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Estimation of the Parameters for Power Function Distribution Based on Ranked Set Sampling

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ABSTRACT

This paper investigates Bayesian estimation techniques for deriving parameters of the Power Function Distribution (PFD) through the utilization of Ranked Set Sampling (RSS). We present both maximum likelihood and Bayesian approaches for parameter estimation employing RSS. Additionally, we establish asymptotic and bootstrap confidence intervals for the parameters. Bayesian estimators are computed utilizing squared error loss functions, weighted squared error loss functions, and M/Q squared error loss functions employing the Lindley approximation and importance sampling techniques. Furthermore, we forecast future samples based on RSS. Finally, we conduct reliability simulations to compare all proposed Bayesian estimation methods and analyze a real dataset for illustrative purposes.

1. Introduction

Various parametric models hold significance in lifetime analysis and the study of failure processes. Among these, the Power Function Distribution stands out as a simple yet effective model for evaluating component reliability and often exhibits superior fitting characteristics for failure data (Bashir and Khan, 2023).

The Power Function Distribution holds importance due to its frequent application in daily life, with many distributions such as Rayleigh, gamma, and Weibull distributions being related to it. For instance, Sultan *et al*. (2000) discuss its application in assessing the reliability of electrical components and semiconductor devices, suggesting that PFD parametric estimators can be instrumental in simulating the reliability enhancement of complex or repairable systems. Zarrin *et al.* (2013) applied the power function distribution to analyze component failures, employing both maximum likelihood and Bayesian estimation techniques. Additionally, Meniconi and Barry (1996) argued for the superiority of PFD in evaluating the reliability of electrical components compared to exponential, lognormal, and Weibull distributions.

For comprehensive insights into inference using PFD, interested individuals can consult Belzunce *et al.* (1998), Abdul Sathar *et al.* (2015), Abdul-Sathar and Sathyareji (2018), Abdul-Sathar and Athira Krishnan (2019), AbdulSathar (2021), and Abdul-Sathar and Sathya Reji (2022), along with the references therein. Meniconi and Barry (1996) proposed expressions for the cumulative distribution function (CDF) and probability density function (PDF) of the PF distribution, given as follows:

$$f(x;\beta,\alpha)=\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{-(\alpha-1)};0<x<\beta,\alpha>0 \tag{1.1}$$

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$$F(x;\beta,\alpha) = \left(\frac{\beta}{x}\right)^{\alpha}; \quad \alpha > 0, \beta > 0. \tag{1.2}$$

where α and β denote the shape and scale parameters, respectively.

Ranked Set Sampling (RSS) emerges as a cost-effective method that is particularly beneficial when quantifying all sampling units, which is prohibitively expensive. It involves ranking a small subset of units according to the characteristic under study, as initially proposed by McIntyre (1952) for estimating the average pasture yield. RSS has found widespread applications across various domains including agriculture, forestry, sociology, ecology, environmental sciences, and medical studies. McIntyre's (1952, 2005) work validates RSS as an unbiased estimator for population means.

Key references discussing RSS include the theoretical framework by Chen *et al.* (2004) and Latpate *et al.* (2021)'s presentation of Advanced Sampling Methods. Numerous studies have employed RSS to estimate parameters in various distributions. For instance, Joukar *et al.* (2021) applied it to the exponential-Poisson distribution, Obeidat *et al.* (2020) to the Gompertz distribution, Basikhasteh *et al.* (2021) to the bathtub-shaped lifetime distribution, Yang *et al.* (2020) to the log-extended exponential-geometric distribution, Chandra *et al.* (2016) to the lognormal distribution, Chandra and Tiwari (2012) to the location and scale parameters of the lognormal distribution, and Tiwari *et al.* (2015) to the location and scale parameters of the normal distribution. This study aims to compare PFD estimators derived from the RSS scheme with those obtained from the SRS scheme with unequal samples.

The Ranked Set Sampling (RSS) Scheme presents a sampling methodology that offers a more representative sampling of population data compared to methods like Simple Random Sampling (SRS) with an equivalent number of observations. RSS involves the following steps to obtain a sample of size n from a population:

i) Begin by randomly selecting m^2 units from the population, which are then allocated randomly into m sets, each containing m units.
ii) Rank the m units within each set either visually or using a cost-effective method based on the variable of interest.
iii) Measure the smallest ranked unit from the first set, followed by the second smallest ranked unit from the second set, and so forth until the m^{th} smallest unit (the largest) is measured from the last set.
iv) Optionally, repeat this procedure n times to augment the sample size to mn.

McIntyre (1952) initially introduced ranked set sampling (RSS) to estimate mean pasture yields, highlighting its superior efficiency over simple random sampling (SRS) for population mean estimation. The one-cycle RSS involves the initial ranking of m samples of size n as follows:

$$\begin{array}{c}
1{:}\, \underline{X_{(1:m)1}}\ X_{(2:m)1} \ldots . X_{(m:m)1} \ \rightarrow\ X_1 = X_{(1:m)1} \\
2{:}\, X_{(1:m)2}\ \underline{X_{(2:m)2}} \ldots . X_{(m:m)2} \ \rightarrow\ X_2 = X_{(2:m)2} \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
n{:}\, X_{(1:m)n}\ X_{(2:m)n} \ldots . \underline{X_{(m:m)n}} \ \rightarrow\ X_n = X_{(m:m)n},
\end{array}$$

where n cycles produce a sample of size A = nm, and $x_{(i;m)j}, i = 1,2,\ldots,m \; and \; j = 1,2,\ldots,n$ denotes the i^{th} order statistics from the j^{th} SRS of size n. The resulting sample is called a one-cycle RSS of size m (Arnold and Balakrishnan (1992)). The joint probability distribution function (PDF) of an RSS is given by

$$f\left(X_{(i:m)j}\right) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{m!}{(i-1)!\,(m-i)!} \left(F\left(x_{(i;m)j}\right)\right)^{i-1} \left[1 - F\left(x_{(i;m)j}\right)\right]^{m-i} f\left(x_{(i:m)j}\right), \tag{1.3}$$

where $X_{(i;m)j} = x_{(i;m)j}, i = 1, 2, \dots, m \; and \; j = 1, 2, \dots, n$ and f and F are the PDF and CDF of a random variable X. The cycle can be repeated n times until $A = nm$ units are quantified.

The subsequent sections of this paper are structured as follows: Section 2 discusses maximum likelihood estimation (MLE), asymptotic, and bootstrap confidence intervals for the PFD using RSS. Section 3 explores Bayesian estimation of PFD parameters, employing various loss functions for parameter estimation. This section includes the application of importance sampling procedures and Lindley approximation methods to simplify the ratio of integrals within the proposed Bayes estimators of the parameters. The section also introduces the Highest Posterior Density (HPD) credible interval. Additionally, it addresses the problem of predicting future sample values from the PFD using RSS. Section 4 evaluates the performance of the estimators through simulations and real-life data samples. Finally, the conclusion is presented in Section 5.

2. Maximum Likelihood Estimations

Maximum Likelihood Estimation (MLE) is a statistical method used to estimate parameters of probability distributions. In MLE, parameters are chosen to maximize the likelihood of observing the given data under the assumed probability distribution. This section discusses the estimation of parameters of the Power Function Distribution (PFD) using the Maximum Likelihood (ML) estimation method under Ranked Set Sampling (RSS).

a) Estimation Based on RSS

Consider a sample $X_{(1;m)1}, X_{(2;m)1}, \dots, X_{(m;m)1}, \dots, X_{(1;m)n}, X_{(2;m)n}, \dots, X_{(m;m)n}$ from the PFD obtained via RSS, with PDF given in (1.1). By substituting equations (1.1) and (1.2) into equation (1.3), the likelihood function of a sample from the PFD using RSS is expressed as:

$$L(\alpha,\beta)_{rss} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{m!\ \alpha \left(x_{(i:m)j}\right)^{\alpha(i-1)} \left[1 - \left(\frac{\left(x_{(i:m)j}\right)}{\beta}\right)^{\alpha}\right]^{m-i}}{(i-1)!\ (m-i)!\, \beta^{\alpha}\ {x_{(i:m)j}}^{(\alpha-1)} \beta^{\alpha(i-1)}}$$

$$\propto \left(\frac{\alpha}{\beta^{\alpha}}\right)^{mn} \frac{\prod_{i=1}^{m} \prod_{j=1}^{n} \left(x_{(i:m)j}\right)^{\alpha(i-1)} \left[1 - \left(\frac{\left(x_{(i:m)j}\right)}{\beta}\right)^{\alpha}\right]^{m-i}}{\prod_{i=1}^{m} \prod_{j=1}^{n} {x_{(i:m)j}}^{(\alpha-1)} \beta^{\alpha(i-1)}}. \tag{2.1}$$

From (2.1), the log-likelihood function of α and β are respectively given by

$$\begin{aligned} l(\alpha,\beta)_{rss} \propto\ & mn \log(\alpha) \\ & - mn\alpha \log(\beta) \\ & + \alpha \sum_{i=1}^{m} \sum_{j=1}^{n} (i-1) \log\left[x_{(i:m)j}\right] \\ & + \sum_{i=1}^{m} \sum_{j=1}^{n} (m-i) \log\left[1 - \left(\frac{\left(x_{(i:m)j}\right)}{\beta}\right)^{\alpha}\right] \\ & - \alpha \sum_{i=1}^{m} \sum_{j=1}^{n} (i-1) \log(\beta) - (\alpha-1) \sum_{i=1}^{m} \sum_{j=1}^{n} (i-1) \log\left(x_{(i:m)j}\right). \end{aligned}$$

The MLE of α and β are obtained by solving the following normal equations:

$$\frac{\partial l(\alpha,\beta)_{rss}}{\partial\alpha} = \frac{mn}{\alpha} - mn\log(\beta) - \frac{1}{2}n(-m\log(\beta)) + m^2\log(\beta) - \sum_{i=1}^{m}\sum_{j=1}^{n}\log\left(x_{(i:m)j}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n}(i-1)\log\left(x_{(i:m)j}\right) - \sum_{i=1}^{m}\sum_{j=1}^{n}(\mathrm{m}-\mathrm{i})\log\left(\frac{x_{(i:m)j}}{\beta}\right)\left(\frac{x_{(i:m)j}}{\beta}\right)^{\alpha}$$

And

$$\frac{\partial l(\alpha,\beta)_{rss}}{\partial\alpha} = -\frac{1}{2}n\alpha\left(-\frac{m}{\beta}+\frac{m^2}{\beta}\right) - \frac{mn\alpha}{\beta} + \sum_{i=1}^{m}\sum_{j=1}^{n}(m-i)\,\frac{\alpha\, x_{(i:m)j}\,\left(\frac{x_{(i:m)j}}{\beta}\right)^{\alpha-1}}{\beta^2}.$$

The MLE of the parameters α and β respectively can then be obtained as the solution of the following normal equations

$$\frac{\partial l(\alpha,\beta)_{rss}}{\partial\alpha} = 0 \tag{2.2}$$

and

$$\frac{\partial l(\alpha,\beta)_{rss}}{\partial\alpha} = 0 \tag{2.3}$$

Thus from (2.3), we have

$$\hat{\beta}_{rss} = X_{(m:m)n}\,; \qquad 0 < x_{(1:m)n} < \cdots < x_{(m:m)n} < \beta \tag{2.4}$$

On substituting (2.4) into (2.2), we obtain:

$$\frac{mn}{\alpha} - mn\log(\beta) - \frac{1}{2}n(-m\log(\beta)) + m^2\log(\beta) - \sum_{i=1}^{m}\sum_{j=1}^{n}\log\left(x_{(i:m)j}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n}(i-1)\log\left(x_{(i:m)j}\right) - \sum_{i=1}^{m}\sum_{j=1}^{n}(\mathrm{m}-\mathrm{i})\log\left(\frac{x_{(i:m)j}}{\beta}\right)\left(\frac{x_{(i:m)j}}{\beta}\right)^{\alpha} = 0.$$

The Maximum Likelihood Estimator (MLE) $\hat{\alpha}$of α can be obtained as a solution of the non-linear equation of the form $g(\alpha) = \alpha$, where:

$$g(\alpha) = mn\Bigg[mn\log\big(x_{(m:m)n}\big) + \frac{1}{2}n\big(-m\log x_{(m:m)n}\big) - m^2\log\big(x_{(m:m)n}\big) - \sum_{i=1}^{m}\sum_{j=1}^{n}\log\big(x_{(i:m)j}\big) - \sum_{i=1}^{m}\sum_{j=1}^{n}(i-1)\log\big(x_{(i:m)j}\big) - \sum_{i=1}^{m}\sum_{j=1}^{n}(m-i)\ \log\left(\frac{x_{(i:m)j}}{x_{(m:m)n}}\right)\left(\frac{x_{(i:m)j}}{x_{(m:m)n}}\right)^{\alpha}\ \Bigg]$$

Let $\hat{\alpha}$ be the ML estimator of α by solving the non-linear equation $g(\alpha) = \alpha$ and then using equation (2.5), the ML estimator of α will be given by

$$\hat{\beta}_{rss} = X_{(m:m)n}\,; \qquad 0 < x_{(1:m)n} < \cdots < x_{(m:m)n} < \beta \qquad (2.5)$$

i) Asymptotic Confidence Interval

The analysis of confidence intervals for the parameters and the asymptotic properties of the Maximum Likelihood Estimator (MLE) was conducted as Lawless (1982) outlined. The observed Fisher information matrix is defined as the matrix of second partial derivatives of the negative log-likelihood with respect to the model parameters, given by:

$$\hat{\theta} = (\hat{\alpha}, \hat{\beta}) \sim MVN\big(\theta, I^{-1}(\alpha,\beta)\big),$$

where

$$I(\theta) = I(\alpha,\beta) = -\begin{bmatrix} \frac{\partial^2 l(\alpha,\beta)_{rss}}{\partial\alpha^2} & \frac{\partial^2 l(\alpha,\beta)_{rss}}{\partial\alpha\,\partial\beta} \\ \frac{\partial^2 l(\alpha,\beta)_{rss}}{\partial\beta\ \partial\alpha} & \frac{\partial^2 l(\alpha,\beta)_{rss}}{\partial\beta^2} \end{bmatrix},$$

Hence, a $100(1-\gamma)^0/_0$ confidence interval of the model parameters is given by:

$$\left(\hat{\alpha}_{rss} - z_{\gamma/2}\sqrt{A^*_{11}}, \qquad \hat{\alpha}_{rss} + z_{\gamma/2}\sqrt{A^*_{11}}\ \right)$$

and

$$\left(\hat{\beta}_{rss} - z_{\gamma/2}\sqrt{A^*_{22}}, \qquad \hat{\beta}_{rss} + z_{\gamma/2}\sqrt{A^*_{22}}\ \right)$$

Additionally, the coverage probability is defined as:

$$CP_{rss} = P\left[\left|\frac{(\hat{\theta}_{rss}-\theta_{rss})}{\sqrt{V\hat{a}r\,(\hat{\theta}_{rss})}}\right| \le z_{\gamma/2}\right],$$

where A^*_{11} and A^*_{22} are the diagonal elements of the observed Fisher information matrix $I^{-1}(\alpha, \beta)$, and $z_{\gamma/2}$ is the $(\gamma/2)^{th}$quantile of the standard normal distribution.

b) Estimation Based on SRS

Consider $X_1, X_2, \ldots, X_m$ as a simple random sample (SRS) from a Power Function Distribution (PFD) with parameters (α, β), and the PDF given by equation (1). The likelihood function (L) and the log-likelihood function (l) are defined as:

$$L(\alpha,\beta)_{srs} = \prod_{i=1}^{m} f_X(x_i) = \frac{\alpha^m}{\beta^{m\alpha}}\ e^{(\alpha-1)\sum_{i=1}^{m}\log[x_i]}$$

And

$$l(\alpha,\beta)_{srs} = m\log[\alpha] - m\alpha\log[\beta] + (\alpha-1)\sum_{i=1}^{m}\log[x_i]\,; \qquad x_i > \beta.$$

Calculate the first derivatives of $l(\alpha,\beta)_{srs}$ with respect to α and β. The Maximum Likelihood Estimates (MLEs) of α and β can be obtained by solving the following equations:

$$\frac{\partial l(\alpha,\beta)_{srs}}{\partial\alpha} = \frac{m}{\alpha} - m\log[\beta] + \sum_{i=1}^{m}\log[x_i]\,,$$

$$\frac{\partial l(\alpha,\beta)_{srs}}{\partial\beta} = -\frac{m\alpha}{\beta}$$

and

$$\frac{\partial l(\alpha,\beta)_{srs}}{\partial\alpha\partial\beta} = \frac{\partial l(\alpha,\beta)_{srs}}{\partial\beta\partial\alpha} = -\frac{m}{\beta}.$$

Similarly, the interval estimate of the asymptotic distribution can be calculated using the maximum likelihood estimator values of the parameters α and β. We know that the expected Fisher information matrix $J(\theta)$, when $\theta = (\alpha,\beta)$ is given respectively as:

$$J(\alpha,\beta) = J(\theta) = -\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = -\begin{bmatrix} \frac{\partial^2 \log f_\theta(x)}{\partial\alpha^2} & \frac{\partial^2 l \log f_\theta(x)}{\partial\alpha\,\partial\beta} \\ \frac{\partial^2 \log f_\theta(x)}{\partial\beta\ \partial\alpha} & \frac{\partial^2 \log f_\theta(x)}{\partial\beta^2} \end{bmatrix},$$

where $f_\theta(x)$ denotes the joint PDF of $X_1, X_2, \ldots, X_m$. We consider the confidence intervals (CIs) under the SRS scheme based on maximum likelihood estimation. The second derivates of $l(\alpha,\beta)_{srs}$can be expressed as:

$$\frac{\partial^2 \log[l(\alpha,\beta)_{srs}]}{\partial\alpha^2} = \frac{m}{\alpha^2} = J_{11},$$

$$\frac{\partial^2 \log[l(\alpha,\beta)_{srs}]}{\partial\beta^2} = \frac{m\alpha}{\beta^2} \ = J_{22}$$

and

$$\frac{\partial^2 \log[l(\alpha,\beta)_{srs}]}{\partial\alpha\,\partial\beta} = \frac{\partial^2 \log[l(\alpha,\beta)_{srs}]}{\partial\beta\,\partial\alpha} = 0 = J_{12} = J_{21}.$$

Belzunce *et al.* (1998) obtained the ML estimates of the parameters of PFD and are given respectively by:

$$\hat{\alpha}_{(srs)mle} = \frac{m}{\sum_{j=1}^{m}\left(X_j - X_{(1)}\right)}$$

and

$$\hat{\beta}_{(srs)mle} = X_{(m)},$$

where $X_{(m)} = Max(X_1, X_2, \dots, X_m)$ and $X_{(1)} = Min(X_1, X_2, \dots, X_m)$.

The asymptotic distribution of the Maximum Likelihood Estimators (MLEs) of α and β, denoted by $\hat{\alpha}_{(srs)mle}$ and $\hat{\beta}_{(srs)mle}$ respectively, is a bivariate normal distribution. In other words, if $m \to \infty$ then we have:

$$\left[\left(\hat{\alpha}_{(srs)mle} - \alpha\right), \left(\hat{\beta}_{(srs)mle} - \beta\right)\right] \to N_2(0, J(\theta)\ ),$$

where $J^{-1}(\theta)$ is the inverse matrix of $J(\theta)$. One may replace the parameters that may appear in $J^{-1}(\theta)$with their corresponding MLEs to obtain an estimator of $J^{-1}(\theta)$, which can be denoted by $\hat{J}^{-1}(\theta)$.

Therefore, the $100(1-\gamma)\%$ asymptotic two-sided equi-tailed (ATE) confidence intervals (CIs) for α and β are given by:

$$\left(\hat{\alpha}_{srs} - z_{\gamma/2}\sqrt{A_{11}}, \qquad \hat{\alpha}_{srs} + z_{\gamma/2}\sqrt{A_{11}}\ \right)$$

and

$$\left(\hat{\beta}_{srs} - z_{\gamma/2}\sqrt{A_{22}}, \qquad \hat{\beta}_{srs} + z_{\gamma/2}\sqrt{A_{22}}\ \right)$$

respectively, where A_{11} and A_{22} are the diagonal elements of $J^{-1}(\theta)$, and $z_{\gamma/2}$ is the $\left(\gamma/2\right)^{th}$ quantile of the standard normal distribution.

Also, the coverage probability for $\theta_{(srs)mle}$ is given as:

$$CP_{srs} = P\left[\left|\frac{\left(\hat{\theta}_{(srs)mle} - \theta_{(srs)mle}\right)}{\sqrt{\widehat{Var}\left(\hat{\theta}_{(srs)mle}\right)}}\right| \le z_{\gamma/2}\right],$$

where $\hat{\theta}_{(srs)mle}$ represents the variance-covariance matrix of $\theta_{(srs)mle}$.

c) Bootstrap Confidence Interval

In this section, we discuss the percentile Bootstrap (Boot-p) confidence interval proposed by Efron (1982). The Boot-p confidence interval can be described as follows:

i) Select a random sample (whether RSS or SRS) from the population and obtain the Maximum Likelihood Estimator $\hat{\theta}$ of the parameter $\theta = (\alpha, \beta)$ as discussed in section 2.

ii) Based on the specified sampling scheme (RSS or SRS), generate a bootstrap random sample from the Power Function Distribution (PFD) with parameters $\hat{\theta}$.

iii) Obtain the Maximum Likelihood Estimator of the model parameters based on the bootstrap sample and denote this bootstrap estimate by $\hat{\theta}^*$
iv) Repeat the second and third steps above N times to obtain $\hat{\theta}^*{}_1, \hat{\theta}^*{}_2, \dots, \hat{\theta}^*{}_N$.
v) Arrange the above estimates in ascending order to obtain the ordered estimates $\hat{\theta}^*{}_{(1)}, \hat{\theta}^*{}_{(2)}, \dots, \hat{\theta}^*{}_{(N)}$.
vi) A confidence interval with confidence level $100(1-\gamma)\%$ is constructed using the $100\left({}^{\gamma}/_{2}\right)$ and $100\left(1-{}^{\gamma}/_{2}\right)$ empirical percentiles of the bootstrap estimates obtained in the previous step.

Selection of the loss function is an important part of the Bayesian setup, and we also use different loss functions to measure the estimators. Three loss functions are used to obtain the point Bayes estimators. The first one is the **squared error loss function** (SELF) (refer to Berger (1985) and Box and Tiao (1973)), where the estimator is the posterior mean. Let $\hat{\theta}$ be an estimate of θ, then the SELF is defined as:

$$L_1(\theta, \hat{\theta}) = (\hat{\theta} - \theta)^2.$$

The Bayes estimator of θ using SELF (Ali et al. (2013)) is given by:

$$\hat{\theta}_s = \left[E_\theta\left({}^{\theta}/_{X}\right)\right] \tag{2.6}$$

The SELF is a symmetrical loss function that assigns equal losses, overestimation, and underestimation. The SELF is a frequently used symmetric loss function because it does not lead to extensive numerical computation. The second loss function is named the **weighted squared error loss function** (WSELF), which gives the squared error loss function a weight. Also, the weight will affect the estimated parameter value. The weighted squared error loss function (WSELF) is an asymmetric loss function that is a weighted version of the symmetric SELF. It is defined as:

$$L_2(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\theta}.$$

The Bayes estimator of θ using WSELF (Ali *et al.* (2013)) is given by:

$$\hat{\theta}_w = \left[E_\theta\left({}^{\theta}/_{X}\right)^{-1}\right]^{-1} \tag{2.7}$$

Also, the third loss function is named as **modified/quadratic squared error loss function** (M/Q SELF). A well-known asymmetric loss function and is a modified version of the commonly used symmetric loss function, SELF and is given by:

$$L_3(\hat{\theta}, \theta) = \left(1 - \frac{\hat{\theta}}{\theta}\right)^2.$$

The Bayes estimator of θ using M/Q SELF (Ali et al. (2013)) is given by:

$$\hat{\theta}_{M/Q} = 1 - \frac{\left[E_\theta\left(\theta/_X\right)^{-1}\right]^{-2}}{\left[E_\theta\left(\theta/_X\right)^{-2}\right]}. \tag{2.8}$$

3. Bayes Estimation

The Bayesian method has proven very useful when the sample size is small. Hierarchical models are one of the central tools of Bayesian analysis. Bayesian models consist of the likelihood function and the prior distribution. The likelihood function is constructed from the data's sampling distribution, which describes the probability of observing the data before the experiment. The prior distribution describes the uncertainty about the parameters of the likelihood function. After observing the data, the prior distribution is updated to the posterior distribution.

Parameter estimation using Ranked Set Sampling (RSS) differs from Simple Random Sampling (SRS) in derivations and offers distinct advantages. In SRS, individual items are randomly selected without specific ordering, and estimators are derived based on standard statistical methods. In contrast, RSS involves selecting samples in ordered sets based on a criterion like the magnitude of the variable estimators in RSS.

In this section, Bayesian estimates and credible intervals of the parameters are obtained using Markov chain Monte Carlo Methods (MCMCs) using RSS. In Bayes' estimation, the unknown parameter is treated as a random variable and assumes a distribution known as the prior distribution. Loss functions and selection of the prior distribution of parameters are two important components of Bayesian analysis. Here, we use the piecewise independent gamma priors for the parameters α and β and are given by:

$$\pi(\alpha,\beta) \propto \alpha^{(a_1-1)}\, \beta^{(a_2-1)}\, e^{-(b_1\alpha+b_2\beta)};\ \alpha,\beta > 0, \tag{3.1}$$

where $a_i, b_i; i = 1,2$ are the hyper parameters.
Using (2.1) and (3.1), the joint posterior distribution based on RSS is given by:

$$\Pi(\alpha,\beta)_{rss} \propto \alpha^{(\phi_3-1)}\, e^{-\psi_3\alpha}\, \beta^{(\phi_4-1)}\, e^{-\psi_4\beta}\, e^{-(T)}, \tag{3.2}$$

where

$$\phi_3 = a_1 + nm, \psi_3 = b_1 + \frac{n}{2}(m^2\log[\beta] - m\log[\beta]) - \sum_{i=1}^{m}\sum_{j=1}^{n} i\ \log[x_{ij}],\ \phi_4 = a_2 - \alpha mn, \psi_4 = b_2$$

and

$$T = (m-1)\sum_{i=1}^{m}\sum_{j=1}^{n}\log\left(1-\left(\frac{x_{ij}}{\beta}\right)^{\alpha}\right) - \sum_{i=1}^{m}\sum_{j=1}^{n}\log(x_{ij}).$$

The Bayes estimators of the parameter $\theta = (\alpha,\beta)$ based on RSS using SELF, WSELF and M/Q SELF are given as follows:

$$\hat{\theta}_{(rss)}{}^{self} = \frac{\int_0^\infty\int_0^\infty \theta\, \Pi(\alpha,\beta)_{rss}\, \partial\alpha\, \partial\beta}{\int_0^\infty\int_0^\infty \Pi(\alpha,\beta)_{rss}\, \partial\alpha\, \partial\beta},$$

$$\hat{\theta}_{(rss)}{}^{wself} = \left[\frac{\int_0^\infty\int_0^\infty \theta^{-1}\, \Pi(\alpha,\beta)_{rss}\, \partial\alpha\, \partial\beta}{\int_0^\infty\int_0^\infty \Pi(\alpha,\beta)_{rss}\, \partial\alpha\, \partial\beta}\right]^{-1} \ and$$

$$\hat{\theta}_{(rss)}{}^{\frac{m}{q}self} = \left[\frac{\int_0^\infty \int_0^\infty \left(1 - \frac{(\theta^{-1})^2}{\theta^{-2}}\right) \Pi(\alpha,\beta)_{rss}\, \partial\alpha\, \partial\beta}{\int_0^\infty \int_0^\infty \Pi(\alpha,\beta)_{rss}\, \partial\alpha\, \partial\beta}\right],$$

where $\Pi(\alpha,\beta)_{rss}$ is given (3.2). Various methods are used here to evaluate the estimates obtained in this section and the preceding sections. First, we use Lindley's method, which explains how to calculate the ratio of integrals, which cannot be further simplified to closed forms. We also use important sampling methods to address the above ratio of integrals, which are discussed in the following sections.

a) Lindley Approximation Method

In this section, we calculate the Bayesian estimation utilizing the loss function for the α and β parameters of the Probability Density Function (PDF) based on Ranked Set Sampling (RSS). Approximation methods are employed to resolve the ratio of integrals. One of the simplest methods is the Lindley approximation method, as published by Lindley (1980). Below is the Lindley approximation method for deriving Bayesian estimates of α and β within the function $I(x)$.

$$I(x) = \frac{\int \psi(\alpha,\beta)\, e^{l(\alpha,\beta)_{rss}+\pi(\alpha,\beta)}\, \partial(\alpha,\beta)}{\int\, e^{l(\alpha,\beta)_{rss}+\pi(\alpha,\beta)}\, \partial(\alpha,\beta)}.$$

The approximate calculation of the ratio of integrals is provided below:

$$I(x) \approx \psi(\alpha,\beta) + \frac{1}{2}\left[\left(\Lambda_{\beta\beta} + 2\Lambda_\beta \cap_\beta\right) \vee_{\beta\beta} + \left(\Lambda_{\alpha\beta} + 2\Lambda_\alpha \cap_\beta\right) \vee_{\alpha\beta} + \left(\Lambda_{\beta\alpha} + 2\Lambda_\beta \cap_\alpha\right) \vee_{\beta\alpha} + (\Lambda_{\alpha\alpha} + 2\Lambda_\alpha \cap_\alpha) \vee_{\alpha\alpha}\right] + \frac{1}{2}\left[\left(\Lambda_\beta \vee_{\beta\beta} + \Lambda_\alpha \vee_{\beta\alpha}\right)\varphi_1 + \left(\Lambda_\beta \vee_{\alpha\beta} + \Lambda_\alpha \vee_{\alpha\alpha}\right)\varphi_2\right],$$

where $\psi(\alpha,\beta)$ is a function of $\alpha\ and\ \beta, L(\alpha,\beta)$ is the log-likelihood, $\pi(\alpha,\beta)$ is the log joint prior, $\wedge\,(\alpha,\beta)$ is the derivatives of $\alpha\ and\ \beta$ and $\vee\,(\alpha,\beta)$ is the inverse of the likelihood function. Here, we denote:

$$\varphi_1 = L_{\beta\beta\beta} \vee_{\beta\beta} + L_{\beta\alpha\beta} \vee_{\beta\alpha} + L_{\alpha\beta\beta} \vee_{\alpha\beta} + L_{\alpha\alpha\beta} \vee_{\alpha\alpha}$$
$$\varphi_2 = L_{\alpha\beta\beta} \vee_{\beta\beta} + L_{\beta\alpha\alpha} \vee_{\beta\alpha} + L_{\alpha\beta\alpha} \vee_{\alpha\beta} + L_{\alpha\alpha\alpha} \vee_{\alpha\alpha}$$

$$\Lambda_\alpha = \frac{\partial\psi(\alpha,\beta)}{\partial\alpha}, \Lambda_\beta = \frac{\partial\psi(\alpha,\beta)}{\partial\beta}, \Lambda_{\alpha\beta} = \frac{\partial\psi(\alpha,\beta)}{\partial\alpha\,\partial\beta}, \Lambda_{\beta\alpha} = \frac{\partial\psi(\alpha,\beta)}{\partial\beta\,\partial\alpha}, \Lambda_{\alpha\alpha} = \frac{\partial\psi(\alpha,\beta)}{\partial\alpha^2}, \Lambda_{\beta\beta} = \frac{\partial\psi(\alpha,\beta)}{\partial\beta^2},\ \cap_\alpha = \frac{\partial\pi(\alpha,\beta)}{\partial\alpha},$$

$$\cap_\beta = \frac{\partial\pi(\alpha,\beta)}{\partial\beta}, L_{\alpha\alpha} = \frac{\partial^2 l(\alpha,\beta)_{rss}}{\partial\alpha^2}, L_{\beta\beta} = \frac{\partial^2 l(\alpha,\beta)_{rss}}{\partial\beta^2}, L_{\alpha\alpha\alpha} = \frac{\partial^3 l(\alpha,\beta)_{rss}}{\partial\alpha^3}, L_{\alpha\alpha\beta} = \frac{\partial^3 l(\alpha,\beta)_{rss}}{\partial\alpha^2\,\partial\beta}, L_{\beta\beta\alpha} = \frac{\partial^3 l(\alpha,\beta)_{rss}}{\partial\beta^2\,\partial\alpha}, L_{\beta\alpha\beta} = \frac{\partial^3 l(\alpha,\beta)_{rss}}{\partial\beta\,\partial\alpha\,\partial\beta}, L_{\alpha\alpha\beta}$$
$$= \frac{\partial^3 l(\alpha,\beta)_{rss}}{\partial\alpha^2\,\partial\beta}, L_{\alpha\beta\beta} = \frac{\partial^3 l(\alpha,\beta)_{rss}}{\partial\alpha\,\partial\beta^2}, L_{\beta\alpha\alpha} = \frac{\partial^2 l(\alpha,\beta)_{rss}}{\partial\beta\,\partial\alpha^2}\ and\ \vee_{ij} = \left(-\frac{1}{L_{ij}}\right).$$

i) Parameters Using RSS

To begin, we can calculate the Bayesian estimate of β ourselves. Using this SELF:

$$\psi(\alpha,\beta)=\beta, \Lambda_\beta=1, \Lambda_{\alpha\alpha}=\Lambda_{\beta\beta}=\Lambda_{\alpha\beta}=\Lambda_{\beta\alpha}=\Lambda_\alpha=0.$$

Here is a Bayesian estimate of β used by SELF:

$$\hat{\beta}_{self}=\hat{\beta}+\cap_\beta \vee_{\alpha\beta}+\cap_\beta \vee_{\beta\beta}+\frac{1}{2}\left[\vee_{\alpha\beta}\,\varphi_1+\vee_{\beta\beta}\,\varphi_2\right].$$

Next, the estimate of α is:

$$\hat{\alpha}_{self}=\hat{\alpha}+\cap_{\alpha\beta} \vee_{\alpha\beta}+\cap_\alpha \vee_{\alpha\alpha}+\frac{1}{2}\left[\vee_{\beta\alpha}\,\varphi_1+\vee_{\alpha\alpha}\,\varphi_2\right].$$

Subsequently, the estimate of the WSELF of β is given as follows:

$$\psi(\alpha,\beta)=\beta^{-1}, \Lambda_\beta=-\beta^{-2}, \Lambda_{\beta\beta}=2\beta^{-3}, \Lambda_{\alpha\alpha}=\Lambda_{\alpha\beta}=\Lambda_{\beta\alpha}=\Lambda_\alpha=0.$$

$$\hat{\beta}_{wself}=\hat{\beta}^{-1}+\frac{1}{2}\left[2\cap_\beta \vee_{\alpha\beta}+\left(\Lambda_{\beta\beta}+2\cap_\beta\right)\vee_{\beta\beta}\right]+\frac{1}{2}\left[\Lambda_\beta \vee_{\beta\alpha}\,\varphi_1+\Lambda_\beta \vee_{\beta\beta}\,\varphi_2\right].$$

Additionally, the Bayesian estimate of α is given as follows:

$$\hat{\alpha}_{wself}=\hat{\alpha}^{-1}+\frac{1}{2}\left[2\cap_\alpha \vee_{\alpha\beta}+\left(\Lambda_{\alpha\alpha}+2\cap_\alpha\right)\vee_{\alpha\alpha}\right]+\frac{1}{2}\left[\Lambda_\alpha \vee_{\beta\alpha}\,\varphi_1+\Lambda_\alpha \vee_{\alpha\alpha}\,\varphi_2\right].$$

Another estimate of the β used by M/Q SELF is given as follows:

$$\psi(\alpha,\beta)=\left(1-\left(\frac{(\beta^{-1})^2}{\beta^{-2}}\right)\right), \Lambda_\beta=\Lambda_{\beta\beta}=\Lambda_{\alpha\alpha}=\Lambda_{\alpha\beta}=\Lambda_{\beta\alpha}=\Lambda_\alpha=0.$$

$$\hat{\beta}_{\frac{m}{q}self}=\left(1-\left(\frac{\left(\hat{\beta}^{-1}\right)^2}{\hat{\beta}^{-2}}\right)\right)+\frac{1}{2}\left[\cap_\beta \vee_{\alpha\beta}+\cap_\beta \vee_{\beta\beta}\right].$$

Similarly, the Bayesian estimate of α is:

$$\hat{\alpha}_{\frac{m}{q}self}=\left(1-\left(\frac{(\hat{\alpha}^{-1})^2}{\hat{\alpha}^{-2}}\right)\right)+\frac{1}{2}\left[\cap_\alpha \vee_{\alpha\beta}+\cap_\alpha \vee_{\alpha\alpha}\right].$$

b) Importance Sampling Procedure

In this section, we explore the significance of the sampling method in computing the ratio of integrals for determining parameter estimates in the Probability Function Distribution (PFD) using the combined posterior distribution based on Ranked Set Sampling (RSS). Additionally, we calculate the Highest Posterior Density (HPD) credible intervals for the parameters of interest.

i) Combined Posterior Distribution with RSS

The combined posterior distribution of the parameters is expressed as the product of the terms:

$$\Pi(\alpha,\beta)_{rss}\propto h(\alpha,\beta)_{rss}\, f_3(\alpha/\beta)\, f_4(\beta/\alpha).$$

Here, the functions $h(\alpha,\beta)_{rss}$, $f_3(\alpha/\beta)$, and $f_4(\beta/\alpha)$ are defined as follows:

$$h(\alpha,\beta)_{rss}=\frac{e^{-T}\,\Gamma(a_2+\alpha nm)\Gamma(a_1+nm)}{e^{(a_2+\alpha nm)\log(b_2)}\times e^{(a_1+nm)\log\left(b_1+\frac{n}{2}(m^2\log[\beta]-m\log(\beta))-\sum_{i=1}^{m}\sum_{j=1}^{n} i\,\log\left[x_{ij}\right]\right)}}, \tag{3.3}$$

$$f_3(\alpha/\beta)\propto\ \alpha^{a_1+nm-1}e^{-\alpha\left(b_1+\frac{n}{2}(m^2\log[\beta]-m\log(\beta))-\sum_{i=1}^{m}\sum_{j=1}^{n} i\,\log[x_{ij}]\right)} \tag{3.4}$$

and

$$f_4(\beta/\alpha)\propto\ \beta^{a_2+\alpha nm-1}e^{-\beta b_2}. \tag{3.5}$$

Thus, the Bayes estimates of the parameters $\alpha\ and\ \beta$ using the importance sampling procedure involve the following steps:

i) Generate α_2 from the distribution $f(\alpha/\beta; \phi_3, \psi_3)$.
ii) Using the generated α_2 value, generate β_2 from the distribution $f(\beta/\alpha; \phi_4, \psi_4)$.
iii) Repeat steps 1 and 2, M_1 times to obtain the important sample procedures $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_{M_1}, \beta_{M_1})$.

Subsequently, the Bayesian estimators of the $\theta = (\alpha, \beta)$ under different loss functions such as SELF, WSELF and M/Q SELF can be expressed as follows;

$$\hat{\theta}_{self} = \frac{\sum_{s=1}^{M_1} \theta_s \; h(\alpha,\beta)_{rss}}{\sum_{s=1}^{M_1} h(\alpha,\beta)_{rss}},$$

$$\hat{\theta}_{wself} = \left[\frac{\sum_{s=1}^{M_1} (\theta_s)^{-1} \; h(\alpha,\beta)_{rss}}{\sum_{s=1}^{M_1} h(\alpha,\beta)_{rss}}\right]^{-1}$$

and

$$\hat{\theta}_{\frac{m}{q}self} = \left[\frac{\sum_{s=1}^{M_1}\left(1 - \left(\frac{\left(\theta_s^{-1}\right)^2}{\theta_s^{-2}}\right)\right) h(\alpha,\beta)_{rss}}{\sum_{s=1}^{M_1} h(\alpha,\beta)_{rss}}\right].$$

Here $h(\alpha, \beta)_{rss}, s = 1, 2$ are as defined by equation (3.3).

c) Highest Posterior Distribution

In this section, we establish the Highest Posterior Density (HPD) credible intervals for the parameters α and β based on Ranked Set Sampling (RSS). The methodology for constructing HPD credible interval was introduced by Chen and Shao (1999).

i) Credible Interval using RSS

Let $(\alpha, \beta) = \left(\hat{\alpha}^{\delta}, \hat{\beta}^{\delta}\right)$, where $\alpha^{(\delta)}$ and $\beta^{(\delta)}$ for δ = 1, 2, ..., M_2 are posterior samples generated from equations (15) to (16) for α and β. Let $\hat{\alpha}^{\delta}$ and $\hat{\beta}^{\delta}$ be the ordered values of $\hat{\alpha}^{(\delta)}$ and $\hat{\beta}^{(\delta)}$, respectively. Define

$$\omega_{\delta} = \frac{h\left(\alpha^{\delta}, \beta^{\delta}\right)_{rss}}{\sum_{i=1}^{M_2} h(\alpha^{\delta}, \beta^{\delta})_{rss}}, \delta = 1, 2, \ldots, M_2.$$

The d^{th} quantile of $\hat{\beta}$ can be estimated as:

$$\hat{\beta}^{d} = \begin{cases} \hat{\beta}_{(1)} & if \quad d = 0 \\ \hat{\beta}_{(i)} & if \quad \sum_{j=1}^{\delta-1} \omega_j < t < \sum_{j=1}^{\delta} \omega_j, \end{cases}$$

where $\left(\hat{\beta}^{j}/_{M_2}, \hat{\beta}^{(j+[(1-\gamma)M_2])}/_{M_2}\right), j = 1, 2, \ldots, M_2$, represents the greatest integer function. Similarly, the $100(1 - \gamma)$ % HPD credible interval for α can be constructed.

d) Prediction

This section discusses the prediction method using RSS when both parameters of the Probability Function Distribution (PFD) are unknown. Let $u_1, u_2, \ldots, u_w$ be an independent sample of size w. To predict the future sample $u_w, for\ w = 1, 2, \ldots, n_1$ based on RSS, the density function of u_w is given by:

$$G\left({}^{u_{w}}/_{\alpha,\beta}\right) = w\binom{n_1}{w}\left(F(u_w)\right)^{w-1}\left(\bar{F}(u_w)\right)^{n_1-w} f(u_w)$$

$$= w\binom{n_1}{w}\left(\frac{\alpha}{u_w}\right)\sum_{k=0}^{n_1-w}\binom{n_1-w}{k}(-1)^k \times e^{\left(-\alpha\left[(k+w)\log\left(\frac{\beta}{u_w}\right)\right]\right)}. \quad (3.6)$$

i) Prediction Intervals Based on RSS

Using equations (3.2) and (3.6), the predictive density function of u_w is given by:

$$f\left({}^{u_w}/_{\underline{x}}\right) = \int_0^{\beta}\int_0^{\infty} G\left({}^{u_w}/_{\alpha,\beta}\right) \Pi(\alpha,\beta)_{rss}\, \partial\alpha\, \partial\beta, \quad (3.7)$$

where the integration is over the entire domain. Hence, the predictive survival function is given by:

$$P\left[u_w > t/\underline{x}\right] = \int_t^{\infty} f\left({}^{u_w}/_{\underline{X}}\right) \partial u_w. \quad (3.8)$$

Therefore, the lower and upper $100(1-\gamma)\ {}^0\!/_0$ prediction bounds [L(x), U(x)] for u_w are obtained by equations $\frac{(1+\gamma)}{2}$ and $\frac{(1-\gamma)}{2}$, respectively. From equation (3.7), the predictive Bayesian estimator using different loss functions of $u_w, w = 1, 2, \ldots, n_1$ can be obtained as:

$$u_w{}^{self} = \int u_w\, f\left({}^{u_w}/_{\underline{x}}\right) \partial u_w,$$

$$u_w{}^{wself} = \left[\int u_w{}^{-1} f\left({}^{u_w}/_{\underline{x}}\right) \partial u_w\right]^{-1}$$

and

$$u_w{}^{self} = \int\left[1 - \frac{(u_w{}^{-1})^{-2}}{u_w{}^2}\right] f\left({}^{u_w}/_{\underline{x}}\right) \partial u_w,$$

where $f\left({}^{u_w}/_{\underline{x}}\right)$ is given by equation (3.7).

4. Simulation Study

In this section, we evaluate the performance of the proposed estimators using simulated data. Data were generated from a Probability Function Distribution (PFD) with parameters $(\alpha,\beta) = (0.5; 1.5)$ respectively. The hyperparameters are set as $a_1 = 2,\ b_1 = 2, a_2 = 2.5\ and\ b_2 = 2.5$ respectively. The outcomes of the simulation study to assess the estimators' performance are presented in Tables 1-5.

According to Table 1, the proposed estimators' Mean Squared Error (MSE) decreases as the number of samples increases. However, in Table 2, the MSE of the proposed estimators increases with the number of samples. Moreover, Tables 3-5 demonstrate that the MSE of the proposed Bayes estimators increases with the number of samples. Additionally, Table 5 shows that the prediction of $u_w$under different loss functions decreases as the number of samples increases. We compute the Bayes estimators using Squared Error Loss Function (SELF), Weighted Squared Error Loss Function (WSELF), and Modified/Quadratic Squared Error Loss Function (M/Q SELF). We replicate the process 1000 times and compute the average bias and MSE.

Table 1: Bias and MSE for MLE and AIL and CP for CI of the parameters α and β of PFD using SRS.

$\alpha = 0.5\ and\ \beta = 0.5$									
m	Estimate	MLF		**Confidence Interval**					
		Bias	**MSE**	**Bootstrap**		**Asymptotic**		**HPD**	
				$\boldsymbol{AIL}$	**CP**	**AIL**	**CP**	**AIL**	$\boldsymbol{CP}$
20	$\hat{\boldsymbol{\alpha}}_{\boldsymbol{srs}}$	0.1137	0.0784	0.1003	0.9399	0.5641	0.9409	0.2552	0.9483
20	$\hat{\boldsymbol{\beta}}_{\boldsymbol{srs}}$	0.0609	0.0059	0.1870	0.9514	0.2399	0.9566	0.3037	0.9198
$\boldsymbol{\alpha = 1.5\ and\ \beta = 0.5}$									
m	Estimate	MLF		**Confidence Interval**					
		Bias	**MSE**	**Bootstrap**		**Asymptotic**		**HPD**	
				$\boldsymbol{AIL}$	**CP**	**AIL**	**CP**	**AIL**	$\boldsymbol{CP}$
40	$\hat{\boldsymbol{\alpha}}_{\boldsymbol{srs}}$	0.1362	0.0073	0.5119	0.9493	0.3229	0.9338	0.3705	0.9448
40	$\hat{\boldsymbol{\beta}}_{\boldsymbol{srs}}$	0.0291	0.0013	0.0858	0.9678	0.2972	0.9501	0.3066	0.9332

Table 2: Bias and MSE for MLE, AIL, and CP for CI of the parameters α and β of PFD using RSS.

$\alpha = 0.5\ and\ \beta = 0.5$									
m×n	Estimate	MLF		**Confidence Interval**					
		Bias	**MSE**	**Bootstrap**		**Asymptotic**		**HPD**	
				$\boldsymbol{AIL}$	**CP**	**AIL**	**CP**	**AIL**	$\boldsymbol{CP}$
2×10	$\hat{\boldsymbol{\alpha}}_{\boldsymbol{rss}}$	0.3175	0.1042	0.3029	0.9321	0.3049	0.9066	0.1625	0.9026
2×10	$\hat{\boldsymbol{\beta}}_{\boldsymbol{rss}}$	0.3311	0.1431	0.7169	0.9277	0.4951	0.9182	0.2210	0.9235
$\alpha = 1.5\ and\ \beta = 1.5$									
n×m	Estimate	MLF		**Confidence Interval**					
		Bias	**MSE**	**Bootstrap**		**Asymptotic**		**HPD**	
				$\boldsymbol{AIL}$	**CP**	**AIL**	**CP**	**AIL**	$\boldsymbol{CP}$
4×10	$\hat{\boldsymbol{\alpha}}_{\boldsymbol{rss}}$	0.1455	0.2119	0.5007	0.9196	0.3164	0.9169	0.3641	0.9244
4×10	$\hat{\boldsymbol{\beta}}_{\boldsymbol{rss}}$	0.5768	0.4195	0.1154	0.9218	0.2360	0.9180	0.2106	0.9046

We can report the following points from the numerical results of Tables 1-5.

i) As the sample size gets large, the MLE method tends to perform almost the same based on RSS and SRS.
ii) With the increase in sample size and number of samples, it can be observed that, in all cases, the bias and MSE of the estimates increase, as expected.
iii) The average length of the approximate confidence intervals decreases as the sample size increases, while the coverage probability remains around 0.95.
iv) It can be observed that the Bayesian estimator outperforms the MLE regarding bias and MSE. The Bayesian estimators incorporate more information in the form of prior information compared to MLE.

Table 3: The Lindley approximation method using RSS are the average and MSE of Bayes estimators of $\alpha\ and\ \beta$.

$\boldsymbol{\alpha = 0.5\ and\ \beta = 0.5}$							
n×m	**Estimate**	**SELF**		**WSELF**		**M/Q SELF**	
		Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
2×10	$\hat{\boldsymbol{\alpha}}_{rss}$	-0.51423	0.20112	-0.51444	0.29234	-0.10055	0.01197
2×10	$\hat{\beta}_{rss}$	-0.39043	0.15244	-0.39062	0.15259	-0.10086	0.10173
$\boldsymbol{\alpha = 1.5\ and\ \beta = 1.5}$							
4×10	$\hat{\boldsymbol{\alpha}}_{rss}$	-0.42313	0.29217	-0.42031	0.37904	-0.40468	0.16377
4×10	$\hat{\boldsymbol{\beta}}_{rss}$	-0.38667	0.15951	-0.38723	0.15994	-0.38728	0.14461

Table 4: The importance sampling procedure using RSS are the average and MSE of Bayes estimators of $\alpha\ and\ \beta$

$\boldsymbol{\alpha = 0.5\ and\ \beta = 0.5}$							
n×m	**Estimate**	**SELF**		**WSELF**		**M/Q SELF**	
		Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
2×10	$\hat{\boldsymbol{\alpha}}_{rss}$	0.49976	0.22487	0.36999	0.22689	0.46052	0.21764
2×10	$\hat{\boldsymbol{\beta}}_{rss}$	0.49999	0.22481	0.51000	0.22801	0.49990	0.22481
$\boldsymbol{\alpha = 1.5\ and\ \beta = 1.5}$							
4×10	$\hat{\boldsymbol{\alpha}}_{rss}$	0.14995	0.24976	0.14949	0.23349	0.46984	0.22972
4×10	$\hat{\boldsymbol{\beta}}_{rss}$	0.15000	0.22499	0.15100	0.26010	0.14998	0.24990

Table 5: The Average and MSE of Bayes estimators of u_w of prediction using RSS.

$\alpha = 0.5$ *and* $\beta = 0.5$							
n×m	**Estimate**	**SELF**		**WSELF**		**M/Q SELF**	
		Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
2×10	$(\boldsymbol{u_w})_{rss}$	0.49570	0.24572	0.52259	0.27310	0.40986	0.16799
2×10	$(\boldsymbol{u_w})_{rss}$	0.49987	0.24987	0.50978	0.25987	0.48890	0.23902
$\alpha = 1.5$ *and* $\beta = 1.5$							
4×10	$(\boldsymbol{u_w})_{rss}$	0.14918	0.22254	0.15000	0.22500	0.14783	0.11853
4×10	$(\boldsymbol{u_w})_{rss}$	0.14995	0.22486	0.14648	0.21458	0.14539	0.21138

5. Numerical Example

In this section, we examine the ranked set sampling data provided by Hettmansperger and McKean (2010), representing the lifetimes of an insulating fluid. Table 6 presents the breakdown time (in minutes) of an electrically insulating fluid under two different voltage levels, 30 and 32 kV.

We conducted various tests and estimation techniques on the two data samples obtained from the study by Hettmansperger and McKean (2010). These datasets reflect real-life scenarios, illustrating the breakdown times of an electrically insulating fluid under different voltage conditions. The values in each row under "Times to Breakdown (Minutes)" likely signify the durations until breakdown for individual instances or trials under the specified voltage.

We assessed the adequacy of the Probability Function Distribution (PFD) for the first sample using the Kolmogorov-Smirnov (KS) test. The test yielded a p-value of 0.259626, which suggests that the distribution fits well ($p > 0.05$). Similarly, for the second sample, the KS test yielded a p-value of 0.163194, indicating a good fit of the distribution to the data ($p > 0.05$).

Next, we applied Maximum Likelihood Estimation (MLE) to fit the PFD to the first dataset and estimated the parameters to $\alpha = 0.0051$ and $\beta = 0.6894$. Similarly, for the second dataset, the MLE estimated the parameters to $\alpha = 0.0046$ and $\beta = 0.3182$.

Table 6: The breakdown times (in minutes) of an electrical insulating fluid

Voltage	**Times to Breakdown (Minutes)**
30 kV	17.05 22.66 21.02 175.88 139.07 144.12 20.46 43.40 194.90 47.30 7.74
32 kV	0.40 82.85 9.88 89.29 215.10 2.75 0.79 15.93 3.91 0.27 0.69 100.58 27.80 13.95 53.24

Table 7-8 presents the estimates derived from sections 2-3, focusing on the annual wage data. The tables illustrate the Maximum Likelihood Estimation (MLE) and Bayesian estimators of α and β for various loss functions. Hyper-parameter values $a_1 = 2,\ b_1 = 2, a_2 = 2.5\ and\ b_2 = 2.5$ were selected for Bayes estimation.

The MLE and Bayes estimates under different loss functions are detailed in Tables 7-8. Notably, the results obtained from the actual data align with those from the simulation study, demonstrating the robustness and reliability of the estimators. Moreover, in scenarios where samples are available only in the form of ranks, the ability to predict the rank is highlighted, showcasing the versatility and applicability of the proposed methodologies.

Table 7: MLE and Bayes estimate of parameters using the real data for lifetimes of an insulating fluid.

Level 1: α = 0.0051 and β = 0.6894									
n×m	**Estimation**						**Prediction**		
	-	**MLE**	**AIL**	**CP**	**LAM**	**ISP**	$(\boldsymbol{u_w})_{rss}$	**L(X)**	**U(X)**
1×11	$\hat{\boldsymbol{\alpha}}_{rss}$	0.2484	-	-	-	-	-	-	-
1×11	$\hat{\boldsymbol{\beta}}_{rss}$	0.4212	-	-	-	-	-	-	-
1×11	$\hat{\boldsymbol{\alpha}}_{CI}$	-	2.286	0.9227	-	-	-	-	-
1×11	$\hat{\boldsymbol{\beta}}_{CI}$	-	2.045	0.9423	-	-	-	-	-
1×11	$\hat{\boldsymbol{\alpha}}_{self}$	-	-	-	0.7768	0.5171	-	-	-
1×11	$\hat{\boldsymbol{\beta}}_{self}$	-	-	-	0.9236	0.4980	-	-	-
1×11	$\hat{\boldsymbol{\alpha}}_{wself}$	-	-	-	0.8955	0.9902	-	-	-
1×11	$\hat{\boldsymbol{\beta}}_{wself}$	-	-	-	0.6697	0.4035	-	-	-
1×11	$\hat{\boldsymbol{\alpha}}_{\frac{m}{q}self}$	-	-	-	0.1877	0.5268	-	-	-
1×11	$\hat{\boldsymbol{\beta}}_{\frac{m}{q}self}$	-	-	-	0.1820	0.5023	-	-	-
1×11	$(\boldsymbol{u_w})_{self}$	-	-	-	-	-	0.5815	0.2692	1.1263
1×11	$(\boldsymbol{u_w})_{wself}$	-	-	-	-	-	0.2236	0.6560	1.3269
1×11	$(\boldsymbol{u_w})_{\frac{m}{q}self}$	-	-	-	-	-	0.3241	0.4773	1.2258

Table 8: MLE and Bayes estimate of parameters using the real data for lifetimes of an insulating fluid.

Level 2: α = 0.0046 and β = 0.3182									
n×m	**Estimation**						**Prediction**		
	-	**MLE**	**AIL**	**CP**	**LAM**	**ISP**	$(u_w)_{rss}$	**L(X)**	**U(X)**
1×11	$\hat{\alpha}_{rss}$	0.1100	-	-	-	-	-	-	-
1×11	$\hat{\beta}_{rss}$	0.6317	-	-	-	-	-	-	-
1×11	$\hat{\alpha}_{CI}$	-	2.362	0.9347	-	-	-	-	-
1×11	$\hat{\beta}_{CI}$	-	2.005	0.9028	-	-	-	-	-
1×11	$\hat{\alpha}_{self}$	-	-	-	0.9439	0.4626	-	-	-
1×11	$\hat{\beta}_{self}$	-	-	-	0.3915	0.1838	-	-	-
1×11	$\hat{\alpha}_{wself}$	-	-	-	0.7072	0.3248	-	-	-
1×11	$\hat{\beta}_{wself}$	-	-	-	0.9105	0.1659	-	-	-
1×11	$\hat{\alpha}_{\frac{m}{q}self}$	-	-	-	0.7899	0.6241	-	-	-
1×11	$\hat{\beta}_{\frac{m}{q}self}$	-	-	-	0.3370	0.7582	-	-	-
1×11	$(u_w)_{self}$	-	-	-	-	-	0.6295	0.3136	1.0562
1×11	$(u_w)_{wself}$	-	-	-	-	-	0.6358	0.2568	1.3926
1×11	$(u_w)_{\frac{m}{q}self}$	-	-	-	-	-	0.3301	0.9244	1.4502

In conclusion, the utilization of Bayesian modeling has yielded significant benefits, demonstrating its potential applicability in various situations beyond the scope of this study.

6. Conclusions

In conclusion, this paper delves into the estimation of Probability Density Function (PFD) parameters using both Maximum Likelihood Estimation (MLE) and Bayesian estimators within the context of Ranked Set Sampling (RSS) design. The practicality and effectiveness of the proposed Bayesian modeling, coupled with Markov Chain Monte Carlo (MCMC) estimation, are demonstrated through comprehensive analyses of simulated and real data.

Specifically, the study highlights that Bayesian estimators under various loss functions, such as Squared Error Loss (SEL), Weighted Squared Error Loss (WSEL), and Modified/Quadratic Squared Error Loss (M/Q SEL), entail the computation of ratios of two integrals. Furthermore, the highest posterior density interval, as computed using the method of Chen and Shao (1999), adds another layer of robustness to the analysis.
Analysis of both simulated and real data indicates that Bayesian estimators outperform MLE in terms of efficiency under different loss functions. Particularly, Bayesian estimators based on ranked set sampling with samples exhibit superior efficiency compared to those based on simple random sampling.

In summary, the results from simulation experiments underscore the efficacy of estimators based on ranked set sampling, affirming their superiority over MLE with Bayesian estimates derived from simple random sampling. This study contributes to advancing the understanding and application of Bayesian modeling in statistical estimation, particularly in scenarios involving ranked set sampling.

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