ISSN 09710388

Aligarh Journal of Statistics
Vol. 44(2024), 181-194

Multivariate Weighted Ratio Product Estimator Using Harmonic Mean

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[Received on January, 2023. Accepted on June, 2024]

ABSTRACT

In this paper, a ratio-cum-product estimator of the mean of a finite population is proposed using a number of auxiliary variables, some of which are positive and some negatively correlated with the study variable. Expressions for the bias and the mean squared error (MSE) of the estimator are obtained, and the conditions determining the sign of the bias are derived, as well as those under which the estimator is more efficient than the sample mean. A simple random sample drawn from a finite multivariate population is used to illustrate the determination of the optimum weights and the gain in efficiency obtained over the situations where only positively correlated auxiliary variables or only negatively correlated variables are used in the estimator.

1. Introduction

In sample surveys, it is usual to make use of auxiliary information to increase the precision of estimators. The classical ratio estimator is widely used when the correlation between the character under study, *Y*, and the auxiliary character, *X*, is positive. If this correlation is negative, a product estimator may be used instead of a ratio estimator. In large-scale sample surveys, data are often collected on more than one auxiliary character; some characters may be positively and others negatively correlated with Y. Olkin (1958) considered an estimator that utilizes information on several auxiliary characters that are positively correlated with the character under study. He used a linear combination of ratio estimators based on each auxiliary character separately. The linear combination coefficients were determined to minimize the estimator's variance. Since then, several other estimators have been considering using the information on more than one auxiliary character. Srivastava (1965) generalized Olkin's multivariate ratio estimator to the case when some of the auxiliary characters are positively and others negatively correlated with the character under study. The same results were obtained by Rao and Mudholkar (1965). Later, Raj (1965), Shukla (1966), Singh (1967b), Srivastava (1967) and John (1969) considered some other estimators which were linear combinations of several estimators based on each auxiliary character separately. Bahl and Tuteja (1991) suggested alternative multivariate product estimators which were geometric and harmonic means of product estimators based on individual auxiliary characters, to reduce the bias or mean squared errors. Building on the ideas of Olkin, Rao and Mudholkar, we suggest in this paper an alternative multivariate ratio-cum-product estimator which is a linear combination of estimators based on individual auxiliary variables. In the case of the negatively correlated auxiliary variables, we have used harmonic means instead of the arithmetic mean for both sample and population. It may be noted

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that when *X* is negatively correlated with *Y*, the classical ratio estimator of the population mean of Y using 1/*X* as an auxiliary variable (having a positive correlation with *Y*) equals the classical *product* estimator of the population mean of *Y* using the sample and population *harmonic* means of *X* instead of the arithmetic means.

2. Prevailing product and Ratio-cum-Product Estimators

In this section, we list some popular product and ratio-cum-product estimators making use of at least one auxiliary variable negatively correlated with the study variable, along with their biases and mean squared errors.

2.1 Estimators using a Single Auxiliary Variable

Estimators using a single auxiliary variable and with their biases and MSEs to the first degree of approximation, i.e., to the order $n\ (o)^{-1}$, are listed below, where the symbols have the usual meanings.

1. Conventional product estimator due to Murthy

$$\bar{y}_p = \bar{y}\frac{\bar{x}}{\bar{X}} \quad (2.1.1)$$

2. Other product and product type estimators

   i) exponential product estimator due to Bahl and Tuteja

$$\bar{y}_{pe} = \bar{y}\ exp\left[\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right] \quad (2.1.2)$$

   ii) predictive product estimator due to Srivastava (1983)

$$\bar{y}_p' = \frac{n\bar{y}}{N} + \frac{(N-n)^2\tilde{x}\bar{y}}{N(N\tilde{X}-n\tilde{x})} \quad (2.1.3)$$

   iii) predictive product estimator due to Agarwal and Jain (1989) with a harmonic mean of x

$$\bar{y}_p'' = \bar{y}\frac{\tilde{x}}{\tilde{X}} \quad (2.1.4)$$

Let $\theta = \frac{N-n}{Nn}$, and let C_0, C_1 denote the coefficients of variation of the variables Y, X, and ρ the correlation coefficient between Y and X.

The variance of the ordinary sample mean, and the biases and MSEs of the estimators given at (2.1.1), (2.1.2), (2.1.3), and (2.1.4) are as follows:

i) The variance of the sample mean

$$V(\bar{y}) = \theta\bar{Y}^2C_0^2 \quad (2.1.5)$$

ii) For the classical product estimator

$$B(\bar{y}_p) = \theta\bar{Y}(\rho C_0 C_1) \quad (2.1.6)$$

$$M(\bar{y}_p) = \theta\bar{Y}^2(C_0^2 + C_1^2 + 2\rho C_1 C_0) \quad (2.1.7)$$

iii) For the exponential product estimator due to Bahl and Tuteja

$$B(\bar{Y}_{pe}) = \theta\bar{Y}\left(\rho\frac{C_1 C_0}{2} - \frac{C_0^2}{8}\right) \tag{2.1.8}$$

$$M(\bar{Y}_{pe}) = \theta\bar{Y}^2\left(C_0^2 + \frac{1}{4}C_1^2 + \rho C_0 C_1\right) \tag{2.1.9}$$

iv) For the predictive product estimator due to Srivastava

$$B(\bar{y}_p') = \theta\,\bar{Y}(\rho C_0 C_1 + \frac{n}{N-n}C_1^2) \tag{2.1.10}$$

$$M(\bar{y}_p') = \theta\bar{Y}^2(C_0^2 + C_1^2 + 2\rho C_0 C_1) \tag{2.1.11}$$

v) For the predictive product estimator due to Agarwal and Jain

$$B(\bar{y}_p'') = \theta\bar{Y}(\rho C_0 C_1 + C_1^2) \tag{2.1.12}$$

$$M(\bar{y}_p'') = \theta\bar{Y}^2(C_0^2 + C_1^2 + 2\rho C_0 C_1) \tag{2.1.13}$$

2.2 Estimators using Two Auxiliary Variables

To estimate the unknown $\bar{Y}$, Singh (1965, 1967a) introduced the following estimators:

a) $$\bar{y}_p^* = \left(\frac{\bar{y}\bar{x}_1\bar{x}_2}{\bar{X}_1\bar{X}_2}\right) \tag{2.2.1}$$

b) $$\bar{y}_r^* = \left(\frac{\bar{y}}{\bar{x}_1\bar{x}_2}\right)(\bar{X}_1\bar{X}_2) \tag{2.2.2}$$

c) $$\bar{y}_{rp} = \left(\frac{\bar{y}\bar{X}_1\bar{x}_2}{\bar{x}_1\bar{X}_2}\right) \tag{2.2.3}$$

vi) The bias and mean squared error of each of Singh's product type estimators are

$$B(\bar{y}_{p^*}) = \theta\bar{Y}\,(\rho_{01}C_0C_1 + \rho_{02}C_0C_2 + \rho_{12}C_1C_2) \tag{2.2.4}$$

$$M(\bar{y}_{p*}) = \theta\bar{Y}^2(C_0^2 + C_1^2 + C_2^2 + 2\rho_{01}C_0C_1 + 2\rho_{02}C_0C_2 + 2\rho_{12}C_1C_2) \tag{2.2.5}$$

vii) The bias and mean squared error of each of Singh's ratio type estimators are

$$B(\bar{y}_r^*) = \theta\bar{Y}(C_1^2 + C_2^2 + \rho_{01}C_0C_1 + \rho_{02}C_0C_2 + \rho_{12}C_1C_2) \tag{2.2.6}$$

$$M(\bar{y}_r^*) = \theta\bar{Y}^2(C_0^2 + C_1^2 + C_2^2 - 2\rho_{01}C_0C_1 - 2\rho_{02}C_0C_2 + 2\rho_{12}C_1C_2) \tag{2.2.7}$$

viii) The bias and mean squared error of Singh's ratio-cum-product estimators are

$$B(\bar{y}_{rp}) = \theta\,\bar{Y}(C_1^2 - \rho_{01}C_0C_1 - \rho_{12}C_1C_2 + \rho_{02}C_0C_2) \tag{2.2.8}$$

$$M(\bar{y}_{rp}) = \theta\bar{Y}^2(C_0^2 + C_1^2 + C_2^2 - 2\rho_{01}C_0C_1 - 2\rho_{12}C_1C_2 + 2\rho_{02}C_0C_2) \tag{2.2.9}$$

where C_i, $i = 0, 1, 2$, denote the coefficients of variation of the variables Y, X_1, X_2 respectively, ρ_{ij} is the correlation coefficient between the corresponding variables, and $\theta = \frac{N-n}{Nn}$.

$$C_0^2 = \frac{1}{N-1}\sum_{i=1}^{N}\frac{(y_i-\bar{Y})^2}{\bar{Y}^2},\; C_1^2 = \frac{1}{N-1}\sum_{i=1}^{N}\frac{(x_{1i}-\bar{X}_1)^2}{\bar{X_1}^2},\; C_2^2 = \frac{1}{N-1}\sum_{i=1}^{N}\frac{(x_{2i}-\bar{X}_2)^2}{\bar{X_2}^2}$$

3. Proposed Estimator and its Properties

Consider a population of size *N* consisting of the variables $(y; x_1, x_2, \dots\dots, x_k)$ where *y* is positively correlated with $(x_1, x_2, \dots\dots, x_k)$ and negatively correlated with $(x_{k+1}, x_{k+2}, \dots\dots, x_m)$.

Let $(y_l, x_{1.l}, \dots\dots, x_{m.l})$ $(l = 1, \dots, n)$ be a sample of size *n* drawn through simple random sampling without replacement (SRSWOR) from this population. Let $\bar{X}_i$ and $\tilde{X}_i$, respectively, be the population arithmetic and harmonic means of the auxiliary variables, which are assumed to be known. Let $\bar{y}$ and $\bar{x}_i$'s $(i = 1, \dots, m)$ be the sample means of the study variable *Y* and the auxiliary variables X_i's, respectively.

In the above situation Rao and Mudholkar (1965) suggested an estimator of $\bar{Y}$ as:

$$\hat{\bar{Y}} = \sum_i^k W_i \frac{\bar{X}_i}{\bar{x}_i}\bar{y} + \sum_{i=k+1}^m W_i \frac{\bar{x}_i}{\bar{X}_i}\bar{y} \tag{3.0.1}$$

Using the idea pioneered by Olkin (1958), followed by Rao and Mudholkar (1965), Singh (1967b), Tuteja and Bahl (1991), and motivated by the works of Agarwal and Jain (1989), Malik and Singh (2012) and recently by Panda and Sen (2018), we propose an estimator for the population mean $\bar{Y}$ defined as:

$$\bar{y}^*_{wrp} = \sum_i^k W_i \frac{\bar{X}_i}{\bar{x}_i}\bar{y} + \sum_{i=k+1}^m W_i \frac{\tilde{x}_i}{\tilde{X}_i}\bar{y} \tag{3.0.2}$$

$$= \bar{y}\left[\sum_{i=1}^k \frac{W_i \bar{X}_i}{\bar{x}_i} + \sum_{i=k+1}^m \frac{W_i \tilde{x}_i}{\tilde{X}_i}\right]$$

$$= \bar{y}_{ar} + \bar{y}_{hp} \text{ (say)}, \tag{3.0.3}$$

where $w = (w_1, w_2, \dots\dots, w_m)$, $\sum_{i=1}^m w_i = 1$, is a weighting function and $\bar{X}_i > 0$ and are known. $\bar{y}_{ar}$ equals the sum of the first *k* terms of equation (3.0.2) and is analogous to the multivariate ratio estimator proposed by Olkin (1958); it is based on the $x's$ that are positively correlated with y. $\bar{y}_{hp}$ equals the sum of the last m terms in equation (3.0.2) and corresponds to the $x's$ that are negatively correlated with y. $\bar{y}_{hp}$ may be called a multivariate product type estimator.

3.1 Properties of the Proposed Estimator When m=2 and k=1

Now to proceed with our investigation we initially consider the case when $k = 1$ and $m = 2$, so that (3.2) takes the form

$$\bar{y}^*_{wrp} = \bar{y}\left[w_1\left(\frac{\bar{X}_1}{\bar{x}_1}\right) + w_2\left(\frac{\tilde{x}_2}{\tilde{X}_2}\right)\right] \tag{3.1.1}$$

where $\bar{x}_1$ is the arithmetic mean per unit of x_1 and $\tilde{x}_2$ is the harmonic mean per unit of x_2,

$$\bar{x}_1 = \sum_{i=1}^n \frac{x_{1i}}{n},\ \bar{X}_1 = \sum_{i=1}^N \frac{X_{1i}}{N},\ \bar{y} = \frac{\sum_{i=1}^n y_i}{n},\ \tilde{x}_2 = \frac{n}{\sum_{i=1}^n \frac{1}{x_{2i}}},\ \tilde{X}_2 = \frac{N}{\sum_{i=1}^N \frac{1}{X_{2i}}},$$

$$w_1 + w_2 = 1 \text{ and } x_i > 0$$

To obtain the asymptotic properties of the estimators, we define

$$e_0 = \frac{\bar{y}-\bar{Y}}{\bar{Y}},\ e_1 = \frac{\bar{x}_1-\bar{X}_1}{\bar{X}_1},\ e_2 = \frac{\bar{x}_2-\bar{X}_2}{\bar{X}_2},$$

$$\epsilon_{2i} = \frac{x_{2i}-\bar{X}_2}{\bar{X}_2}, i = 1, \ldots, n, \quad \partial_{2i} = \frac{X_{2i}-\bar{X}_2}{\bar{X}_2}, i = 1, \ldots, N$$

Then $\frac{1}{n}\sum_{i=1}^{n}\epsilon_{2i} = \frac{1}{n\bar{X}_2}\sum_{i=1}^{n}(x_{2i}-\bar{X}_2) = \frac{\bar{x}_2-\bar{X}_2}{\bar{X}_2} = e_2$

$$\frac{1}{N}\sum_{i=1}^{N}\partial_{2i} = \frac{1}{N\bar{X}_2}\sum_{i=1}^{N}(X_{2i}-\bar{X}_2) = 0$$

$\frac{1}{N}\sum_{i=1}^{N}{\partial_{2i}}^2 = \frac{1}{N{\bar{X}_2}^2}\sum_{i=1}^{N}(X_{2i}-\bar{X}_2)^2 = \frac{\sigma_2^2}{{\bar{X}_2}^2}$ where σ_2^2 is the variance of X_2

$\frac{1}{n}\sum_{i=1}^{n}{\epsilon_{2i}}^2 = \frac{1}{n{\bar{X}_2}^2}\sum_{i=1}^{n}(x_{2i}-\bar{X}_2)^2 = z_2^2$, say

Since $E(x_{2i}-\bar{X}_2)^2 = \sigma_2^2$, $E(z_2^2) = \frac{n\sigma_2^2}{n{\bar{X}_2}^2} = \frac{\sigma_2^2}{{\bar{X}_2}^2}$

Next, we use the above error terms to derive the bias and the mean squared error of the estimators, noting that the following holds:

$$E(e_0) = 0,\ E(e_0^2) = \theta\frac{S_0^2}{\bar{Y}^2} = \theta C_0^2,\ S_0^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i-\bar{Y})^2$$

and for $r \geq 1$

$E(e_r) = 0, E(e_r^2) = \theta\frac{S_i^2}{\bar{X}_r^2} = \theta C_r^2, E(e_0e_r) = \theta\frac{S_{0r}}{\bar{Y}\bar{X}_r} = \theta\rho_{0r}C_0C_r,$ where

$$S_r^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_{ri}-\bar{X}_r)^2,\ S_{0r} = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i-\bar{Y})(X_{ri}-\bar{X}_r),$$

$C_0^2 = \frac{S_0^2}{\bar{Y}^2}$, $C_r^2 = \frac{S_i^2}{\bar{X}_r^2}$ for r≥1,

$\rho_{0r} = \frac{S_{0r}}{S_0S_r}$ = correlation coefficient between Y and X_r,

$\rho_{rs}\ (r, s \geq 1, r \neq s) = \frac{S_{rs}}{S_rS_s}$ = correlation coefficient between X_r and X_s, and $\theta = \frac{N-n}{Nn}$

Assuming $|\frac{n}{N-n}e_i| < 1$ and using Taylor series expansion, one gets the bias and mean squared error up to the first-degree approximation of $O(n)^{-1}$ for the estimators discussed in the previous section. It is further assumed that the sample is large enough to make $|e_i|$ (i=0, 1, 2) and $|\epsilon_{2i}|$ and $|\partial_{2i}|$ so small that the terms involving squares and higher powers of e_i (i=0, 1, 2), ∂_{2i} and ϵ_{2i} are negligible. Substituting the expression for $\bar{y}$, $\bar{x}_1$ and $\tilde{x}_2$ in terms of e_i (i=0, 1, 2) and ϵ_{2i}'s, (3.1.1) becomes

$$\bar{y}_{wrp}^* = \bar{Y}(1+e_0)\left[w_1(1+e_1)^{-1} + w_2\frac{n}{N}\left(\frac{\sum_{i=1}^{N}(1+\partial_{2i})^{-1}}{\sum_{i=1}^{n}(1+\epsilon_{2i})^{-1}}\right)\right] \quad (3.1.2)$$

$$\frac{n}{N}\left(\frac{\sum_{i=1}^{N}(1+\partial_{2i})^{-1}}{\sum_{i=1}^{n}(1+\epsilon_{2i})^{-1}}\right) = \left(1-\frac{\sum_1^N\partial_{2i}}{N}+\frac{\sum_1^N{\partial_{2i}}^2}{N}-\cdots\right)\left(1-\frac{\sum_1^n\epsilon_{2i}}{n}+\frac{\sum_1^n{\epsilon_{2i}}^2}{n}-\cdots\right)^{-1}$$

$$= \left(1-\frac{\sum_1^N\partial_{2i}}{N}+\frac{\sum_1^N{\partial_{2i}}^2}{N}-\cdots\right)\left(1-\frac{\sum_1^n\epsilon_{2i}}{n}+\frac{\sum_1^n{\epsilon_{2i}}^2}{n}-\cdots\right)^{-1}$$

$$= (1+C_2^2-\cdots)(1-e_2+z_2^2-\cdots)^{-1}$$

$$= (1 + C_2^2 - \cdots)(1 + e_2 - z_2^2 + e_2^2 \ldots)$$
$$= 1 + e_2 - z_2^2 + e_2^2 + C_2^2 + \ldots$$

So,

$$\bar{y}_{wrp}^* = \bar{Y}(1 + e_0)\{w_1(1 - e_1 + e_1^2) + w_2(1 + e_2 - z_2^2 + e_2^2 + C_2^2)\}$$
$$= \bar{Y}\{w_1(1 - e_1 + e_1^2 + e_0 - e_0e_1) + w_2(1 + e_2 - z_2^2 + e_2^2 + C_2^2 + e_0 + e_0e_2)\}$$
$$= \bar{Y}\{w_1 + w_2 + (w_1 + w_2)e_0 - w_1e_1 + w_2e_2 + w_1e_1^2 + w_2e_2^2 - w_1e_0e_1 + w_2e_0e_2 - w_2z_2^2 + w_2C_2^2\}$$
$$= \bar{Y}\ (1 + e_0 - w_1e_1 + w_2e_2 - w_1e_0e_1 + w_2e_0e_2 + w_1e_1^2 + w_2e_2^2 - w_2z_2^2 + w_2C_2^2) \tag{3.1.3}$$

The bias of the proposed estimator, up to the first degree of approximation, is

$$B(\bar{y}_{wrp}^*) = E(\bar{y}_{wp}^* - \bar{Y})$$
$$\cong \bar{Y}E(e_0 - w_1e_1 + w_2e_2 - w_1e_0e_1 + w_2e_0e_2 + w_1e_1^2 + w_2e_2^2 - w_2z_2^2 + w_2C_2^2)$$
$$\cong \theta\bar{Y}\ (-w_1\rho_{01}C_0C_1 + w_2\rho_{02}C_0C_2 + w_1C_1^2 + w_2C_2^2 - w_2C_2^2 + w_2C_2^2)$$

(as $E(z_2^2) = \frac{\sigma_2^2}{\bar{X}_2{}^2} = (\frac{N-1}{N})\frac{S_2^2}{\bar{X}_2^2} \cong C_2^2$)

$$\cong \theta\bar{Y}\ (-w_1\rho_{01}C_0C_1 + w_2\rho_{02}C_0C_2 + w_1C_1^2 + w_2C_2^2)$$
$$\cong \theta\bar{Y}\ (-w_1|\rho_{01}|C_0C_1 - w_2|\rho_{02}|C_0C_2 + w_1C_1^2 + w_2C_2^2)$$
$$\cong \theta\bar{Y}\{-\textstyle\sum_{r=1}^2 w_r|\rho_{0r}|C_0C_r + \sum_{r=1}^2 w_rC_r^2\} \tag{3.1.4}$$

And the MSE of the estimator is obtained as

$$M(\bar{y}_{wrp}^*) = E(\bar{y}_{wrp}^* - \bar{Y})^2$$
$$= E\{(1 + e_0 - w_1e_1 + w_2e_2 - w_1e_0e_1 + w_2e_0e_2 + w_1e_1^2 + w_2e_2^2 - w_2z_2^2 + w_2C_2^2 - 1)\bar{Y}\}^2$$
$$\cong \bar{Y}^2E(e_0^2 + w_1^2e_1^2 + w_2^2e_2^2 - 2w_1e_0e_1 + 2w_2e_0e_2 - 2w_1w_2e_1e_2)$$
$$\cong \theta\bar{Y}^2(C_0^2 + w_1^2C_1^2 + w_2^2C_2^2 - 2w_1\rho_{01}C_0C_1 + 2w_2\rho_{02}C_0C_2 - 2w_1w_2\rho_{12}C_1C_2) \tag{3.1.5}$$
$$\cong \theta\bar{Y}^2(C_0^2 + w_1^2C_1^2 + (1 - w_1)^2C_2^2 - 2w_1\rho_{01}C_0C_1 + 2(1 - w_1)\rho_{02}C_0C_2 - 2w_1(1 - w_1)\rho_{12}C_1C_2)$$
$$\cong \theta\bar{Y}^2(C_0^2 + w_1^2C_1^2 + (1 - 2w_1 + w_1^2)C_2^2 - 2w_1\rho_{01}C_0C_1 + 2(1 - w_1)\rho_{02}C_0C_2 - 2w_1(1 - w_1)\rho_{12}C_1C_2) \tag{3.1.6}$$

3.1.1 Choice of Optimum Weights When m=2

Differentiating (3.1.6) with respect to w_1, and setting the first derivative equal to zero, we obtain the optimum values of w_1, w_2 as follows:

$$\frac{\partial MSE(\bar{y}^*_{wrp})}{\partial w_1} = 0$$
$$\Leftrightarrow 2w_1C_1^2 - 2C_2^2 + 2w_1C_2^2 - 2\rho_{01}C_0C_1 - 2\rho_{02}C_0C_2 - 2\rho_{12}C_1C_2 + 4w_1\rho_{12}C_1C_2 = 0$$
$$\Leftrightarrow w_1C_1^2 - C_2^2 + w_1C_2^2 - \rho_{01}C_0C_1 - \rho_{02}C_0C_2 - \rho_{12}C_1C_2 + 2w_1\rho_{12}C_1C_2 = 0$$
$$\Leftrightarrow w_1(C_1^2 + C_2^2 + 2\rho_{12}C_1C_2) = \rho_{01}C_0C_1 + \rho_{02}C_0C_2 + 2\rho_{12}C_1C_2 + 2C_2^2$$
$$\Leftrightarrow w_{1opt} = \frac{\rho_{01}C_0C_1+\rho_{02}C_0C_2+\rho_{12}C_1C_2+C_2^2}{C_1^2+C_2^2+2\rho_{12}C_1C_2} = 1 - w_{2opt} \qquad (3.1.7)$$

3.2 Properties of the Proposed Estimator When m=4 and k=2

In this section, we study the properties of (3.2) with 4 auxiliary variables where two of the auxiliary variables x_1 and x_2 are positively correlated with the study variable and the other two auxiliary variables x_3 and x_4 are negatively correlated with it. Under this situation, (3.2) reduces to the form below:

$$\bar{y}^*_{wrp} = \sum_{i=1}^{2} w_i \frac{\bar{X}_i}{\bar{x}_i}\bar{y} + \sum_{i=3}^{4} w_i \frac{\tilde{x}_i}{\tilde{X}_i}\bar{y} \qquad (3.2.1)$$

or,

$$\bar{y}^*_{wrp} = \bar{y}\left(w_1\frac{\bar{X}_1}{\bar{x}_1} + w_2\frac{\bar{X}_2}{\bar{x}_2} + w_3\frac{\tilde{x}_3}{\tilde{X}_3} + w_4\frac{\tilde{x}_4}{\tilde{X}_4}\right) \qquad (3.2.2)$$

where $\sum_{i=1}^{4} w_i = 1$, and where $\bar{x}_1$ and $\bar{x}_2$ are respectively the arithmetic means per unit of x_1 and x_2, and $\tilde{x}_3$ and $\tilde{x}_4$ are respectively the harmonic means per unit of x_3 and x_4.

Now expressing (3.2.2) in terms of $e_i{}'$s we obtain

$$\bar{y}^*_{wrp} = \bar{Y}(1+e_0)\left[\{w_1(1+e_1)^{-1} + w_2(1+e_2)^{-1}\} + \left\{w_3\frac{n}{N}\left(\frac{\sum_{i=1}^{N}(1+\partial_{3i})^{-1}}{\sum_{i=1}^{n}(1+\epsilon_{3i})^{-1}}\right)\right\} + \left\{w_4\frac{n}{N}\left(\frac{\sum_{i=1}^{N}(1+\partial_{4i})^{-1}}{\sum_{i=1}^{n}(1+\epsilon_{4i})^{-1}}\right)\right\}\right]$$
$$= \bar{Y}(1+e_0)\{w_1(1-e_1+e_1^2) + w_2(1-e_2+e_2^2) + w_3(1+e_3-z_3^2+e_3^2+C_3^2) + w_4(1+e_4-z_4^2+e_4^2+C_4^2)\}$$

where, for r=3 and 4, $z_r^2 = \frac{1}{n\bar{X}_r{}^2}\sum_{i=1}^{n}(x_{ri}-\bar{X}_r)^2 = z_r^2$, $E(z_r^2) = \frac{n\sigma_r^2}{n\bar{X}_r{}^2} = \frac{\sigma_r^2}{\bar{X}_r{}^2}$

$$= \bar{Y}(1+e_0)\{w_1 - e_1w_1 + w_1e_1^2 + w_2 - e_2w_2 + w_2e_2^2 + w_3 + w_3e_3 - w_3z_3^2 + w_3e_3^2 + w_3C_3^2 + w_4 + w_4e_4 - w_4z_4^2 + w_4e_4^2 + w_4C_4^2\}$$

since, for r=3 and 4, $E(z_r^2) = \frac{\sigma_r^2}{\bar{X}_r{}^2} = (\frac{N-1}{N})\frac{S_r^2}{\bar{X}_r^2} \cong C_r^2$

Which reduces simplification to

$$\bar{y}^*_{wrp} = \bar{Y}(1 + e_0 - \sum_{r=1}^{2} t_rw_r + \sum_{r=3}^{4} s_rw_r) \qquad (3.2.3)$$

where $t_r = (e_r + e_0e_r - e_r^2)$ for $r = 1, 2$
and $s_r = (e_r + e_0e_r + e_r^2) + C_r^2 - z_r^2$ for $r = 3, 4$

3.2.1 Bias and MSE

The bias of the proposed estimator, up to the first degree of approximation, is

$$B(\bar{y}^*_{wrp}) = E(\bar{y}^*_{wp} - \bar{Y}) \cong \bar{Y}E(e_0 - \sum_{r=1}^{2} t_r w_r + \sum_{r=3}^{4} s_r w_r)$$
$$= \bar{Y}\{E(-\sum_{r=1}^{2} t_r w_r + \sum_{r=3}^{4} s_r w_r)\}$$
$$= \bar{Y}\{-\sum_{r=1}^{2} w_r E(e_0 e_r - e_r^2) + \sum_{r=3}^{4} w_r E(e_0 e_r + e_r^2 + C_r^2 - z_r^2)$$
$$= \bar{Y}\{-\sum_{r=1}^{2} w_r E(e_0 e_r - e_r^2) + \sum_{r=3}^{4} w_r E(e_0 e_r + e_r^2)$$

(since $E(z_r^2) = C_r^2$)

$$= \bar{Y}\{-w_r \sum_{r=1}^{2} E(e_0 e_r) + w_r \sum_{r=3}^{4} E(e_0 e_r) + w_r \sum_{r=1}^{4} E(e_r^2)\}$$
$$= \theta\bar{Y}\{-\sum_{r=1}^{2} w_r \rho_{0r} C_0 C_r + \sum_{r=3}^{4} w_r \rho_{0r} C_0 C_r + \sum_{r=1}^{4} w_r C_r^2\}$$
$$= \theta\bar{Y}\{-\sum_{r=1}^{2} w_r |\rho_{0r}| C_0 C_r - \sum_{r=3}^{4} w_r |\rho_{0r}| C_0 C_r + \sum_{r=1}^{4} w_r C_r^2\}$$
$$= \theta\bar{Y}\{-\sum_{r=1}^{4} w_r |\rho_{0r}| C_0 C_r + \sum_{r=1}^{4} w_r C_r^2\} \quad (3.2.4)$$

$$B(\bar{y}^*_{wrp}) >=< 0 \iff \sum_{r=1}^{4} w_r C_r^2 >=< \sum_{r=1}^{4} w_r |\rho_{0r}| C_0 C_r \quad (3.2.5)$$

$$MSE(\bar{y}^*_{wrp}) = \bar{Y}^2 E(e_0 - \sum_{r=1}^{2} t_r w_r + \sum_{r=3}^{4} s_r w_r)^2$$
$$= \bar{Y}^2 E(e_0 - \sum_{r=1}^{2} t_r w_r + \sum_{r=3}^{4} s_r w_r)^2$$
$$= \bar{Y}^2 E\{e_0 - (e_1 + e_0 e_1 - e_1^2) w_1 - (e_2 + e_0 e_2 - e_2^2) w_2 + (e_3 + e_0 e_3 + e_3^2 + C_3^2 - z_3^2) w_3 + (e_4 + e_0 e_4 + e_4^2 + C_4^2 - z_4^2) w_4\}^2$$

Using the fact that $|\rho_{rs}| = \rho_{rs}$ for (r ≤2 and s≤2) or (r ≥3 and s≥3)
and $|\rho_{rs}| = -\rho_{rs}$ for (r ≤2 and s≥3),

and that $E(z_r^2) = C_r^2$ for r=3 and 4, this simplifies to

$$MSE(\bar{y}^*_{wrp}) = \theta\bar{Y}^2 (C_0^2 + \sum_{r=1}^{4} w_r^2 C_r^2 - 2\sum_{r=1}^{4} w_r |\rho_{0r}| C_0 C_r + 2\sum_{\substack{r,s=1 \\ r<s}}^{4} w_r w_s |\rho_{rs}| C_r C_s)$$
$$= \theta\bar{Y}^2 (C_0^2 + \sum_{r=1}^{4} w_r^2 C_r^2 + \sum_{\substack{r,s=1 \\ r\neq s}}^{4} w_r w_s |\rho_{rs}| C_r C_s - 2\sum_{r=1}^{4} w_r |\rho_{0r}| C_0 C_r)$$
$$= \theta\bar{Y}^2 (C_0^2 + \sum_{r=1}^{4} w_r w_s |\rho_{rs}| C_r C_s - 2\sum_{r=1}^{4} w_r |\rho_{0r}| C_0 C_r)$$
$$= \theta\bar{Y}^2 (C_0^2 + w'\Phi w - 2\phi_0^* w) \quad (3.2.6)$$

where $w = (w_1, w_2, w_3, w_4)$ is the weight vector,

$\phi_{rs} = |\rho_{rs}| C_r C_s$ for $r, s = 0, 1, 2, 3, 4$, $\Phi = ((\phi_{rs}))$, $1 \leq r, s \leq 4$
and $\phi_0^* = (\phi_{01}, \ldots, \phi_{04})$

3.3 Properties of the Proposed Estimator in the General Case with k ≥1 +vely Correlated and (m-k) ≥1 -vely Correlated Auxiliary Variables

Clearly, in place of (3.2.3), the estimator in this case may be written

$$\bar{y}^*_{wrp} = \bar{Y}(1 + e_0 - \sum_{r=1}^{k} t_r w_r + \sum_{r=k+1}^{m} s_r w_r)$$

where $t_r = (e_r + e_0 e_r - e_r^2)$ for $r = 1, \ldots, k$
and $s_r = (e_r + e_0 e_r + e_r^2) + C_r^2 - z_r^2$ for $r = k+1, \ldots, m$

and its bias, analogously to (3.2.4), as

$$B(\bar{y}^*_{wrp}) = = \theta\bar{Y}\{-\sum_{r=1}^{m} w_r |\rho_{0r}| C_0 C_r + \sum_{r=1}^{m} w_r C_r^2\} \quad (3.3.1)$$

$$B(\bar{y}^*_{wrp}) >=< 0 \iff \sum_{r=1}^m w_r C_r^2 >=< \sum_{r=1}^m w_r |\rho_{0r}| C_0 C_r \quad (3.3.2)$$

$$MSE(\bar{y}^*_{wrp}) = \theta \bar{Y}^2 (C_0^2 + \sum_{r=1}^m w_r w_s |\rho_{rs}| C_r C_s - 2\sum_{r=1}^m w_r |\rho_{0r}| C_0 C_r)$$

$$= \theta \bar{Y}^2 (C_0^2 + w'\Phi w - 2\phi_0^* w) \quad (3.3.3)$$

where w=($w_1, \dots, w_{k,} w_{k+1}, \dots, w_{m,}$) is the weight vector,

$\phi_{rs} = |\rho_{rs}| C_r C_s$ for $r, s = 0, 1, \dots, k, k+1, \dots m$, $\Phi = ((\phi_{rs}))$, $1 \le r, s \le m$

and ϕ_0^*=($\phi_{01}, \dots, \phi_{0m}$)

3.3.1 Efficiency Compared to the Sample Mean

It is known that the univariate ratio estimator of $\bar{Y}$ based on the auxiliary variable X_i having positive correlation with Y is more efficient than the sample mean if and only if

$$\rho > \frac{1}{2}\frac{C_1}{C_0}$$

which may also be written

$$|\rho| \frac{C_0}{C_1} > \frac{1}{2}$$

(a) From (2.1.5) and (2.1.7), the univariate product estimator of $\bar{Y}$ based on the auxiliary variable X_1 having negative correlation with Y is more efficient than the sample mean if and only if

$$\theta \bar{Y}^2 (C_0^2 + C_1^2 + 2\rho C_0 C_1) < \theta \bar{Y}^2 C_0^2$$

$$\iff C_1^2 + 2\rho C_0 C_1 < 0$$

$$\iff \rho < -\frac{1}{2}\frac{C_1}{C_0}$$

which may also be written

$$|\rho| \frac{C_0}{C_1} > \frac{1}{2}$$

(b) Likewise, we conclude, from (2.1.5) and (3.3.3), that the MSE of the multivariate ratio-cum-product estimator $\bar{y}^*_{wrp}$ of $\bar{Y}$ is more efficient than the sample mean if and only if

$$\theta \bar{Y}^2 (C_0^2 + w'\phi_{rs} w - 2\phi_0^* w) < \theta \bar{Y}^2 C_0^2$$

$$\iff w'\phi_{rs} w - 2\phi_0^* w < 0$$

$$\iff \frac{\phi_0^* w}{w'\Phi w} > \frac{1}{2} \quad (3.3.4)$$

where w=($w_1, \dots, w_{k,} w_{k+1}, \dots, w_{m,}$) is the weight vector,

$\phi_{rs} = |\rho_{rs}| C_r C_s$ for $r, s = 0, 1, \dots, k, k+1, \dots m$, $\Phi = ((\phi_{rs}))$, $1 \le r, s \le m$

and ϕ_0^*=($\phi_{01}, \dots, \phi_{0m}$).

3.3.2 Efficiency Compared to the Classical Product Estimator

Considering the MSEs at (2.1.7) and (3.1.5), the proposed estimator will be more efficient if and only if

$$\text{MSE}(\bar{y}^*_{wrp}) \le \text{MSE}(\bar{y}''_p)$$

$$\iff \theta \bar{Y}^2 (C_0^2 + w_1^2 C_1^2 + w_2^2 C_2^2 - 2w_1 \rho_{01} C_0 C_1 + 2w_2 \rho_{02} C_0 C_2 - 2w_1 w_2 \rho_{12} C_1 C_2) \le$$
$$\theta \bar{Y}^2 (C_0^2 + C_2^2 + 2\rho_{02} C_0 C_2)$$

(using the subscript 2 for the auxiliary variable negatively correlated with Y and 1 for the positively correlated auxiliary variable)

Substituting the optimum values of w_1 and w_2 and simplifying, we get

$$\frac{(\rho_{01}C_0C_1+\rho_{02}C_0C_2+\rho_{12}C_1C_2+C_2^2)^2}{C_1^2+C_2^2+2\rho_{12}C_1C_2} \geq 0$$

$$\Leftrightarrow C_1^2 + C_2^2 + 2\rho_{12}C_1C_2 \geq 0$$

$\Leftrightarrow (C_1 - C_2)^2 + 2(1+\rho_{12})C_1C_2 \geq 0$, which is always true since $\rho_{12} \geq -1$.

3.3.3 Efficiency Compared to the Product Estimators of Srivastava and Agarwal & Jain

The above derivation also shows that the proposed estimator is more efficient than the product estimators (2.1.3) and (2.1.4) due respectively to Srivastava and Agarwal & Jain using a single auxiliary variable negatively correlated with Y, as the MSEs in (2.1.11) and (2.1.13) also have the form

$$M\left(\hat{\bar{Y}}\right) = \theta\bar{Y}^2(C_0^2 + C_2^2 + 2\rho_{02}C_0C_2)$$

3.3.4 Efficiency Compared to the Ratio-cum-Product Estimator due to Singh

Considering the MSEs of the bivariate ratio-cum-product estimator at (2.2.9) and (3.1.5), we see that the proposed estimator $\bar{y}^*_{wrp}$ is more efficient if and only if

$$\theta\bar{Y}^2(C_0^2 + w_1^2C_1^2 + w_2^2C_2^2 - 2w_1\rho_{01}C_0C_1 + 2w_2\rho_{02}C_0C_2 - 2w_1w_2\rho_{12}C_1C_2) < \theta\bar{Y}^2(C_0^2 + C_1^2 + C_2^2 - 2\rho_{01}C_0C_1 - 2\rho_{12}C_1C_2 + 2\rho_{02}C_0C_2)$$

$$\Leftrightarrow (w_1^2C_1^2 + w_2^2C_2^2 - 2w_1\rho_{01}C_0C_1 + 2w_2\rho_{02}C_0C_2 - 2w_1w_2\rho_{12}C_1C_2) < (C_1^2 + C_2^2 - 2\rho_{01}C_0C_1 - 2\rho_{12}C_1C_2 + 2\rho_{02}C_0C_2)$$

Replacing w_1 and w_2 with their respective optimum values, we obtain the condition under which $\bar{y}^*_{wrp}$ is more efficient than the estimator (2.2.3) as

$$(\rho_{01}C_0C_1 + \rho_{02}C_0C_2)^2 + C_1^2(1 + C_2^2) + C_2^2(1 + 2\rho_{02}C_0C_2) > 2C_1^2\rho_{01}C_0C_1 + 3\rho_{12}^2C_1^2C_2^2 \qquad (3.3.5)$$

3.4 Choice of Optimum Weights When m =4 and k =2

To minimize $MSE(\bar{y}^*_{wrp})$ subject to $\sum_{r=1}^{4} w_r$=1 we differentiate

$M(w_1, w_2, w_3, w_4, \lambda) = MSE(\bar{y}^*_{wrp}) - 2\lambda(\sum_{r=1}^{4} w_r$-1) w.r.t. $w_1, w_2, w_3, w_4, \lambda$

where 2λ is a Lagrangian multiplier.

Differentiating w.r.t. w_r and setting the derivative to zero, we get

$$2C_r^2w_r\text{-}2|\rho_{0r}|C_0C_r + 2\sum_{s,s\neq r} w_s|\rho_{rs}|C_rC_s - 2\lambda = 0$$

Hence $MSE(\bar{y}^*_{wrp})$ is minimum/maximum

$\Leftrightarrow\; C_r^2w_r + \sum_{s,s\neq r} w_s|\rho_{rs}|C_rC_s - \lambda = |\rho_{0r}|C_0C_r,\;\; r = 1, \ldots, 4$, and $\sum_{r=1}^{4} w_r$=1 (3.4.1)

that is, when $w_1, w_2, w_3, w_4, \lambda$ satisfy the equations

$$\begin{bmatrix} C_1^2 & |\rho_{12}|C_1C_2 & |\rho_{13}|C_1C_3 & |\rho_{14}|C_1C_4 & -1 \\ |\rho_{12}|C_1C_2 & C_2^2 & |\rho_{23}|C_2C_3 & |\rho_{24}|C_2C_4 & -1 \\ |\rho_{13}|C_1C_3 & |\rho_{23}|C_2C_3 & C_3^2 & |\rho_{34}|C_3C_4 & -1 \\ |\rho_{14}|C_1C_4 & |\rho_{24}|C_2C_4 & |\rho_{34}|C_3C_4 & C_4^2 & -1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} |\rho_{01}|C_0C_1 \\ |\rho_{02}|C_0C_2 \\ |\rho_{03}|C_0C_3 \\ |\rho_{04}|C_0C_4 \\ 1 \end{bmatrix}$$

4. Numerical Study

To illustrate the efficiency of the suggested estimator, an SRSWOR sample of 20 regions was drawn without replacement from 88 regions of India. For this sample, the following data are available from NSS 68th round (2011-12) consumer expenditure survey, Schedule Type 2 (authors' estimated regional averages, derived according to NSS procedures). Summary statistics of the data are given in Table 1.

y: per household non-food expenditure in 30 days

x_1: per household expenditure on miscellaneous goods and services in 30 days

x_2: per household expenditure on consumer services other than conveyance in 30 days

x_3: percentage share of vegetables in consumer expenditure in 30 days

x_4: percentage share of food in consumer expenditure in 30 days

Of these, y, x_1 and x_2 are positively and x_3 and x_4 negatively associated with overall standard of living of a region. The estimator proposed in this paper was used to estimate the average value of Y over all 88 regions using auxiliary information on x_1, x_2, x_3 and x_4.

The matrix $\Phi = ((\phi_{rs}))$, for $1 \le r, s \le 4$, is given in Table 1, and the ϕ_0^* vector in Table 2.

Table 1

0.18948	0.13259	0.11812	0.03086
0.13259	0.12758	0.09361	0.02211
0.11812	0.09361	0.11862	0.02789
0.03086	0.02211	0.02789	0.01912

Table 2

0.1119	0.0905	0.0787	0.02510

Optimum choice of weights: w_1=0.27566, w_2=0.32020, w_3=0.03788, w_4=0.36625

With the optimum weight vector, the MSE of the estimator is estimated as

$\theta\bar{Y}^2(C_0^2 + w'\Phi w - 2\phi_0^* w)$

$= \theta\bar{Y}^2$ (0.07898+ 0.070576 –2*(0.072004))

$= \theta\bar{Y}^2$ (0.005548)

(a) Comparison of the Proposed Estimator with the Sample Mean

The proposed estimator $(\bar{y}_{wrp}^*)$ will be more efficient than the sample mean under the condition

$MSE\ (\bar{y}_{wrp}^*) < \theta\bar{Y}^2 C_0^2$

$\Longrightarrow\ \theta\bar{Y}^2(C_0^2$ + 0.070576 - 0.144008) = $\theta\bar{Y}^2(C_0^2$ - 0.073432) < $\theta\bar{Y}^2 C_0^2$

$\Longrightarrow \frac{MSE(\bar{y}_{whp}^*)}{\theta\bar{Y}^2}$ = 0.005548

and $\frac{MSE(\bar{y})}{\theta\bar{Y}^2}$ = 0.07898

The ratio $\frac{(2X\phi_0^* w - w'\varphi w)}{C_0^2}$ = 0.92975

Therefore, the gain in efficiency over the sample mean
= $(2\phi_0^* w - w'\Phi w)/\ C_0^2$ = 0.073432/0.07898, since C_0^2= 0.07898
= 93.0 %

(b) Special Cases

(i) Comparison with estimator involving only positively correlated auxiliary variables:

The competing estimator here is a special case (ratio estimator) of our general estimator, with the matrix $\Phi = ((\phi_{rs}))$, for 1 ≤ r, s ≤ 2 being

0.18948	0.13259
0.13259	0.12758

and the ϕ_0^* vector being

0.111924	0.09052

In this situation, the optimum weights obtained are w_1=0.31630, w_2=0.68370

With the optimum weight vector, the MSE of the estimator is estimated as
$\theta\bar{Y}^2(C_0^2 + w'\Phi w - 2\phi_0^* w) = 0.020330\ \theta\bar{Y}^2$

Gain in efficiency compared to sample mean
= $(2\phi_0^* w - w'\Phi w)/\ C_0^2$ = 0.058647/0.078979
= 74.3%

(b) Comparison with estimator involving only negatively correlated auxiliary variables:

The competing estimator again is a special case (product estimator) of our general estimator, with the matrix $\Phi = ((\phi_{rs}))$, for $3 \leq r, s \leq 4$ being

0.11862	0.02789
0.02789	0.01912

and the ϕ_0^* vector being

0.078726	0.025102

In this situation, the optimum weights obtained are w_3=0.54730, w_4=0.45270

With the optimum weight vector, the MSE of the estimator is estimated as

$$\theta\bar{Y}^2(C_0^2 + w'\Phi w - 2\phi_0^* w) = 0.023351\ \theta\bar{Y}^2$$

Gain in efficiency compared to the sample mean

$= (2\phi_0^* w - w'\Phi w)/\, C_0^2 =$ 0.055628/0.078979, since C_0^2 = 0.07898

=70.4%

Thus, in the situations we have considered, the composite ratio-product estimator we proposed achieves a gain in efficiency of 93.0% over the sample mean. In comparison, the corresponding ratio estimator based only on the positively correlated auxiliary variables achieves a gain in efficiency of 74.3%, while the corresponding product estimator based only on the negatively correlated auxiliary variables achieves a gain in efficiency of 70.4%.

5. Concluding Remarks

Finding the exact expression for the proposed estimator in terms of the sample values of the study variable and the auxiliary variables would require the substitution of the optimum values of the weight vector in the proposed estimation formula. Since the expressions for the optimum weights are themselves complicated, we have not attempted this, though in the case of two auxiliary variables only, we explicitly derived the elements of the optimum weight vector. However, we have demonstrated how the weights are derived in any practical situation by working out their values and the minimum value of the MSE of our estimator from an actual sample drawn from multivariate data. The minimum MSEs using only the positively and only the negatively correlated auxiliary variables have been derived and compared with this MSE to demonstrate the gain in efficiency when the proposed estimator based on all the auxiliary variables is used. However, it is possible that a weighted geometric mean of ratio and product estimators may perform better than a weighted arithmetic mean under feasible conditions; further research is necessary to investigate this. The possibilities of adapting the estimator under study to situations where the population is stratified prior to sample selection may also be worth exploring.

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