)	0.1191	0.1372	0.1650	0.1707	0.2043
	UB(β)	14.7268	15.1192	16.8771	17.1054	12.7369
	LB(λ)	0.0000	0.0000	1.00E-04	2.00E-04	6.00E-04
	UB(λ)	0.7373	0.7172	0.7138	0.7128	0.7043
	LB(θ)	0.0743	0.0780	0.0734	0.0718	0.0956
	UB(θ)	6.0498	4.7711	3.8901	3.7347	3.1214

Table 3: Biases, MSEs and CI for MLEs based on 1000 simulations for initials α=0.5, β=2.0, λ=0.25 and θ=0.75.

	Parameter	Sample size (n)				
		100	150	200	250	300
Bias	α	0.3946	0.356	0.2942	0.1816	0.1129
	β	0.5860	0.6921	0.8971	0.7535	0.7256
	λ	-0.0875	-0.0774	-0.0571	-0.0610	-0.0481
	θ	1.1440	0.8814	0.6567	0.6061	0.4717
MSE	α	7.0368	6.0773	5.6126	2.2243	0.5989
	β	19.2874	24.0471	21.6389	19.9637	15.8538
	λ	0.0512	0.0464	0.0443	0.0409	0.0382
	θ	3.2465	2.1677	1.4864	1.2650	0.8940
CI	LB(α)	0.0291	0.0334	0.0417	0.0426	0.0460
	UB(α)	6.7845	4.8032	3.9699	3.6678	2.8750
	LB(β)	0.2379	0.2681	0.2991	0.3236	0.3553
	UB(β)	16.7212	18.3535	18.2153	18.1154	15.5797
	LB(λ)	0.0000	1.00E-04	2.00E-04	4.00E-04	8.00E-04
	UB(λ)	0.6227	0.6227	0.6250	0.6255	0.6126
	LB(θ)	0.1285	0.1138	0.1119	0.1175	0.1283
	UB(θ)	4.9413	4.2594	3.9102	3.5058	3.1460

Table 4: Biases, MSEs and CI for MLEs based on 1000 simulations for initials α=1.5, β=0.5, λ=0.75 and θ=1.0.

	Parameter	Sample size (n)				
		100	150	200	250	300
Bias	α	0.7989	0.4707	0.4092	0.2202	0.3614
	β	-0.0475	-0.0335	-0.0169	-0.0156	-0.0135
	λ	0.0476	-0.0176	0.0012	-0.0234	0.0196
	θ	0.2688	0.1881	0.1226	0.1051	0.0828
MSE	α	15.1603	7.8033	5.074	2.6494	3.4775
	β	0.0361	0.0255	0.0196	0.0151	0.0137
	λ	0.8774	0.4377	0.2960	0.2255	0.2488
	θ	0.3810	0.1975	0.1303	0.0928	0.083
CI	LB(α)	0.3156	0.3908	0.4552	0.5055	0.5467

UB(α)	12.8578	10.2843	7.5952	5.7343	6.9485
LB(β)	0.2197	0.2558	0.2775	0.3076	0.3147
UB(β)	0.8835	0.8743	0.8220	0.7668	0.7491
LB(λ)	0.0375	0.0825	0.1258	0.1723	0.2083
UB(λ)	3.0381	2.3742	2.0649	1.7631	1.8971
LB(θ)	0.5024	0.5302	0.5707	0.6376	0.6007
UB(θ)	2.5408	2.1062	1.8668	1.7187	1.6616

5.6 Application

Employing two real data sets, we demonstrate the application of the OIChW distribution in this sub-section. The data sets employed for the application of the OIChW model are given as follows

i) Data set

Data set I

The initial dataset pertains to the breaking stress (measured in GPa) of carbon fibers that are 50 mm in length. This particular dataset has been utilized in prior research studies conducted by (Nichols and Padgett, 2006) and (Cordeiro and Lemonte, 2011). The data itself is presented below.

“0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90”

Data set II

The information provided is derived from a test that was conducted to study the lifespan of 59 conductors (Nelson and Doganaksoy, 1995). The experiment was designed to accelerate the ageing process of the conductors, and the data obtained from it shows the time it took for each conductor to fail. The cause of failure was attributed to electro-migration, which is the movement of atoms within the conductor that disrupts the circuit. The failure times are expressed in hours, and there were no instances of censoring, meaning that all the observations were complete.

“6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923.”

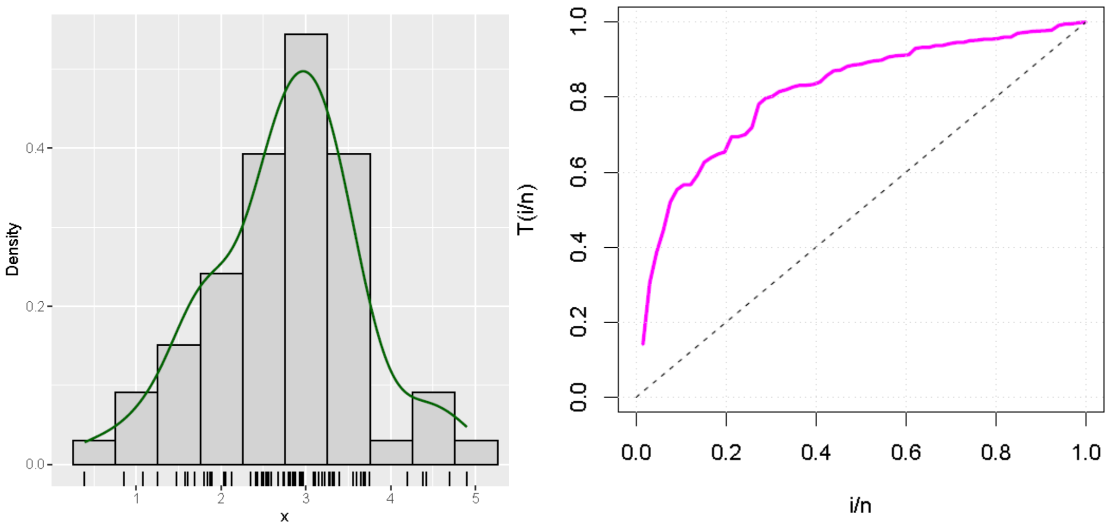

Figure 2:. Histogram and TTT plots of dataset-I.

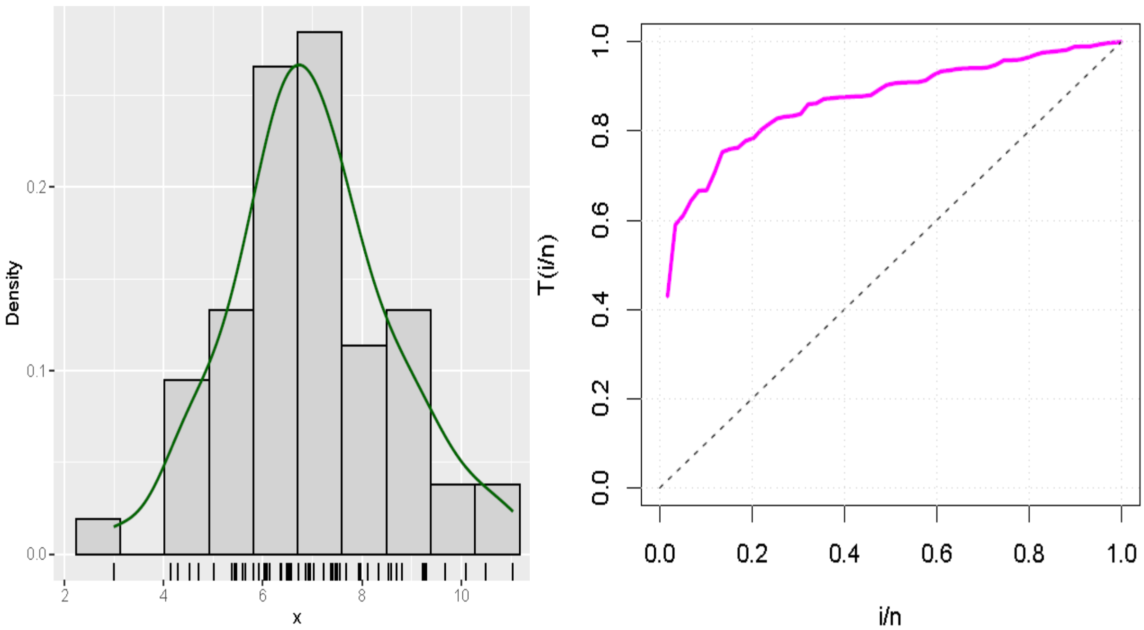

Figure 3: Histogram and TTT plots of dataset-II.

ii) Model Analysis

We have computed some well-known goodness-of-fit statistics to analyze data sets I and II, and the fitted models are evaluated using the log-likelihood value (-2logL), Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (AD), Kolmogrov-Smirnov (KS) with p-values and Cramer-von Mises (CVM). All the essential computations are carried out in R-software (Wickham and Grolemund, 2016). For the comparison of fitting capability, we have selected some models such as Weibull (Weib), Inverse Weibull (IWeib), Inverse Chen (IChen) (Srivastava and Srivastava, 2014), odd Chen Weibull (OChenW) (Anzagra *et al.*, 2022), exponentiated Chen (ExpChen) (Day *et al*., 2017) and exponentiated exponential inverse Weibull (EEIW) (Chaudhary and Sapkota, 2021).

We have presented the KS plots and probability-probability (PP) plots for both datasets in Figures 4 and 5, and our analysis indicates that the suggested model can effectively fit the real datasets. The estimated values of the parameters (Par) and their associated standard errors (SE) for both datasets were presented in Tables 5 and 6, which were calculated using the MLE method. Additionally, Tables 7 and 8 showcase model selection criteria, such as log-likelihood, HQIC, and AIC, and goodness of fit statistics, such as KS, AD, and CVM, for both data sets with p-values p(KS), p(CVM) and p(AD) respectively. Our observations show that the OIChW model has the least statistics compared to the Weib, IWeib, IChen, OChenW, ExpChen, and EEIW distributions, along with corresponding highest p-values, indicating that the OIChW distribution is more flexible and provides a good fit. Furthermore, we have provided graphical illustrations of the fitted models in Figures 6 and 7, which support our findings that the OIChW model outperforms the other candidate models.

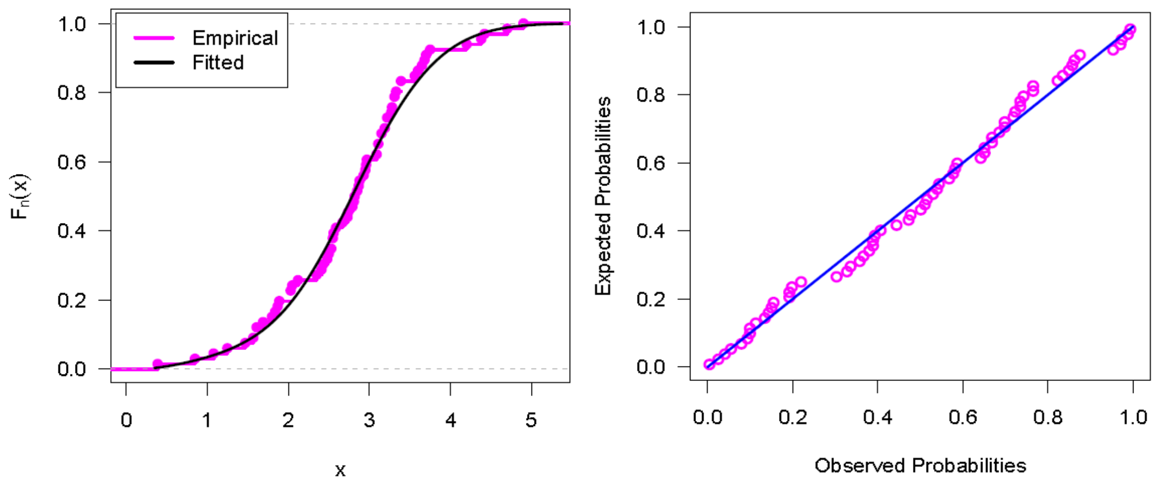

Figure 4: KS and PP plots of OIChW distribution (dataset-I).

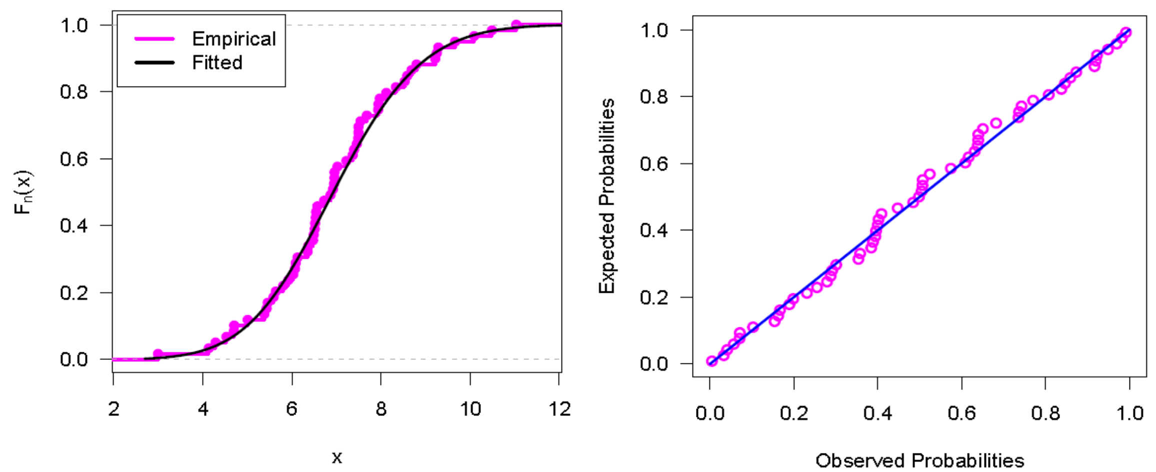

Figure 5: KS and PP plots of OIChW distribution (dataset-II).

Table 5: MLEs with SE (dataset-I)

Model	Par	SE	Par	SE	Par	SE	Par	SE
OICh $(\alpha,\beta,\lambda,\theta)$	1.9790	0.6375	0.0968	0.0453	0.7327	0.7552	2.7612	0.5550
IWeib (α,λ)	1.6480	0.1226	3.2263	0.4193	--	--	--	--
Weib (λ,θ)	3.0623	0.1149	3.4412	0.3305	--	--	--	--
IChen (δ,λ)	1.4331	0.1775	0.9370	0.0655	--	--	--	--
OChenW $(\alpha,\beta,\lambda,\theta)$	2.2404	0.3119	0.2452	0.0494	0.2464	0.0652	2.0315	0.3816
ExpChen (α,λ,θ)	2.3397	0.9614	0.8429	0.0968	0.1404	0.0762	--	--
EEIW $(\alpha,\beta,\delta,\theta)$	11.4994	0.0013	0.7519	0.0013	100.1052	0.0128	0.5945	0.0732

Table 6: MLEs with SE (dataset-II)

Model	Par	SE	Par	SE	Par	SE	Par	SE
OICh $(\alpha,\beta,\lambda,\theta)$	3.8348	2.5596	0.1748	0.2084	0.0639	0.1654	2.6237	0.7950
IWeib (α,λ)	3.0199	0.0000	223.5288	30.5772	--	--	--	--
Weib (λ,θ)	7.6130	0.2231	4.6988	0.4550	--	--	--	--
IChen (δ,λ)	212.6202	8.7208	2.9941	0.0795	--	--	--	--
OChenW $(\alpha,\beta,\lambda,\theta)$	6.3129	0.6961	0.2714	0.0543	0.2663	0.0762	2.6550	0.5419
ExpChen (α,λ,θ)	7.0224	4.0988	0.6005	0.0656	0.1006	0.0669	--	--
EEIW $(\alpha,\beta,\delta,\theta)$	39.3196	0.3554	1.0246	0.0203	124.1549	4.1201	0.7968	0.1455

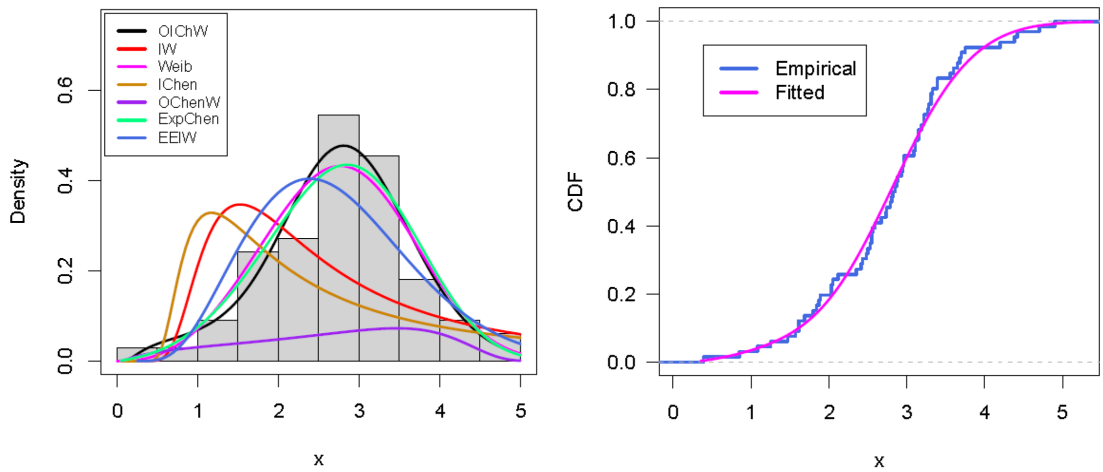

Figure 6: Fitted PDF (left) and fitted CDF vs empirical CDF (right) (dataset-I).

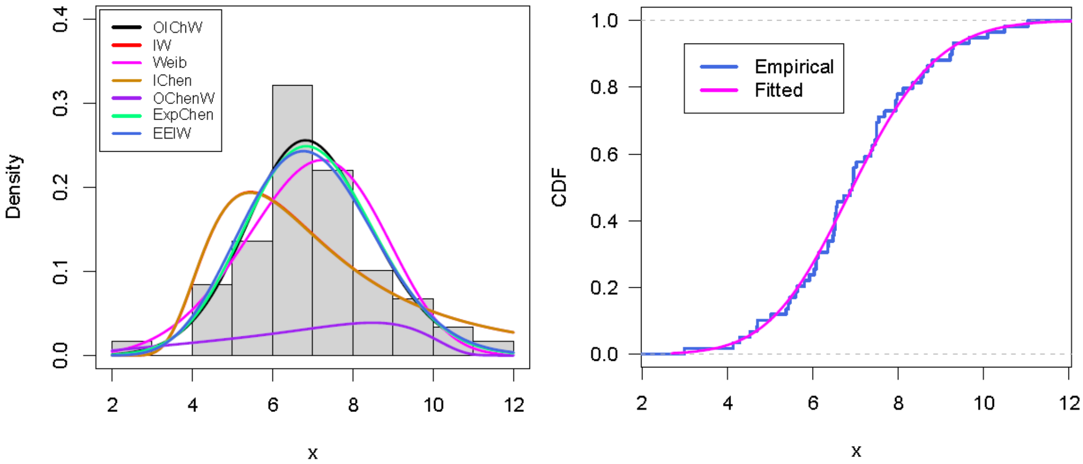

Figure 7: Fitted PDF (left) and fitted CDF vs empirical CDF (right) (dataset-II).

Table 7: Some model selection and goodness-of-fit statistics (dataset-I).

Model	**-2logL**	**AIC**	**HQIC**	**KS**	**p(KS)**	**CVM**	**p(CV)**	**AD**	**p(AD)**
OIChW	169.8343	177.8343	181.2953	0.0684	0.9171	0.0482	0.8889	0.3108	0.9296
IWeib	242.3898	246.3898	248.1203	0.2303	0.0018	1.1562	0.0010	6.5040	6.00E-04
Weib	172.1352	176.1352	177.8656	0.0823	0.7625	0.0837	0.6725	0.4859	0.7606
IChen	275.2038	279.2038	280.9342	0.3066	0.0000	1.8585	0.0000	9.8547	0.0000
OChenW	389.6795	397.6795	401.1405	0.2228	0.0028	0.9500	0.0031	5.3362	0.0020
ExpChen	171.1812	177.1812	179.7769	0.0814	0.7743	0.0698	0.7547	0.4306	0.8174
EEIW	183.5118	191.5118	194.9728	0.1477	0.1121	0.3052	0.1305	1.5421	0.1668

Table 8: Some model selection and goodness-of-fit statistics (dataset-II).

Model	**-2logL**	**AIC**	**HQIC**	**KS**	**p(KS)**	**CVM**	**p(CVM)**	**AD**	**p(AD)**
OIChW	222.3036	230.3036	233.5475	0.0615	0.9690	0.0293	0.9792	0.1703	0.9965
IWeib	251.6153	255.6153	257.2373	0.1664	0.0677	0.5250	0.0341	3.2210	0.0213
Weib	224.9946	228.9946	230.6165	0.0957	0.6177	0.0842	0.6695	0.4779	0.7687
IChen	252.0245	256.0245	257.6465	0.1680	0.0633	0.5349	0.0322	3.2758	0.0200
OChenW	428.6546	436.6546	439.8985	0.2495	0.0010	0.9732	0.0027	5.3844	0.0019
ExpChen	222.5402	228.5402	230.9732	0.0655	0.9477	0.0342	0.9616	0.1958	0.9915
EEIW	223.0528	231.0528	234.2967	0.0628	0.9627	0.0386	0.9415	0.2235	0.9826

6. Conclusion

We have created a new family of distributions called the odd inverse Chen-G family by utilizing the T-X approach. We also provide some key properties of this family of distributions. One member of this family, the OIChW distribution, has a reverse-J, increasing, or inverted bathtub shaped hazard function, which we obtained by using the Weibull distribution as the baseline distribution. We have examined some statistical characteristics of this distribution and estimated its associated parameters through the MLE method. We conducted a Monte Carlo simulation to evaluate the estimation procedure and found that the biases and MSEs decrease as the sample size increases, even for small samples. To apply the OIChW distribution, we used two real medical data sets and compared it with six other existing models using model selection criteria and goodness of fit test statistics. Our results showed that the OIChW distribution performs better than the other models, suggesting that it can be applied in various fields such as medical science, reliability engineering, and survival analysis. Furthermore, this family of distributions can be used to create new models in the future.

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