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On Estimation of Parameters of Double Weibull Distribution Using *U*-Statistics Based on Best Linear Functions of Order Statistics as Kernels

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ABSTRACT

In this work, we have obtained estimators for the location and scale parameters of double Weibull distribution using *U*-statistics based on best linear functions of order statistics as kernels when its shape parameter is known. The efficiency comparisons of the proposed *U*-statistics with respect to some standard estimators are also described.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from a population. If the observations are arranged in non-decreasing order as $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, then the random variables $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ are called the order statistics of the sample. In particular $X_{r:n}$ is called the r-th order statistic, for $1 \leq r \leq n$.

Order Statistics is an important branch of Statistics which deals with the theory and applications of ordered random variables and of functions involving them. Research in the area of order statistics has been steadily and rapidly growing especially in the last three decades. The extensive role of order statistics in several areas of statistical inference has made it important and useful for researchers in many fields of investigation.

It is well-known that, order statistics help to provide the most suitable and appropriate solution for many problems arising in life testing experiments, meteorological studies arising in atmospheric pressure, temperature, wind, etc., investigations on flood, reliability studies, longevity studies, breaking strength analysis, detection of outliers, insurance problems, industrial research, health

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care studies and so on. A huge volume of work have been carried out on theories and practices of order statistics, see, Arnold *et al.* (1992), Balakrishnan and Cohen (1991), Balakrishnan *et al.* (2006).

Order statistics enter into the problem of estimation in a variety of ways. Best linear unbiased estimators (BLUEs) of location (μ) and scale (σ) parameters of a distribution by order statistics as developed by Lloyd (1952) is a well-known and extensively applied method of estimation. Moreover the estimates obtained in this case are optimal in the sense that their variances are least among all other linear functions of order statistics which estimate them.

The class of estimators, now termed as *U*-statistics, was first coined by Hoeffding (1948). For some interesting properties of *U*-statistics such as their strong consistency, asymptotic normality etc. see, Serfling (1980). The *U*-statistics are widely considered as nonparametric statistics for estimating the parameters of a distribution and many well-known nonparametric statistics are in fact members of the class of *U*-statistics.

It may be noted that since the introduction of *U*-statistics by Hoeffding (1948), no major parametric estimation procedures using *U*-statistics based on order statistics has been developed other than the results of Thomas and Sreekumar (2008) in which they have developed a new method for estimating the location and scale parameters of a distribution using *U*-statistics based on best linear functions of order statistics as kernels. For the estimation of location and scale parameters of some distributions using *U*-statistics based on best linear functions of order statistics as kernels, one can refer to Thomas and Baiju (2013, 2015), Thomas and Sreekumar (2008) and Sreekumar and Thomas (2007, 2008). In a recent development, Thomas and Anjana (2021) further extended the concept to absolute order statistics.

In order to obtain Lloyd's (1952) BLUEs of μ and σ, one requires the values of means, variances and covariances of the entire order statistics of a random sample of size n arising from the corresponding standard distribution. Unfortunately, in most cases, explicit expressions fail to exist for the means, variances and covariances of these order statistics. In such situations these values are usually computed numerically for $n \leq 20$. If $n > 20$, then there is no way to obtain the desirable optimal estimators *viz.*, BLUEs based on order statistics and statistics practitioners scared if $n > 20$ though BLUEs have optimal properties. Unlike maximum likelihood estimators, the asymptotic theory of BLUEs based on order statistics are not seen developed in a general sense (prevents large sample tests). Instead, if we consider the *U*-statistics based on best linear functions of order statistics as kernels (based on a small sample size, say $m \leq 5$), then they estimates the parameters explicitly and utilize the optimality conditions of both BLUEs and *U*-statistics. Further, evaluation of means, variances and covariances of order statistics of sample sizes up to only $2m - 1$ arising from standard form of the parent distribution alone are necessary to obtain the value of the variance of the *U*-statistics, whatever be the sample size *n*.

The present work has been tri-sectioned, section 1 deals with some basic results, which are very necessary in developing the proposed estimators. Section 2 explores the importance of double Weibull distribution along with the newly introduced *U*-statistics estimators of location and scale parameters based on BLUEs of order statistics as kernels together with their variances. In the last section, relative efficiency comparisons of the proposed *U*-statistics with some suitable other estimators has been made.

2. Some Basic Concepts

A random variable X belongs to location-scale family if its probability density function (pdf) of the form,

$$f(x;\mu,\sigma) = \frac{1}{\sigma} f_0\left(\frac{x-\mu}{\sigma}\right); -\infty < x < \infty, -\infty < \mu < \infty,\ \sigma > 0 \qquad (2.1)$$

where μ and σ in (2.1) are known as the location and scale parameters respectively. Lloyd (1952) has introduced a method of estimation of μ and σ by using best linear functions of order statistics of a random sample of size *m* drawn from (2.1).

Let X_1, X_2, …,X_m be a random sample from (2.1) with corresponding order statistics $X_{1:m}, X_{2:m}, \ldots, X_{m:m}$.

Then the BLUE of μ can be written as

$$h_1(X_1,X_2,\ldots,X_m) = a_{1;m}X_{1:m} + a_{2;m}X_{2:m} + \ldots + a_{m;m}X_{m:m} \qquad (2.2)$$

and the BLUE of σ can be written as

$$h_2(X_1,X_2,\ldots,X_m) = d_{1;m}X_{1:m} + d_{2;m}X_{2:m} + \ldots + d_{m;m}X_{m:m}, \qquad (2.3)$$

where $a_{i;m}$ and $d_{i;m}$ (i = 1, 2,…, m) are appropriate constants. Thomas and Sreekumar (2008) introduced the *U*-statistic for μ based on kernel (2.2) as

$$U_{1;n}^{(m)} = \frac{1}{\binom{n}{m}} \sum_{r=1}^{n} \left[\sum_{i=0}^{m-1} \binom{n-r}{m-i} \binom{r-1}{i-1} a_{i;m} \right] X_{r:n} \qquad (2.4)$$

and the U-statistic for σ based on kernel (2.3) is

$$U_{2;n}^{(m)} = \frac{1}{\binom{n}{m}} \sum_{r=1}^{n} \left[\sum_{i=0}^{m-1} \binom{n-r}{m-i} \binom{r-1}{i-1} d_{i;m} \right] X_{r:n} \qquad (2.5)$$

Clearly $U_{1;n}^{(m)}$ and $U_{2;n}^{(m)}$ are unbiased estimates μ and σ respectively, see Hoeffding (1948).

The variances of $U_{1;n}^{(m)}$ and $U_{2;n}^{(m)}$ are given by

$$\text{Var}\left[U_{1;n}^{(m)}\right] = \frac{1}{\binom{n}{m}} \sum_{c=1}^{m} \binom{m}{c} \binom{n-m}{m-c} \xi_c^{(m)} \qquad (2.6)$$

and

$$\text{Var}\left[U_{2;n}^{(m)}\right] = \frac{1}{\binom{n}{m}} \sum_{c=1}^{m} \binom{m}{c} \binom{n-m}{m-c} \zeta_c^{(m)} \; . \qquad (2.7)$$

where

$$\xi_c^{(m)} = \text{Cov}[h_1(X_1, X_2, \ldots, X_c, X_{c+1}, \ldots, X_m), h_1(X_1, X_2, \ldots, X_c, X_{m+1}, \ldots, X_{2m-c})], \qquad (2.8)$$

and

$$\zeta_c^{(m)} = \text{Cov}[h_2(X_1, X_2, \ldots, X_c, X_{c+1}, \ldots, X_m), h_2(X_1, X_2, \ldots, X_c, X_{m+1}, \ldots, X_{2m-c})]. \qquad (2.9)$$

Clearly $\xi_m^{(m)} = \text{Var}\left[h_1\left(X_1, X_{2,\cdots,} X_m\right)\right]$, $\zeta_m^{(m)} = \text{Var}\left[h_2\left(X_1, X_{2,\cdots,} X_m\right)\right]$.

In order to obtain Var[$U_{1;n}^{(m)}$] and Var[$U_{2;n}^{(m)}$], we have to obtain the values of $\xi_c^{(m)}$ and $\zeta_c^{(m)}$

The main concern now shifted to the evaluation of $\xi_c^{(m)}$ and $\zeta_c^{(m)}$, c = 1, 2, …, m -1, but Thomas and Sreekumar (2008) again fruitfully put forward a method for finding $\xi_c^{(m)}$ and $\zeta_c^{(m)}$. This removed the once prevailed mathematical intractability in using such statistics as estimators.

3. Estimation of Parameters of Double Weibull Distribution Using *U*-Statistics

A random variable X is said to have double Weibull distribution if its pdf is given by,

$$f(x;\mu,\sigma,\lambda)=\frac{\lambda}{2\sigma}\left|\frac{x-\mu}{\sigma}\right|^{\lambda-1}\exp\left\{-\left|\frac{x-\mu}{\sigma}\right|^{\lambda}\right\};x\in(-\infty,\infty), \tag{3.1}$$

where $\lambda>0,\sigma>0$ and $\mu\in(-\infty,\infty)$

The corresponding cumulative distribution function is given by

$$F(x;\mu,\sigma,\lambda)=\frac{1}{2}exp\left\{-\left[-\frac{x-\mu}{\sigma}\right]^{\lambda}\right\};\text{ if } x<\mu$$

$$=1-\frac{1}{2}exp\left\{-\left[\frac{x-\mu}{\sigma}\right]^{\lambda}\right\};\text{ if } x\geq\mu$$

We may write DW (μ, σ, λ) to denote the distribution defined by (3.1)

Weibull distribution has been used as an appropriate model in several situations in reliability theory (see, Johnson and Kotz (1970)). The double Weibull distribution is an extension of Weibull distribution extended with support set R = $(-\infty,\infty)$. Since the shape parameter λ is also involved in DW (μ, σ, λ), it provides an ideal symmetric model to analyze data sets arising from several populations. Balakrishnan and Kocherlakota (1985) has made a systematic study of the double Weibull distribution particularly concerning order statistics and evaluated means ($\alpha_{r:n}$), variances and covariances ($\nu_{r,s:n}$), of order statistics arising from DW(0, 1, λ) for sample sizes up to ten and for some fixed values of λ. They have also tabulated the coefficients of order statistics in the BLUEs of μ and σ involved in DW (μ, σ, λ) together with their variances for sample sizes up to ten and also compared the relative efficiencies of these BLUEs with some standard estimators. Rao and Narasimham (1989) have further extended the estimators of μ and σ based on order statistics for sample sizes up to 20.

It may be noted that the method of moment estimators and maximum likelihood estimators of μ, σ and λ for DW (μ, σ, λ) are not explicitly available. For sample sizes n > 20, it may be difficult for statistics practitioners to use BLUEs of μ and σ based on order statistics because means and covariances of order statistics have not been tabulated in the available literature for n > 20. Hence there is much relevance to the use of *U*-statistics put forward by Thomas and Sreekumar (2008) for estimating μ and σ involved in DW (μ, σ, λ).

If we draw an initial sample of size *m* from DW (μ, σ, λ) for known values of λ and construct the kernels (BLUEs of μ and σ)

$$h_1(X_1,X_2,\ldots,X_m) = a_{1;m}X_{1:m} + a_{2;m}X_{2:m} + \ldots + a_{m;m}X_{m:m}$$

and

$$h_2(X_1,X_2,\ldots,X_m) = d_{1;m}X_{1:m} + d_{2;m}X_{2:m} + \ldots + d_{m;m}X_{m:m},$$

then the U-statistics estimators for μ and σ are $U_{1;n}^{(m)}$ and $U_{2;n}^{(m)}$ respectively and are given in (2.4) and (2.5) respectively. The components $\xi_1^{(m)}, \xi_2^{(m)}, \ldots, \xi_{m-1}^{(m)}$ of Var($U_{1;n}^{(m)}$) and the components $\zeta_1^{(m)}, \zeta_2^{(m)}, \ldots, \zeta_{m-1}^{(m)}$ of Var($U_{2;n}^{(m)}$) can be obtained by the method suggested by Thomas and Sreekumar (2008).

The coefficients of BLUEs of μ and σ involved in DW(μ, σ, λ) for sample sizes n ≤ 10 , λ = 0.5, 0.75, 1 & 3 and their variances are given in Balakrishnan and Kocherlakota (1985). We used those coefficients $a_{i;m}$ and $d_{i;m}$, i = 1, 2, …, *m* and means, variances and covariances of order statistics arising from DW(0, 1, λ) to obtain $\sigma^{-2}\xi_c^{(m)}$ and $\sigma^{-2}\zeta_c^{(m)}$ for c = 1, 2, …, m-1. The values of $\sigma^{-2}\xi_c^{(m)}$, $\sigma^{-2}\zeta_c^{(m)}$ are tabulated in for c = 1, 2, …, m - 1, m = 2 (1) 5 and for λ = 0.5, 0.75, 1 & 3 and are given in Table 1. We have also obtained variances of the *U*-statistics $U_{1;n}^{(m)}$ and $U_{2;n}^{(m)}$ for m = 2 (1) 5, λ=0.5, 0.75, 1 & 3 and n = 10, 15, 20, 30, 50 & 100 and are also given in Table 2.

λ		0.5		0.75		1		3	
m	c	$\sigma^{-2}\xi_c^{(m)}$	$\sigma^{-2}\zeta_c^{(m)}$	$\sigma^{-2}\xi_c^{(m)}$	$\sigma^{-2}\zeta_c^{(m)}$	$\sigma^{-2}\xi_c^{(m)}$	$\sigma^{-2}\zeta_c^{(m)}$	$\sigma^{-2}\xi_c^{(m)}$	$\sigma^{-2}\zeta_c^{(m)}$
2	1	6.00002	1.36886	1.00305	0.49116	0.50000	0.25926	0.22569	0.02816
	2	12.00002	2.91836	2.00610	1.20244	1.00000	0.77778	0.45137	0.55597

3	1	0.31413	0.60838	0.20260	0.21829	0.17964	0.11523	0.06965	0.01252
	2	0.95966	1.23683	0.45894	0.46104	0.36871	0.25926	0.14038	0.08055
	3	2.09429	1.88536	0.83389	0.72825	0.58951	0.43210	0.24928	0.20410
4	1	0.14789	0.29983	0.09670	0.11770	0.09338	0.06403	0.02618	0.00699
	2	0.39240	0.60273	0.21400	0.23921	0.19179	0.13472	0.05343	0.02493
	3	0.74500	0.91749	0.35813	0.36648	0.29819	0.21239	0.09103	0.05395
	4	1.21719	1.26087	0.53532	0.50542	0.41554	0.29862	0.14825	0.10176
5	1	0.04821	0.18538	0.05006	0.07422	0.05664	0.04073	0.01239	0.00447
	2	0.11792	0.37323	0.10829	0.14974	0.11582	0.08379	0.02551	0.01187
	3	0.21286	0.56599	0.17562	0.22719	0.17819	0.12931	0.04192	0.02231
	4	0.34796	0.76663	0.25500	0.30755	0.24456	0.17757	0.06415	0.03732
	5	0.56418	0.97864	0.35752	0.39212	0.31685	0.22898	0.09569	0.05984

The main advantage of these proposed estimators is that, if one uses BLUE based on a small sample size m (say 5) as kernel, it is enough to calculate the means and covariances of standard order statistics up to sample sizes 2m-1(say 9) only. Using this we can explicitly evaluate the estimators and their variances for any sample size n.

Table 1: Values of $\sigma^{-2}\,\xi_c^{(m)}$ and $\sigma^{-2}\,\zeta_c^{(m)}$ for c = 1,2,…,m-1,m = 2(1)5 and λ= 0.5,0.75,1&3

4. Efficiency Comparisons of $\boldsymbol{U_{1:n}^{(m)}}$ and $\boldsymbol{U_{2:n}^{(m)}}$ with other Unbiased Estimators Based on Gini's Mean Difference

It may be recalled that in the work of Balakrishnan & Kocherlakota (1985) or Rao & Narasimham (1989), the relative efficiency comparisons of the estimate of the parameters μ and σ are not carried out. In this context we consider two possible estimators for comparing the efficiencies of $U_{1;n}^{(m)}$ and $U_{2;n}^{(m)}$.

Let $X_{1:2}$ and $X_{2:2}$ are the order statistics of a random sample X_1 and X_2 drawn from an absolutely continuous distribution having pdf of the form given in (2.1), then the population Gini's mean difference G is given by

$$G = E|X_1 - X_2| = E(X_{2:2} - X_{1:2}) = (\alpha_{2:2} - \alpha_{1:2})\sigma$$

The *U*-statistic based on the order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ of a random sample of size *n* from (2.1) using the kernel $(\alpha_{2:2} - \alpha_{1:2})^{-1}(X_{2:2} - X_{1:2})$ is given by

$$T_n = -\frac{2}{n(n-1)}(\alpha_{2:2} - \alpha_{1:2})^{-1}\sum_{i=1}^{n}(n-2i+1)X_{i:n}$$

Clearly T_n is an unbiased estimator of σ and is based on the Gini's mean difference of the sample.

Now we are going to prove the following theorem and the results in the theorem were not seen proved in the available literature till now.

Theorem 4.1

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics of a random sample of size n arising from an absolutely continuous distribution having pdf f(x; μ, σ) with location parameter μ and scale parameter σ. Let T_n be the unbiased estimator of σ based on sample Gini's mean difference and $U^{(2)}_{2:n}$ be the U-statistic estimator of σ using BLUE of σ based on a sample of size 2 as kernel. Then $U^{(2)}_{2:n} = T_n$. If f(x; μ, σ) is symmetric about μ, then $U^{(2)}_{2:n} = U^{(3)}_{2:n}$ and $U^{(2)}_{1:n} = \bar{X}_n$, where $\bar{X}_n$ is the sample mean.

Proof

Let X_1 and X_2 be two independent and identically distributed random variables each distributed with pdf f(x; μ, σ). Let $X_{1:2}$, $X_{2:2}$ be the order statistics of X_1 and X_2. Then the population Gini's mean difference is defined by

$$E|X_1 - X_2| = E(X_{2:2} - X_{1:2}) = (\alpha_{2:2} - \alpha_{1:2})\sigma \quad (4.1)$$

where $\alpha_{1:2}$ and $\alpha_{2:2}$ are the means of the smallest and largest order statistics of a random sample of size 2 drawn from f(x; 0, 1). The sample Gini's mean difference based on the given sample is then given by

$$G_n = \frac{2}{n(n-1)}\sum_{i=1}^{n}(n-2i+1)X_{i:n} \quad (4.2)$$

From (4.1) and (4.2) we obtain

$$T_n = (\alpha_{2:2} - \alpha_{1:2})^{-1} G_n$$

as an unbiased estimator of σ of f(x; μ, σ) based on sample Gini's mean difference.

Using (2.3) and on simplification we obtain the BLUE $\hat{\sigma}_2$ based on the order statistics $X_{1:2}$ and $X_{2:2}$ of a random sample of size 2 drawn from f(x; μ, σ) as

$$\hat{\sigma}_2 = (\alpha_{2:2} - \alpha_{1:2})^{-1}(X_{2:2} - X_{1:2}). \tag{4.3}$$

On using $\hat{\sigma}_2$ in (4.3) as the kernel of degree 2, the *U*-statistic estimator generated is equal to $U_{2:n}^{(2)}$.

As T_n and $U_{2:n}^{(2)}$ are the *U*-statistics generated from the same kernel $(\alpha_{2:2} - \alpha_{1:2})^{-1}(X_{2:2} - X_{1:2})$,

$$U_{2:n}^{(2)} = T_n$$

If the pdf f(x; μ, σ) is symmetric about μ, then its standard form f(x; 0, 1) will be symmetric about 0 and hence we have,

$$\alpha_{r:n} = -\alpha_{n-r+1:n}, \ r = 1, 2, \ldots, [n/2]; \ \alpha_{[n/2]+1:n} = 0, \text{ if n is odd} \tag{4.4}$$

and

$$v_{r,s:n} = v_{n-s+1,n-r+1:n}, \ 1 \le r \le s \le n \tag{4.5}$$

Using (4.4) and (4.5) in (2.3), we get

$$\hat{\sigma}_2 = (2\alpha_{2:2})^{-1}(X_{2:2} - X_{1:2})$$

$$\hat{\sigma}_3 = (2\alpha_{3:3})^{-1}(X_{3:3} - X_{1:3})$$

Hence (2.7) reduces to

$$U_{2:n}^{(2)} = -\frac{1}{2\alpha_{2:2}\binom{n}{2}} \sum_{i=1}^{n} (n - 2i + 1) X_{i:n} \tag{4.6}$$

and

$$U_{2:n}^{(3)} = -\frac{1}{2\alpha_{3:3}\binom{n}{3}} \sum_{i=1}^{n} \left[\binom{n-i}{2} - \binom{i-1}{2} \right] X_{i:n}$$

$$= -\frac{3}{4\alpha_{3:3}\binom{n}{2}} \sum_{i=1}^{n} (n-2i+1) X_{i:n} \tag{4.7}$$

Then using (4.6) in (4.7) we get

$$U_{2:n}^{(3)} = \frac{3\alpha_{2:2}}{2\alpha_{3:3}} U_{2:n}^{(2)} \tag{4.8}$$

From David and Nagaraja (2003, P.49) we have,

$$3\alpha_{2:2} = 2\alpha_{3:3} \tag{4.9}$$

Using (4.8) in (4.9) we get

$$U_{2:n}^{(2)} = U_{2:n}^{(3)}$$

Now using the symmetry property of f(x; 0, 1) and using (2.2), the BLUE of μ based on a random sample of size 2 is

$$\hat{\mu}_2 = 0.5\left(X_{2:2} + X_{1:2}\right) = \overline{X}_2$$

Hence (2.4) becomes

$$U_{1:n}^{(2)} = \overline{X}_n$$

This proves the theorem 4.1.

Then for comparing the class of estimators $\left\{U_{1:n}^{(m)}\right\}$ of μ, we have computed the relative efficiency $RE\left(U_{1:n}^{(m)} \mid T_n\right)$ of *U*-statistic $U_{1:n}^{(m)}$ with respect T_n for λ=0.5, 0.75, 1 & 3, m=3(1)5, n = 10, 15, 20, 30, 50 & 100 and is given in Table 2. The asymptotic relative efficiency of *U*-statistic $U_{1:n}^{(m)}$ with respect T_n is given by,

$$ARE\left(U_{1:n}^{(m)} \mid T_n\right) = \frac{4\xi_1^{(2)}}{m^2 \xi_1^{(m)}} \tag{4.10}$$

The numerical values of asymptotic relative efficiencies are also computed using (4.10) for λ=0.5, 0.75, 1&3, m=3(1)5 and are given in Table 2.

Again, the estimator $T_n^* = U_{2:n}^{(2)}$ is an unbiased estimator for σ and is based on sample Gini's mean difference. For comparing the class of estimators $\{U_{2:n}^{(m)}\}$ of σ, we have computed the relative efficiency $RE\left(U_{2:n}^{(m)} \mid T_n^*\right)$ of *U*-statistic $U_{2:n}^{(m)}$ with respect to T_n^* and are evaluated for λ=0.5, 0.75, 1 & 3 and n = 10, 15, 20, 30, 50 & 100 and are given in Table 2. The asymptotic relative efficiency $ARE\left(U_{2:n}^{(m)} \mid T_n^*\right)$ of *U*-statistic $U_{2:n}^{(m)}$ with respect T_n^* is given by,

$$ARE\left(U_{2:n}^{(m)} \mid T_n^*\right) = \frac{4\zeta_1^{(2)}}{m^2 \zeta_1^{(m)}} \tag{4.11}$$

The numerical values of asymptotic relative efficiencies are also computed using (4.11) for λ=0.5, 0.75, 1&3, m=3(1)5 and are given in Table 2.

From the Table 2, we noticed that the asymptotic relative efficiency increases with kernel size. Further the proposed *U*-statistics are asymptotically normal and hence by an application of the well-known Slutzky's theorem, those *U*-statistics and their variances can be used for constructing test statistics for testing location and scale parameters of double-Weibull distribution for large samples.

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Table 2 : Variances and Relative efficiencies of *U*-statistics estimators of double Weibull distribution for λ = 0.5,0.75,1 & 3, m = 2,3,4 & 5.

``λ	0.5							0.75						
N	5	10	20	30	50	100	∞	5	10	20	30	50	100	∞
$\sigma^{-2}\mathrm{Var}(U_{1:n}^{(2)})$	4.80002	2.40001	1.20000	0.80000	0.48000	0.24000		0.40000	0.20000	0.10000	0.06667	0.04000	0.02000	
$\sigma^{-2}\mathrm{Var}(U_{1:n}^{(3)})$	0.87946	0.35031	0.15719	0.10113	0.05899	0.02888		0.33407	0.16375	0.08130	0.05409	0.03241	0.01618	
$\sigma^{-2}\mathrm{Var}(U_{1:n}^{(4)})$	0.83943	0.31545	0.13678	0.08692	0.05017	0.02437		0.32166	0.15383	0.07570	0.05023	0.03003	0.01498	
$\sigma^{-2}\mathrm{Var}(U_{1:n}^{(5)})$	0.56418	0.17280	0.07196	0.04522	0.02588	0.01249		0.31685	0.14781	0.07219	0.04780	0.02853	0.01421	
$\sigma^{-2}\mathrm{Var}(U_{2:n}^{(3)})$	1.11315	0.55156	0.27472	0.18293	0.10966	0.05479		0.23333	0.10947	0.05322	0.03516	0.02095	0.01042	
$\sigma^{-2}\mathrm{Var}(U_{2:n}^{(4)})$	0.98617	0.48339	0.24057	0.16020	0.09604	0.04800		0.22964	0.10783	0.05249	0.03470	0.02069	0.01029	
$\sigma^{-2}\mathrm{Var}(U_{2:n}^{(5)})$	0.97864	0.47104	0.23324	0.15511	0.09290	0.04640		0.22898	0.10713	0.05215	0.03448	0.02056	0.01023	
$\mathrm{RE}\left(U_{1:n}^{(3)} \mid T_n\right)$	5.45790	6.85110	7.63391	7.91034	8.13757	8.31170	8.48914	1.19735	1.22138	1.22994	1.23245	1.23435	1.23571	2.20038
$\mathrm{RE}\left(U_{1:n}^{(4)} \mid T_n\right)$	5.71815	7.60822	8.77317	9.20421	9.56666	9.84955	10.14250	1.24356	1.30015	1.32099	1.32716	1.33186	1.33526	2.59331
$\mathrm{RE}\left(U_{1:n}^{(5)} \mid T_n\right)$	8.50795	13.8886	16.6749	17.6912	18.5465	19.2160	19.91200	1.26241	1.35307	1.38515	1.39473	1.40205	1.40735	3.20614
$\mathrm{RE}\left(U_{2:n}^{(4)} \mid T_n^*\right)$	1.12877	1.14101	1.14196	1.14190	1.14174	1.14157	1.14137	1.01608	1.01519	1.01379	1.01328	1.01286	1.01254	2.13058
$\mathrm{RE}\left(U_{2:n}^{(5)} \mid T_n^*\right)$	1.13745	1.17095	1.17785	1.17935	1.18033	1.18094	1.18147	1.01899	1.02182	1.02046	1.01984	1.01932	1.01890	2.16222
λ	1							3						

n	5	10	20	30	50	100	∞	5	10	20	30	50	100	∞
$\sigma^{-2}\text{Var}(U_{1:n}^{(2)})$	0.23813	0.11906	0.05953	0.03969	0.02381	0.01191		0.20000	0.10000	0.05000	0.03333	0.02000	0.01000	
$\sigma^{-2}\text{Var}(U_{1:n}^{(3)})$	0.23636	0.11810	0.05904	0.03936	0.02361	0.01181		0.17649	0.08727	0.04352	0.02899	0.01739	0.00869	
$\sigma^{-2}\text{Var}(U_{1:n}^{(5)})$	0.23559	0.11752	0.05872	0.03914	0.02348	0.01174		0.16401	0.07824	0.03867	0.02570	0.01539	0.00769	
$\sigma^{-2}\text{Var}(U_{2:n}^{(3)})$	0.23494	0.11674	0.05829	0.03885	0.02330	0.01165		0.16123	0.07359	0.03594	0.02383	0.01424	0.00710	
$\sigma^{-2}\text{Var}(U_{2:n}^{(4)})$	0.12274	0.05211	0.02411	0.01567	0.00921	0.00454		0.09001	0.03396	0.01465	0.00929	0.00535	0.00259	
$\sigma^{-2}\text{Var}(U_{2:n}^{(5)})$	0.12233	0.05204	0.02411	0.01568	0.00922	0.00454		0.08647	0.03303	0.01442	0.00919	0.00531	0.00258	
$\text{RE}\left(U_{1:n}^{(3)} \mid T_n\right)$	0.12225	0.05203	0.02412	0.01569	0.00923	0.00455		0.08530	0.03240	0.01428	0.00913	0.00529	0.00258	
$\text{RE}\left(U_{1:n}^{(4)} \mid T_n\right)$	1.00747	1.00818	1.00839	1.00844	1.00848	1.00851	1.23704	1.13318	1.14581	1.14891	1.14966	1.15018	1.15052	1.44011
$\text{RE}\left(U_{1:n}^{(5)} \mid T_n\right)$	1.01078	1.01318	1.01386	1.01404	1.01417	1.01427	1.33855	1.21947	1.27809	1.29313	1.29680	1.29935	1.30106	2.15543
$\text{RE}\left(U_{2:n}^{(4)} \mid T_n^{*}\right)$	1.01356	1.01995	1.02129	1.02164	1.02190	1.02207	1.41251	1.24044	1.35897	1.39102	1.39895	1.40448	1.40818	2.91548
$\text{RE}\left(U_{2:n}^{(5)} \mid T_n^{*}\right)$	1.00333	1.00137	1.00001	0.99953	0.99914	0.99884	1.01221	1.04101	1.02816	1.01577	1.01098	1.00690	1.00370	1.00730
	1.00398	1.00170	0.99962	0.99887	0.99825	0.99778	1.01847	1.05525	1.04806	1.02620	1.01748	1.01004	1.00419	1.00832