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An Autoregressive Process with Marshall-Olkin Generalized Esscher Transformed Laplace Marginals

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ABSTRACT

In this article, we introduce a new family of distributions namely, Marshall-Olkin generalized Esscher transformed Laplace distribution, which is a generalization of the three parameter Esscher transformed Laplace distribution. We obtain explicit forms for their density, distribution function and Hazard function. Properties of the distribution are studied and a real application of the distribution is considered. We also derive an AR (1) process with Marshall-Olkin generalized Esscher transformed Laplace distribution as stationary marginal and its properties are studied. The Kumaraswamy generalization of the Marshall-Olkin generalized Esscher transformed Laplace distribution is proposed along with its properties.

1. Introduction

Many generalizations of the various probability distributions that can be used to better reflect the distribution of the original data sets from diverse fields such as biomedical sciences, climatology, environmental, financial, image processing, signal processing and telecommunications are introduced so far and one among them is that introduced by Marshall and Olkin (1997). Such generalizations can be used to better reflect the distribution of the original data set. The recently introduced univariate distributions belonging to the Marshall-Olkin family of distributions are Marshall-Olkin Weibull Ghitany *et al.* (2005), Marshall-Olkin Pareto Alice and Jose (2003), Marshall-Olkin semi-Weibull Alice and Jose (2005), Marshall-Olkin ETL Dais George and Sebastian George (2013), Marshall- Olkin Lomax Ghitany *et al.* (2007), Marshall-Olkin semi Burr and Marshall-Olkin Burr Jayakumar and Thomas (2008) and Marshall-Olkin q-Weibull Jose *et al.* (2010)], bivariate Marshall Olkin Weibull Jose *et al.* (2011),

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Marshall-Olkin Frechet Krishna *et al.* (2013a,b), Marshall- Olkin Morgenstern Weibull Jose *et al.* (2013) and Modified generalized marshall-olkin family of distributions Muhammad Aslam *et al.* (2019).
Tahir *et al.* (2015) provides a detail account of some generalizations of univariate continuous distributions through the introduction of additional parameters. One among them is the Kumaraswamy Marshal-Olkin family proposed by Alizadeh *et al.* (2015) by integrating the Kumaraswamy-G family Cordeiro and de Castro, (2011) as the base line distribution. Based on this method we propose a new family of Marshall-Olkin generalized Esscher transformed Laplace distribution namely, Kumaraswamy Marshall-Olkin generalized Esscher transformed Laplace distribution which introduces more asymmetry and heavier tails in the base distribution.

In the last several decades various forms of skewed Laplace distributions have appeared in the literature see, McGill (1962), Holla and Bhattacharya (1968), Balakrishnan and Ambagaspitiya (1994), Kozumbowski and Podgorski (2000) and Poiraud-Casanova and Thomas-Agnam (2000). One parameter Esscher transformed Laplace distribution is a new class of asymmetric Laplace distributions introduced by Sebastian and Dais (2012) through Esscher transformation. It is a sub-class of one parameter exponential family and an alternative to various types of asymmetric Laplace distributions given in Kozubowski and Podgorski (2000). This class of distributions satisfy several properties compared to the general class of asymmetric Laplace distributions. These distributions are unimodal with mode equal to zero.

A major advantage of the class of ETL (τ) distributions over the general class of asymmetric Laplace distributions is that ETL (τ) belongs to a regular one parameter exponential family and hence families of this type are especially tractable for statistical inference. The various representations, properties and applications of the distribution are studied for more details see, Dais George and Sebastian George (2011) and Sebastian George and Dais George (2012). The Marshall-Olkin generalization of this distribution with application in time series analysis and the distribution of e X, where X follows Esscher transformed Laplace distribution are also studied, see George and George (2013),Sebastian George *et al.* (2016), Dais George *et al.* (2016), Dais George and Rimsha (2017), Rimsha and Dais George (2018), Rimsha and Dais George (2019), Krishnakumari. K and Dais George (2020) and Rimsha and Dais George (2020). The rest of the paper is organized as follows. We introduced a Marshall-Olkin generalized Esscher transformed Laplace distribution by adding a new parameter to the existing distribution in Section 2. In Section 3, we estimate the parameters of the distribution by the method of maximum likelihood and by method of moments. An autoregressive process with Marshall-Olkin generalized Esscher transformed Laplace distribution as marginal is developed and its properties are studied in Section 4. In Section 5, we discuss the real application of the distribution in remission times of bladder cancer patients. Finally we conclude the paper by Section 6.

2. Marshall-Olkin Generalized Esscher Transformed Laplace Distribution

In this paper we introduced Marshall-Olkingeneralized Esscher transformed Laplace distribution, which is a generalization of the three parameter Esscher transformed Laplace distribution. Three parameter Esscher transformed laplace distribution is the location scale family of the one parameter Esscher transformed laplace distribution denoted by ETL (θ, μ, σ). The probability density function and distribution function of the three parameter Esscher transformed Laplace distribution are,

$$f(x,\theta,\mu,\sigma) = \begin{cases} \frac{(1-\theta^2)}{2\sigma}\exp\left[\left(\frac{x-\mu}{\sigma}\right)(1+\theta)\right], & x<\mu \\ \frac{(1-\theta^2)}{2\sigma}\exp\left[\left(\frac{x-\mu}{\sigma}\right)(1-\theta)\right], & x\geq\mu \end{cases} \quad |\theta|\leq 1, \quad \sigma>0 \tag{2.1}$$

and

$$F(x) = \begin{cases} \frac{(1-\theta)}{2}\exp\left[\left(\frac{x-\mu}{\sigma}\right)(1+\theta)\right], & x<\mu \\ 1-\frac{(1+\theta)}{2}\exp\left[\left(\frac{x-\mu}{\sigma}\right)(1-\theta)\right], & x\geq\mu \end{cases} \quad |\theta|<1, \quad \sigma>0 \tag{2.2}$$

The characteristic function of the distribution is,

$$\Phi_{x_\theta}(t) = \frac{e^{it\mu}}{1-\frac{2it\theta\sigma}{1-\theta^2}+\frac{t^2\sigma^2}{1-\theta^2}} \tag{2.3}$$

This distribution being a heavy-tailed distribution is a competing model for data related with biomedical sciences, climatology, environmental, financial, image processing, signal processing and telecommunications. The distribution is infinite divisible, geometric infinite divisible and self-decomposable (for details see, Sebastian George and Dais George (2012) & Sebastian George *et al.* (2016)). According to the method proposed by Marshall and Olkin (1997), if Φ(t) is a characteristic function of some arbitrary distribution, then

$$\Psi(t) = \frac{\beta\Phi(t)}{1-(1-\beta)\Phi(t)}, \quad \beta > 0 \tag{2.4}$$

When β = 1, Ψ(t) = Φ(t). If X is a random variable with characteristic function (2.3), using (2.4) we get a new family of distributions which we shall refer to Marshall-Olkin Generalized Esscher transformed Laplace [MOGETL] distribution. The characteristic function of MOGETL distribution is given by,

$$\Psi(t) = \frac{1}{1+\frac{1}{\beta e^{it\mu}}[1-\frac{2it\theta\sigma}{1-\theta^2}+\frac{t^2\sigma^2}{1-\theta^2}-e^{it\mu}]}, \tag{2.5}$$

and a random variable X with this characteristic function can be represented as $X \stackrel{d}{=} \mathrm{MOGETL}(\beta,\theta,\mu,\sigma)$. The probability density function and cumulative distribution of MOGETL distributions are respectively,

$$f(x) = \frac{\sqrt{\beta e^{it\mu}}}{2\sigma^2}(1-\theta^2)\begin{cases}\exp\left[\left(\frac{x-\mu}{\sigma}\right)(1+\theta)\sqrt{\beta e^{it\mu}}\right], x < \mu \\ \exp\left[\left(\frac{x-\mu}{\sigma}\right)(1-\theta)\sqrt{\beta e^{it\mu}}\right], x \geq \mu\end{cases} |\theta| \leq 1, \sigma > 0 \tag{2.6}$$

and

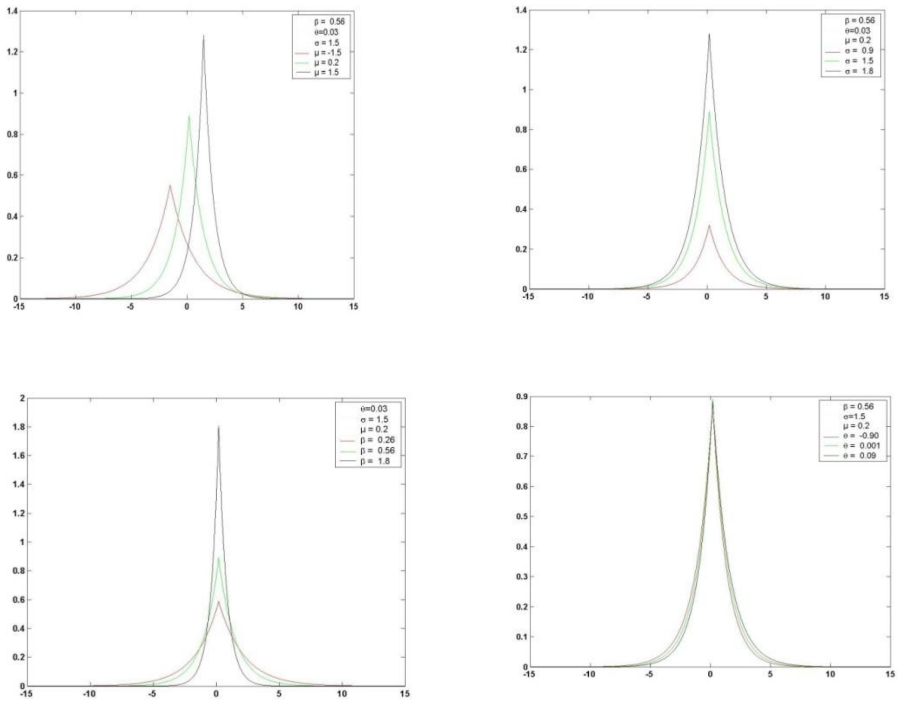
$$F(x) = \begin{cases}\frac{(1-\theta)}{2} \exp\left[\left(\frac{x-\mu}{\sigma}\right)(1+\theta)\right]\sqrt{\beta e^{it\mu}}, x < \mu \\ 1-\frac{(1+\theta)}{2}\exp\left[\left(\frac{x-\mu}{\sigma}\right)(1-\theta)\sqrt{\beta e^{it\mu}}\right], x \geq \mu\end{cases} |\theta| < 1, \quad \sigma > 0 \tag{2.7}$$

Where $\theta = \frac{2\sigma\sqrt{\beta e^{it\mu}}}{\mu+\sqrt{4\sigma^2\beta e^{it\mu}+\mu^2}}$.

The graphs of the probability density function of MOGETL (β, θ, μ, σ) distribution are given in

Figure 1. Densities of Marshall Olkin Generalized Esscher transformed Laplace distribution for

(a) β = 0.26, 0.56, 1.8, θ = 0.03, σ = 1.5 and μ = 0.2 (b) β = 0.56, θ = -0.09, 0.001, 0.09, σ = 1.5 and μ = 0.2 (c) β = 0.56, θ = 0.03, σ = 0.9, 1.5, 1.8 and μ = 0.2 (d) β = 0.56, θ = 0.03, σ =1.5 and μ= -1.5, 0.2, 1.5.



It is seen from the graph that for β < 0, the distribution is left heavy tailed and for β > 0, it is right heavy-tailed. The mean, variance, moment generating function, characteristic function and hazard function of the MOGETL (β, θ, μ, σ) distribution are given by

$$\text{Mean} = \mu + \frac{2\theta\sigma^2}{(1-\theta^2)\sqrt{\beta e^{it\mu}}} \ ,$$

$$\text{Variance} = \frac{2\theta\sigma^2\mu}{(1-\theta^2)\sqrt{\beta e^{it\mu}}} + \left(\frac{\sigma^2}{(1-\theta^2)\sqrt{\beta e^{it\mu}}}\right)^2 (1-\theta^2)\ ,$$

$$\text{Moment Generating Function} = \frac{e^{t\mu}}{1-\frac{2t\theta\sigma^2}{(1-\theta^2)\sqrt{\beta e^{it\mu}}} + \frac{t^2\sigma^2}{(1-\theta^2)\sqrt{\beta e^{it\mu}}}},$$

$$\text{Characteristic Function} = \frac{e^{it\mu}}{1-\frac{2it\theta\sigma^2}{(1-\theta^2)\sqrt{\beta e^{it\mu}}} + \frac{t^2\sigma^2}{(1-\theta^2)\sqrt{\beta e^{it\mu}}}} \quad \text{and}$$

Hazard rate Function, H(x) = $\begin{cases} \frac{\left(1-\theta^2\right)\sqrt{\beta e^{it\mu}}\exp[\alpha(\theta)]}{\sigma(2-(1-\theta)\exp[\alpha(\theta)]}, & x < \mu \\ \frac{(1-\theta)\sqrt{\beta e^{it\mu}}}{\sigma^2} & x \geq \mu \end{cases}$

$$\text{Where } \alpha(\theta) = \left(\frac{x-\mu}{\sigma^2}\right)(1+\theta)\sqrt{\beta e^{it\mu}}.$$

Now we consider the Marshall-Olkin generalized Esscher transformed Laplace distribution with location at origin. Then the pdf and d.f of the MOGETL(β, θ, σ) distribution are

$$f(x) = \frac{\sqrt{\beta}}{2\sigma^2}(-\theta^2)\begin{cases} \exp\left[\frac{x(1+\theta)\sqrt{\beta}}{\sigma^2}\right], x < 0 \\ \exp\left[\frac{-x(1-\theta)\sqrt{\beta}}{\sigma^2}\right], x \geq 0 \end{cases} \quad and \qquad (2.8)$$

$$F(x) = \begin{cases} \frac{(1-\theta)}{2} \exp\left[\frac{x(1+\theta)\sqrt{\beta}}{\sigma^2}\right], x < 0 \\ 1 - \frac{(1+\theta)}{2}\exp\left[x(1-\theta)\sqrt{\beta}\right], x \geq 0 \end{cases}$$

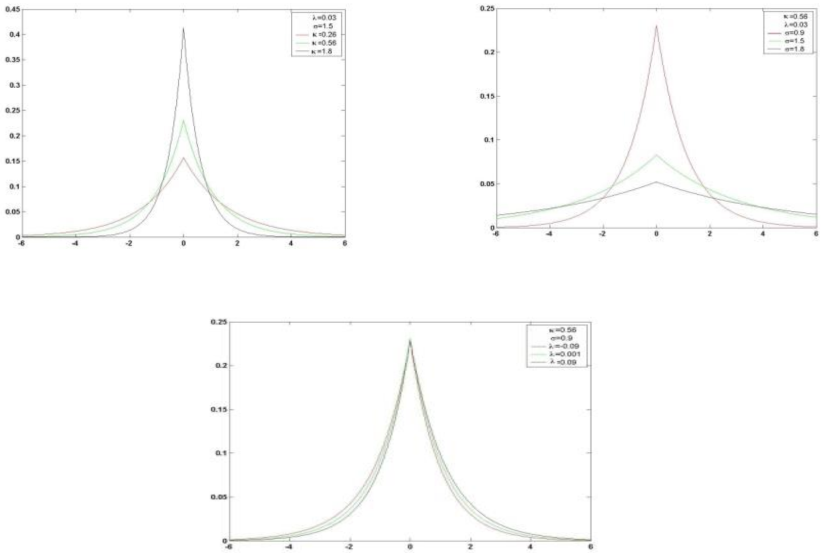
We re-parameterize the distribution given in (2.8), by putting $\sqrt{\beta(1-\theta^2)}$=λ and $\frac{\sqrt{1-\theta}}{\sqrt{1+\theta}} = k$ so that re parameterized model is given by,

$$f(x) = \frac{\lambda}{2\sigma^2}\frac{\kappa}{1+\kappa^2}\begin{cases} \exp\left[\frac{x\lambda}{\kappa\sigma^2}\right], x < 0 \\ \exp\left[\frac{-x\lambda\kappa}{\sigma^2}\right], x \geq 0 \end{cases} \quad and \qquad (2.9)$$

$$F(x) = \begin{cases} \frac{\kappa^2}{1+\kappa^2} \exp\left[\frac{x\lambda}{\kappa\sigma^2}\right], x < 0 \\ 1 - \frac{1}{1+\kappa^2}\exp\left[\frac{-x\kappa\lambda}{\sigma^2}\right], x \geq 0 \end{cases}$$

Figure 2 represents the probability density plots of MOGETL (κ, λ, σ) distribution.

Figure 2: Densities of Marshall Olkin Generalized Esscher transformed Laplace distribution for (a) κ = 0.56, λ= 0.03$ and σ = 0.9, 1.5, 1.8 (b) λ = -0.03, σ = 1.5$ and κ = 0.26, 0.56, 1.8 (c) κ = 0.56, σ = 1.5 and λ = -0.09, 0.001, 0.09.



It is clear that the distribution is unimodal with mode equal to zero and we can notice characteristic peakedness of the density at zero. The mean, variance, hazard rate function, moments, skewness and kurtosis are

The n th arbitrary moment of MOGETL distribution is given by

E(x $_n$) = n!$(\frac{\sigma^2}{\kappa\lambda})^n \frac{1+(-1)^n\kappa^{2(n+1)}}{1+\kappa^2}$.

The mean, variance, hazard rate function, moments, skewness and kurtosis are

$$Mean = \frac{(1-\kappa^2)\sigma^2}{\kappa\lambda}$$

$$Variance = \frac{(1-\kappa^4)(\sigma^2)^2}{\kappa^2\lambda^2}$$

$$HazardRateFunction = \frac{\lambda}{2\sigma^2}\frac{\kappa}{1+\kappa^2}\begin{cases} \dfrac{\exp\left[\frac{x\lambda}{\kappa\sigma^2}\right]}{1-\frac{\kappa}{1+\kappa^2}\exp\left[\frac{x\lambda}{\kappa\sigma^2}\right]}, & x < 0 \\ 1+\kappa^2, & x \geq 0 \end{cases}$$

$$\mu_1 \;=\; 0$$

$$\mu_2 = \frac{(1+\kappa^4)(\sigma^2)^2}{\kappa^2\lambda^2}$$

$$\mu_3 = \frac{2(1-\kappa^4)\sigma^6}{\kappa^3\lambda^3}$$

$$\mu_4 = \frac{(9+6\kappa^4+9\kappa^8)\sigma^8}{k^4\lambda^4}$$

$$Skewness = \frac{4(1-\kappa^6)^2}{(1+\kappa^2)^3} and$$

$$Kurtosis = \frac{9+6\kappa^4+9\kappa^8}{(1+\kappa^4)^2}$$

3. Estimation of Parameters

In this section, for estimating the parameters, we use the method of maximum likelihood and method of moments.

3.1 Maximum Likelihood Estimation

For the easiness of the estimation process, we re-parameterize the distribution given in (2.9), by putting $\kappa^2 = \frac{\eta}{\delta}$ and $\lambda^2 = \eta\delta$ so that the re-parameterized model is given by

$$f(x,\eta,\delta,\sigma) = \frac{1}{\sigma^2}\frac{\eta\delta}{\eta+\delta}\begin{cases}\exp\left[\frac{x\delta}{\sigma^2}\right], x < 0\\ \exp\left[\frac{-x\eta}{\sigma^2}\right], x \geq 0\end{cases} \tag{3.1}$$

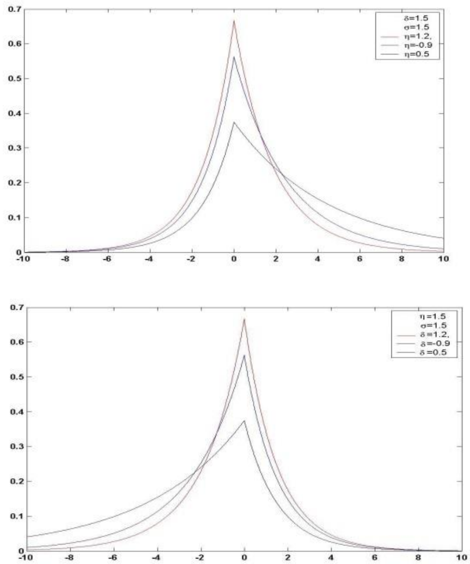
where $\eta > 0$ and $\delta > 0$ are the model parameters. We will use the notation MOGETL (η, δ, σ) to refer this distribution. The skewness and kurtosis of the distribution depends on η and δ.

Skewness = $\frac{4(\delta^3-\eta^3)^2}{(\delta+\eta)^3\delta^2}$ and

Kurtosis = $\frac{9\delta^4+6\delta^2\eta^2+9\eta^4}{\eta^2+\delta^2}$

The probability density plots of the re-parameterize MOGETL distribution is given in.

Figure 3: Densities of Marshall Olkin Generalized Esschertrans formed Laplace distribution for (a) $\delta = 1.5$, $\sigma = 1.5$ and $\eta = 0.5, 0.9, 1.2$ (b) $\eta = 1.5$, $\sigma = 1.5\$$ and $\delta = 0.5, 0.9, 1.2$.



A value of $\eta > \delta$ suggests that the right tail is thinner and there is less probability concentration to the right side of zero than the left side. Similarly, if $\delta > \eta$ the left tail will be thinner and there will be less probability concentration to the left of zero than the right side. When $\delta = \eta$, the distribution becomes symmetric.

Let $D = (X_1, X_2, ..., X_n)$ where X_i' s are independently id following MOETL(η, δ, σ) distribution given by (3), then the log likelihood function is obtained as

$$LL\ (\eta, \delta, \sigma \backslash D)\ =\ \text{-nlog}\ \sigma^2 + \text{nlog} \left(\frac{\eta\delta}{\eta+\delta}\right) + \sum_{(x\epsilon D \backslash xi<0)} \left(\frac{\delta xi}{\sigma^2}\right) + \sum_{(x\epsilon D \backslash xi>0)} \left(\frac{\eta xi}{\sigma^2}\right),$$

$$=\ \text{-nlog}\sigma^2 + \text{nlog} \left(\frac{\eta\delta}{\eta+\delta}\right)\ + \frac{\delta}{\sigma^2} S_l +\ S_r \frac{\eta}{\sigma^2}, \tag{3.2}$$

Where $S_l = \sum_{(x\epsilon D \backslash xi<0)} x_i$ and $S_r = \sum_{(x\epsilon D \backslash xi>0)} x_i$.

First by fixing σ and solving the equations formed by equating the partial derivatives of the log likelihood function with respect to η and δ to zero, we obtain the maximum likelihood estimates of η and δ and then estimating σ by iteration. The ML estimates of η and δ are

$$\hat{\delta} = \frac{n\sigma^2}{-s_l + \sqrt{(-s_l)s_r}} \tag{3.3}$$

and $$\hat{\eta} = \frac{n\sigma^2}{-s_r + \sqrt{(-s_l)s_r}}. \tag{3.4}$$

3.2 Method of Moments

By equating the population and sample moments obtained from (2.8), we get the moment estimates of β, θ and σ are

$$\widehat{\beta} = \frac{4\theta^2\sigma^2}{(1-\theta^2)^{\frac{3}{2}} m_1'}$$

$$\widehat{\theta} = \frac{m_2' - (m_1')^2}{m_2' + (m_1')^2} \qquad \text{and} \tag{3.5}$$

$$\widehat{\sigma} = \sqrt{\frac{\left(\frac{2\theta\sqrt{1-\theta}}{(1-\theta)^{\frac{3}{2}}}\right) m_2'}{\left(\frac{(1+\theta^2)^2-(1-\theta^2)^2}{(1+\theta^2)^2} m_2'\right) m_1'}}.$$

4. Marshall-Olkin Generalized Esscher Transformed Laplace Processes

Autoregressive models with non-Gaussian marginal distribution have received a tremendous attention in the recent two decades. Since the fundamental frame work of Gaver and Lewis (1980), various authors developed autoregressive models with non-Gaussian marginal distribution due to the wide applications of such models in real life situations. Lawrence and Lewis (1981,1985), Dewald and Lewis (1985), Anderson and Arnold (1993) and Jayakumar and Pillai (1993), Dais George, *et al.* (2016) developed autoregressive models with different non-Gaussian marginal distributions. Here we develop an auto-regressive model with Marshall-Olkin generalized Esscher transformed Laplace distribution as marginal. Consider a first order auto-regressive (AR (1)) model defined by the structural relationship,

$$X_n = \begin{cases} \varepsilon_n & w.p \quad \frac{1}{\beta} \\ X_{n-1} + \varepsilon_n & w.p \quad 1-\frac{1}{\beta} \end{cases} \tag{4.1}$$

where $\beta > 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables. This model is equivalent to the autoregressive model discussed in Lawrance and Lewis (1985).

Theorem:

Consider $\{Xn, n \geq 1\}$ given by

$$X_n = \begin{cases} \varepsilon_n & w.p \quad \frac{1}{\beta} \\ X_{n-1} + \varepsilon_n & w.p \quad 1 - \frac{1}{\beta} \end{cases} \tag{4.2}$$

where $\beta > 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables.

A necessary and sufficient condition that $\{X_n\}$ is a stationary process with ETL (θ, σ) marginal is that $\{\varepsilon_n\}$ is distributed as MOGETL (β, θ, σ).

Proof: If $\Phi Xn\ (t)$ is the characteristic function of $\{Xn\}$, by (4.2),

$$\Phi_{X_n}(t) = \frac{1}{\beta}\Phi_{\varepsilon_n}(t) + \left(1 - \frac{1}{\beta}\right)\Phi_{X_{n-1}}(t)\Phi_{\varepsilon_n}(t) \tag{4.3}$$

So that under stationarity assumption, we have

$$\Phi_\varepsilon(t) = \frac{\Phi_X(t)}{\frac{1}{\beta}+\left(1-\frac{1}{\beta}\right)\Phi_X(t)}. \tag{4.4}$$

If $X_n \stackrel{d}{=} ETL(\theta, \sigma)$ then

$$\Phi_X(t) = \frac{1}{1-\frac{2it\theta\sigma}{1-\theta^2}+\frac{t^2\sigma^2}{1-\theta^2}}. \tag{4.5}$$

Substituting this (4.5) in (4.4) and simplifying, we get

$$\Phi_\varepsilon(t) = \frac{1}{1 + \frac{1}{\beta}\left(1 - \frac{2it\theta\sigma}{1-\theta^2} + \frac{t^2\sigma^2}{1-\theta^2} - 1\right)}.$$

Hence $\varepsilon_n \stackrel{def}{=}$ MOET L (β, θ, σ). Conversely, if $\{\varepsilon_n\}$ is a sequence of independently and identically distributed random variables and $X_0 \stackrel{def}{=}$ ET L(θ, σ) then from (4.3) when $n = 1$ we have

$$\phi_{X1}(t) = \frac{1}{1-\frac{2it\theta\sigma}{1-\theta^2}+\frac{t^2\sigma^2}{1-\theta^2}}. \tag{4.6}$$

4.1 Bivariate Process of (Xn, Xn+1)

The joint characteristic function (Xn, Xn+1) in the autoregressive model is

$\phi_{Xn,Xn+1}(t_1, t_2) = E\ e^{(it_1 Xn + it_2 Xn+1)}$

$$= \frac{1}{1+\frac{1}{\beta}\left(1-\frac{2it_2\theta\sigma}{1-\theta^2}+\frac{{t_2}^2\sigma^2}{1-\theta^2}-1\right)}\left[\frac{\frac{1}{\beta}}{1-\frac{2it_1\theta\sigma}{1-\theta^2}+\frac{{t_1}^2\sigma^2}{1-\theta^2}}+\frac{1-\frac{1}{\beta}}{1-\frac{2i(t_1+t_2)\theta\sigma}{1-\theta^2}+\frac{(t_1+t_2)^2\sigma^2}{1-\theta^2}}\right]$$

This process is not time reversible.

Since $\varphi_{Xn,Xn+1}(t_1, t_2) \neq \varphi_{Xn,Xn+1}(t_2, t_1)$.

Therefore Xn is stationary with Xn $\stackrel{\text{def}}{=}$ ET L (θ, σ) and $\varepsilon_n \stackrel{\text{def}}{=}$ ET L (β, θ, σ).

We have

$$E(Xn) = \frac{2\theta\sigma}{1-\theta^2} \tag{4.7}$$

$$V(Xn) = \frac{(1+\theta^2)2\sigma^2}{(1-\theta^2)^2} \text{and} E(\varepsilon_n) = \frac{2\theta\sigma}{\beta(1-\theta^2)}$$

Then $E[X_{n+1}|X_n = x] = (1-\frac{1}{\beta})x + \frac{2\theta\sigma}{(1-\theta^2)}$.

The covariance between X_n and X_{n-k} is obtained on considering the representation (4.2) and simple computation

$$\text{Cov}(X_n, X_{n-k}) = (1-\frac{1}{\beta})^h \text{Cov}(X_{n-k}, X_{n-k}) \tag{4.8}$$

Hence the autocorrelation function of the process is given by $\rho(h) = (1-\frac{1}{\beta})^h$.

4.2 Extension to Higher Order Processes

Now consider the k th order autoregressive model constructed by Lawrence and Lewis (1981) with structure

$$X_n = \begin{cases} \varepsilon_n & \text{with probability } p_0 \\ X_{n-1}+\varepsilon_n & \text{with probability } p_1 \\ X_{n-2}+\varepsilon_n & \text{with probability } p_2 \\ \vdots & \\ X_{n-k}+\varepsilon_n & \text{with probability } p_k \end{cases} \tag{4.8}$$

where $\sum_{i=0}^{k} P_i = 1$, $0 < P_i < 1$, i = 0, 1, 2, ..., k and $\{\varepsilon_n\}$ is a sequence of iid MOGETL (β, θ, σ) independent of $\{X_n, X_{n-1}, \ldots, X_{n-2}\}$. In terms of characteristic function, (4.8) can be written as

$$\varphi_{X_n}(t) = P_0\varphi_{\varepsilon_n}(t) + P_1\varphi_{X_{n-1}}(t)\varphi_{\varepsilon_n}(t) + P_2\varphi_{X_{n-2}}(t)\varphi_{\varepsilon_n}(t) + \ldots + P_k\varphi_{X_{n-k}}(t)\varphi_{\varepsilon_n}(t), \quad (4.9)$$

Assuming stationarity, we get

$$\Phi_\varepsilon(t) = \frac{\Phi_X(t)}{P+(1-P)\Phi_X(t)} \quad (4.10)$$

This shows that the previous results in this section can be applied in higher order cases also.

Sample Path Behavior

The Sample path behavior of MOGETL (β, θ, σ) distribution is studied for various values of β, θ and σ and it is give in Figure 4. Sample paths of MOGETL (β, θ, σ) distribution for
(a) $\theta = 0.6$, $\sigma = 2$, $\beta = 1.5$, (b) $\theta = -0.6$, $\sigma = 2$, $\beta = 1.5$(c) $\theta = 0.9$, $\sigma = 2$, $\beta = 1.2$, (d) $\theta = -0.9$, $\sigma = 2$, $\beta = 1.2$ (e) $\theta = -0.9$, $\sigma = 2$, $\beta = 1.5$, and (f) $\theta = 0.9$, $\sigma = 2$, $\beta = 1.5$.

Figure 4.3 (a)

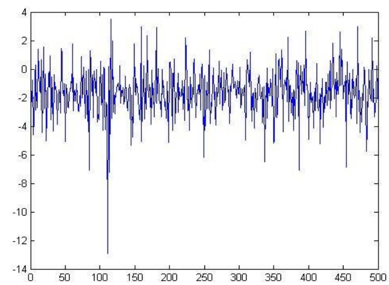


Figure 4.3 (b)

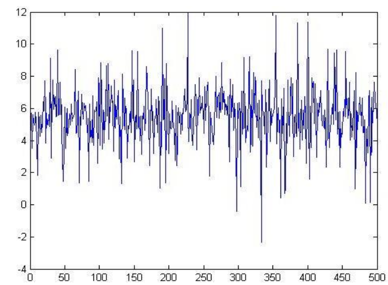


Figure 4.3 (c)

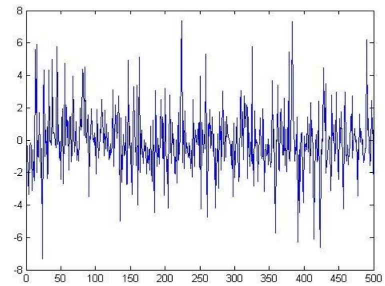


Figure 4.3 (d)

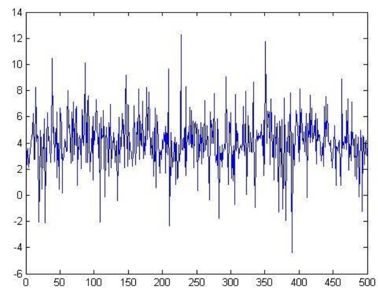


Figure 4.3 (e)

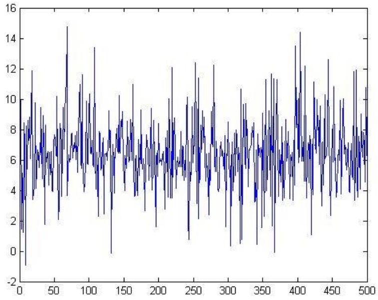
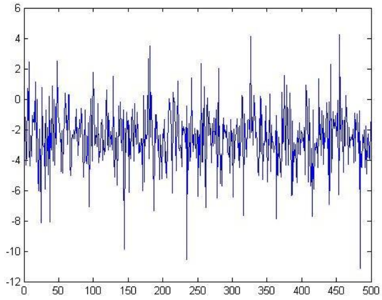


Figure 4.3 (f)



5. Fitting a Real Data to the Model

Now we consider the real data application of MOGETL (β, θ, σ) distribution. In this study we use a secondary data, which is the ordered remission times (in months) of a random sample of 142 bladder cancer patients reported in Lee and Wang (2003). It being a time series data, we identify the time series model for the data. For that, we have to estimate the value of ρ using least square estimation. The least squares estimator from the real data is

$$\hat{\rho} \;=\; \frac{\sum_{k=2}^{n} X_k X_{k-1}}{\sum_{k=2}^{n} {X_{k-1}}^{2}} \qquad = 0.75232$$

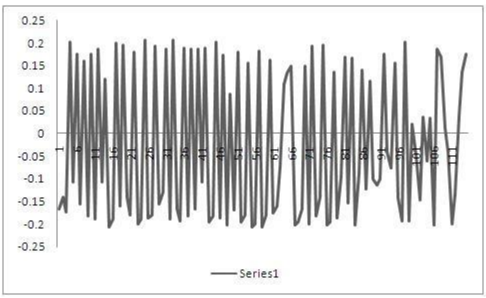
In order to find the distribution of our data, we need to first find the distribution of our error terms.

Therefore, we solve for $\hat{\varepsilon}_k$,

$\hat{\varepsilon}_k = X_k - \hat{\rho} X_{k-1}$.

The time series plot of this stationary data is given in Figure 5.

Figure 5: Time series plot of the stationary data.



The correlogram and the PACF graph of the stationary data are given Figure 6.

Figure 6. (a)The ACF plot

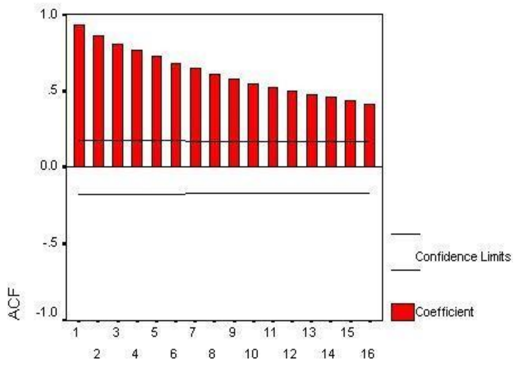
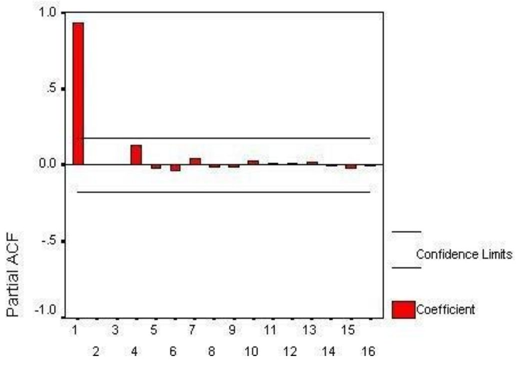
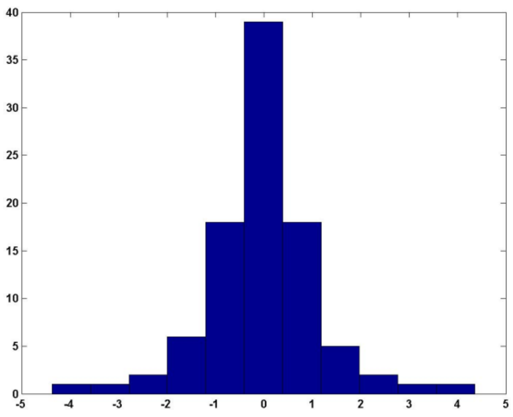


Figure 6. (b) The PACF plot



The decaying pattern of the ACF and the single large spike at lag 1 in the sample PACF suggests an AR (1) model. This time series data is modeled by AR (1) with MOGETL marginal distribution. It can be justified by fitting MOGETL distribution to the error terms. The histogram of the error terms is shown in Figure 7.

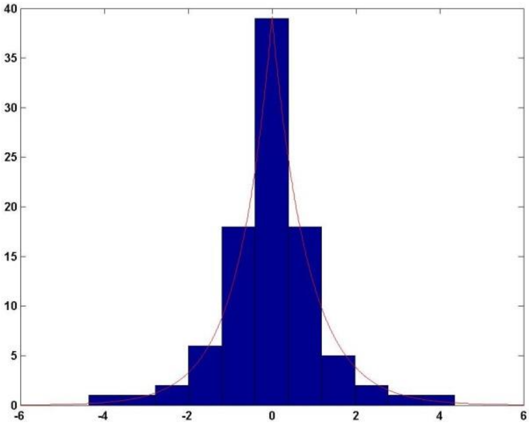
Figure 7: Histogram of the observed data.



The figure resembles the shape of the graph of MOGETL distribution given in Figure 2.

We estimate the parameters of the distribution from the data and it is obtained as β̂ = 1.68762, θ̂ = 0.79325 and σ̂ = 1.26501. The frequency curve of the distribution is superimposed in the histogram and is presented in Figure 8.

Figure 8: Embedded Frequency polygon of the observed data.



From the Figure it is clear that the Marshall-Olkin generalized Esscher transformed Laplace distribution is a good fit for this data. Also we calculate the value of K-S statistic and it is obtained as 0.049074. So we can conclude that MOGETL distribution is suitable for modeling the data on remission times of bladder cancer patients.

6. Kumaraswamy Marshall-Olkin Generalized EsscherTransformed Laplace Distribution

The cumulative distribution function with pdf of Marshall-Olkin generalized Esscher transformed Laplace Distribution is (3.1). Then here we introduce Kw-G distribution with cumulative distribution function and pdf.

$$F^{K_{wG}}(x) = \begin{cases} 1-\left[1-\left(\frac{1-\theta}{2}\exp\left[\frac{x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^a\right]^b, x<0 \\ 1-\left[1-\left(1-\frac{1+\theta}{2}\exp\left[\frac{-x(1-\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^a\right]^b, x\geq 0 \end{cases} \quad and \qquad (6.1)$$

$$f^{K_{wG}}=ab\frac{\sqrt{\beta}}{2\sigma^2}(1-\theta^2)\begin{cases}\exp\left[\frac{x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\left(\frac{1-\theta}{2}\exp\left[\frac{x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^{a-1}\left[1-\left(\frac{1-\theta}{2}\exp\left[\frac{x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^{a}\right]^{b-1}, x<0\\ \exp\left[\frac{-x(1-\theta)\sqrt{\beta}}{\sigma^2}\right]\left(1-\frac{1-\theta}{2}\exp\left[\frac{-x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^{a-1}\left[1-\left(\frac{1-\theta}{2}\exp\left[\frac{-x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^{a}\right]^{b}, x\geq 0\end{cases}$$

(6.2)

Here the parameters, a > 0 and b > 0 introduce asymmetry and heavier tails in the baseline distribution. If a = b = 1, the pdf reduces to the Marshall-Olkin generalized Esscher transformed Laplace Distribution. Survival function,

$$\bar{F}\hat{}(K\_wG)(x) = \begin{cases}\left[1-\left(\frac{1-\theta}{2}\exp\left[\frac{x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^{a}\right]^{b}, x<0\\ \left[1-\left(1-\frac{1+\theta}{2}\exp\left[\frac{-x(1-\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^{a}\right]^{b}, x\geq 0\end{cases} \quad (6.3)$$

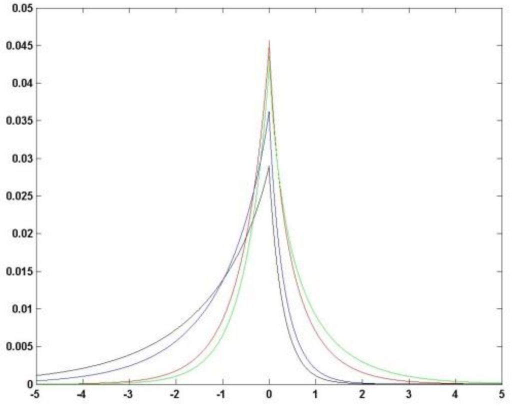
Hazard rate function(hrf):

$$h^{K_{wG}}(x) = ab\frac{\sqrt{\beta}}{2\sigma^2}\begin{cases}\frac{\exp\left[\frac{x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\left(\frac{1-\theta}{2}\exp\left[\frac{x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^{a-1}\left[1-\left(\frac{1-\theta}{2}\exp\left[\frac{x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^{a}\right]^{b-1}}{\left[1-\left(\frac{1-\theta}{2}\exp\left[\frac{x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^{a}\right]^{b}}, x<0\\ \frac{\exp\left[\frac{-x(1-\theta)\sqrt{\beta}}{\sigma^2}\right]\left(1-\frac{1-\theta}{2}\exp\left[\frac{-x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^{a-1}\left[1-\left(\frac{1-\theta}{2}\exp\left[\frac{-x(1+\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^{a}\right]^{b}}{\left[1-\left(1-\frac{1+\theta}{2}\exp\left[\frac{-x(1-\theta)\sqrt{\beta}}{\sigma^2}\right]\right)^{a}\right]^{b}}, x\geq 0\end{cases}$$

(6.4)

The probability density plots of KMOGETL distribution is given in figure9. Densities of Kumaraswamy Marshall Olkin Generalized Esscher transformed Laplace distribution for

(a) a = 0.5, b = .9, β = 5.2, σ = 0.23 and θ = 0.3, 0.5, −0.3, −0.5 (b) a = 0.5, b = .9, θ = 0.3, σ = 0.23 and β = 3.2, 5.2, 8.2, 11.2 (c) a = 0.5, b = .9, β = 5.2, θ = 0.3 and σ = 0.23, 0.53, 0.93, 1.2 (d) θ = 0.3, b = .9, β = 5.2, σ = 0.23 and a = 0.5, 1.5, 2.5, 3.5 (e) a = 1.2, 2.2, 3.2, 4.2, θ = 0.3, β = 5.2, σ = 0.23 and b = 1.23, 2.35, 3.6, 4.9 (f) a = 10, θ = 0.3, β = 5.2, σ = 0.23 and b = 0.5, 1.2, 1.5, 1.9.


0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
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0
-5
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-3
-2
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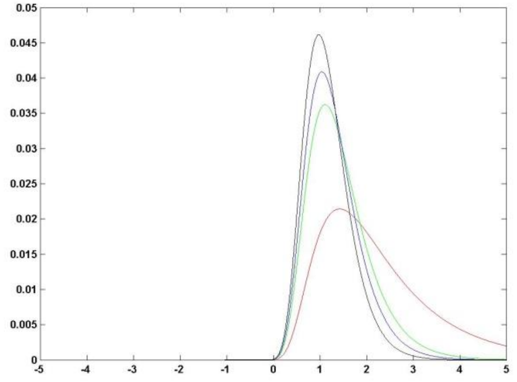
0
1
2
3
4
5


0.05
0.045
0.04
0.035
0.03
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0.02
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0.01
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-5
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-2
-1
0
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2
3
4
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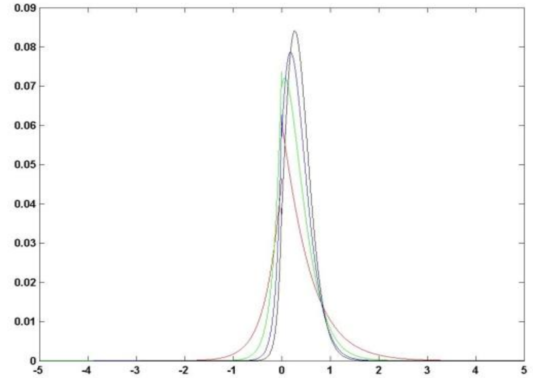
0.09
0.08
0.07
0.06
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0.04
0.03
0.02
0.01
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-5
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-2
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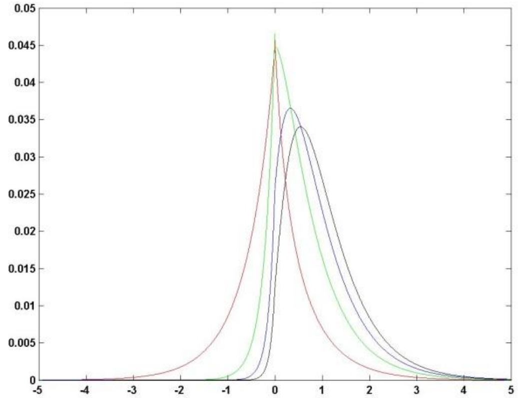
1
2
3
4
5


0.05
0.045
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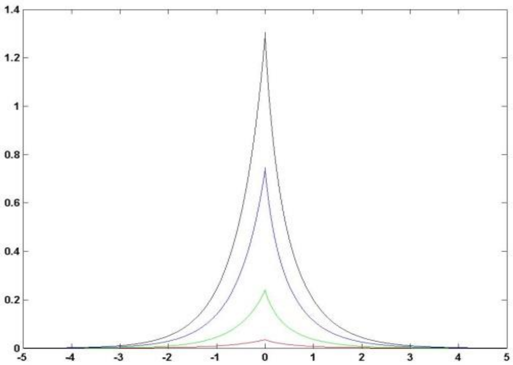
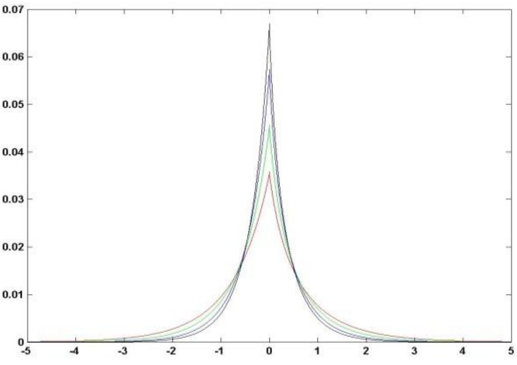
-5 -4 -3 -2 -1 0 1 2 3 4 5


1.4
1.2
1
0.8
0.6
0.4
0.2
0
-5 -4 -3 -2 -1 0 1 2 3 4 5


0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-5 -4 -3 -2 -1 0 1 2 3 4 5

6. Conclusion

In this paper, we introduced the Marshall-Olkin generalized Esscher transformed Laplace distribution and studied its properties. We also developed first order autoregressive processes with this distribution as marginal and it is extended to higher order. We identify the time series model of the real data on remission times of bladder cancer patients and it is obtained as an autoregressive model of order one with Marshall-Olkin generalized Esscher transformed Laplace marginal distribution. We also justify that the marginal distribution is Marshall-Olkin generalized Esscher transformed Laplace using Kolmogorov-Smirnov statistical test. For various values of β, MOGETL distribution provides more flexibility, allowing for asymmetry, peakedness and tail heaviness so that it can be used as an alternative to various asymmetric and heavy-tailed distributions. It can even be used for modeling left heavy-tailed and right heavy-tailed distributions by adjusting β.

References

Alizadeh M., Tahir M.H., Cordeiro G.M., Zubair M. and G.G. Hamedani (2015): The Kumaraswamy Marshal-Olkin family of distributions. *Journal of the Egyptian Mathematical Society*. In press.

Anderson D. N. and B. C. Arnold (1993) : Linnik distributions and processes. *Journal of Applied Probability,* **30**, 330-340.

Alice T. and K. K. Jose (2005): Marshall-Olkin semi-Weibullminification processes. *Recent Advances in Statistical Theory and Applications*, **1**, 6-17.

Balakrishnan N. and R. S. Ambagaspitiya (1994): On skewed Laplace distributions. Report, McMaster University, Hamilton, Ontario, Canada.

Dais George and Sebastian George (2011a): Application of Esscher transformed Laplace distribution in Web server data. *Journal of Digital Information Management*, **9(1)**, 19-26.

Dais George and Sebastian George (2011b): Application of generalized confidence interval in the study of Web performance. *International Journal of Web Applications*, **3(3)**, 138-151.

Dais George and Sebastian George (2013): Marshall-OlkinEsscher transformed Laplace distribution and processes. *Brazelian Journal of Probability and Statistics*, **27(2**), 162-184.

Dais George, Sebastian George and H. Rimsha (2016) : Esscher transformed and geometric Esscher transformed Laplace processes. *Far East Journal of Theoretical Statistics,* **52(6**), 455-483.

Dais George and H. Rimsha (2017) : Estimation of Parameters of Three Parameter Esscher Transformed Laplace Distribution. *International Journal of Statistics and Systems*, **12(4),** 669-675.

Dewald L.S. and P. A.W. Lewis (1985): A new Laplace second-order autoregressive time series model: NLAR (2). *IEEE Transactions on Information Theory***,** **31**, 645-651.
Gaver D. P. and P. A. W. Lewis (1980) : First order autoregressive Gamma sequences and point processes. *Advances in Applied Probability*, **12:** 727- 745.
Ghitany M. E., Al-Hussaini E.K. and R.A. Al-Jarallah (2005) : Marshall-Olkin Extended Weibull distribution and its applications to censored data. *Journal of Applied Statistics*, **32**, 1025-1034.
Ghitany, M. E., Al-Awadhi F. A. and A. Alkhalfan L. (2007): Marshall-Olkin extended lomax distribution and its application to censored data, *Commun. Statist. Theory Methods*, **36**, 1855-1866.
Holla, M.S. and S.K. Bhattacharya (1968): On a compound Gaussian distribution, *Ann. Inst. Statist. Math.*, **20**, 331-336.
Jayakumar, K. and M. Thomas (2008) : On a generalization to MarshallOlkin scheme and its application to Burr type XII distribution, *Statistical Papers*, doi: 10.1007/s00362-006-0024-5.
Jayakumar, K. and R. N. Pillai (1993) : The First-order autoregressive Mittag-Leffler Processes. *Journal of Applied Probability*, **30,** 462-466.
Jose, K. K., Naik, S.R. and M.M. Ristic (2010): Marshall-Olkin q Weibull distribution and maximin processes. *Statistical Papers*, **51**, 837-851.
Jose, K.K., Ristic, M.M. and J. Ancy (2011): Marshall-Olkin Bivariate Weibull distributions and processes. *Statistical Papers*, **52**, 789-798.
Jose, K. K. and S. Rani (2013): Marshall-Olkin Morgenstern-Weibull distribution: generalizations and applications. *Journal of Economic Quality Control*, **28(2**), 105-116.
Krishna, E., Jose, K. K., Alice, T., and M. M. Ristic (2013a) : Marshall-Olkin Frechet distribution. *Communications in Statistics-Theory and methods*, **42**, 4091-4107.
Krishna, E., Jose, K.K. and M. M. Ristic (2013b): Applications of Marshall Olkin Frechet distribution. *Communications in Statistics-Simulation and Computation*, **42,** 76-89.
Krishnakumari K. and Dais George (2020): Discrete Esscher Transformed Laplace Distribution and Its Application. *Stochastic Modeling and Applications*, **24(2)**, 149-159.
Kozubowski, T.J. and K. Podgorski (2000): Assymetric Laplace distributions. *Math. Sci.*, **25**, 37-46.
Lawrance, A.J. and P. A. W. Lewis (1981): A New Autoregressive Time Series Model in Exponential Variables [NEAR(1)]. *Adv. Appl. Prob.*,**13(4)**, 826-845.
Lawrance, A. J. and P. A. W. Lewis (1985): Modeling and residual analysis of nonlinear autoregressive time series in exponential variables. *Journal of Royal Statistical Society B*, **47,** 165-202.
McGill W. J. (1962) : Random uctuations of response rate. *Psychometrica*, **27**, 3-17.

Marshall, A. W. and I. Olkin (1997): A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families, *Biometrica*, **84**, 641-652.
Marshall, A. W. and I. Olkin (2007) : Life Distributions. Structure of Nonparametric, Semiparametric and Parametric Families. *Springer Series in Statistics*.
Manoel Santos-Neto, Marcelo Bourguignon, Luz M Zea, Abraao DC Nascimento and Gauss M Cordeiro, (2014): The Marshall-Olkin extended Weibull family of distributions, *Journal of Statistical Distributions and Applications.*
Muhammad Aslam, Zawar Hussain and Zahid Asghar (2019): Modified generalized marshall-olkin family of distributions, *International Journal of Advanced Statistics and Probability*, **7(1),** 18-27.
Poiraud-Casanova S. and C. Thomas-Agnan (2000): About monotone regression quantiles. *Statist. Probab. Lett.,* **48**.
Rimsha H and Dais George (2018): A generalized Esscher transformed Laplace distribution and its application. *International Journal of Statistics and Applied Mathematics*, **3(2**), 122-126.
Rimsha H and Dais George (2019): q-Esscher transformed Laplace distributions. *Communications In Statistics-Theory And Methods*, **48(7)**, 1563- 1578.
Rimsha H and Dais George (2020): Multivariate Escher Transformed Laplace Distribution and Its Generalization. *Journal of Statistical Theory and Applications,***19(2)**, 325-331.
Sebastian George and Dais George (2012) : Esscher transformed Laplace distributions and its applications. *Journal of Probability and Statistical Sciences*, **10(2),** 135-152.
Sebastian George, Dais George, and Ligi George (2016): Log-Esscher Transformed Laplace Distribution and Its Applications. *Journal of Probability and Statistical Science.* **14(1),** 29-44.
Tahir, M.H. and S. Nadarajah (2015): Parameter induction in continuous univariate distributions: Well-established G families. *Anais da Academia Brasileira de Ciencias (Annals of the Brazilian Academy of Sciences),* **87(2)**, 539-568.

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