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Estimation of Reliability Function of Lomax Distribution Using Information Theoretic Approach

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ABSTRACT

In describing the system behavior accurately through various mathematical models include parametric dynamic models, statistical probability distribution functions etc, parameter estimation plays a critical role. The Lomax distribution has played a very important role in different contexts. It was originally introduced for modeling business failure data but its limits has been extended. It also has been used for reliability modeling and life testing. Also, a new measure called Kullback-Leibler divergence for survival function is used and is a very much easier to compute in continuous distributions than the K-L divergence. It measures the distance between an empirical and a prescribed survival function. In this paper, we have estimated, Maximum Likelihood Estimator, Uniform Minimum Variance Unbiased Estimator and Kullback-Leibler Divergence for survival function of the Reliability Function of the Lomax Distribution. Also, Mean Square Error (MSE) values have been generated. R^{th} order raw moments and Mean Square Errors are presented in the form of theorems. Comparisons among UMVUE, MLE and KLS have been made to find out the best estimator. Detailed simulations show a greater performance of the KLS estimation method than the commonly used Uniform Minimum Variance Unbiased Estimation and Maximum Likelihood method in Lomax scale parameter estimation as this distance converges to zero with increasing sample size. The numerical results obtained from simulation has been illustrated.

1. Introduction

Reliability engineering that stresses trustworthiness in the life cycle administration of an item is a sub-discipline of system engineering. From the past several decades estimation of product reliability has attracted worldwide

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attention. The estimation method generally begins with parameter estimation in perspective of test data. Reliability is firmly identified with accessibility, which is commonly portrayed as the capacity of a segment or framework to work at a predefined minute or interim of time.

Lomax (1954) proposed a very useful distribution and names it as Lomax Distribution and used it for the various analyses such as for business failure life time data, reliability engineering, survival analysis, queuing theory, actuarial science and Internet traffic modelling. Lomax distribution is regarded as the heavy tailed distribution.

The cumulative density function for two parameter Lomax distribution is defined by

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \; x > 0, \alpha, \lambda > 0. \tag{1.1}$$

The corresponding probability density function is

$$f(x) = \frac{\alpha}{\lambda} * \left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)}, \; x > 0, \alpha, \lambda > 0. \tag{1.2}$$

The survival function for two parameter Lomax distribution is given by:

$$R(x) = \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \; \alpha, \lambda > 0. \tag{1.3}$$

where, λ is the scale parameter and α is the shape parameter of the distribution.

In the past few decades, a considerable amount of research effort has also been devoted to developing and voluminous literature is available covering inferential problems related to various probability distributions or life time models. Al-Noor and Al-Amer (2014) has proposed some estimation methods for Burr Type *XII* distribution to study the reliability function and shape parameter. Asrabadi (1990) have studied estimation in the Pareto distribution. Devi *et al*. (2017) obtained the entropy of Lomax probability distribution and its order statistics. Dixit *et al*. (2010) have derived efficient estimators in the Pareto distribution. He *et al*. (2014) discuss the estimation for Pareto distribution. Jalali and Watkins (2009) considered two-parameter Burr type *XII* distribution and studied three related aspects associated with ML estimation of parameters. Johnson *et al*. (1994) introduced the estimation of the parameters and the reliability of the distribution of Whipple and Rayleigh. Kern (1983) have derived minimum variance unbiased estimation in the Pareto distribution. Kumar *et al*. (2018) have estimated the probability density function of Lomax distribution. Soliman (2002) have computed the reliability estimation in generalized life-model and compared it with application to the Burr-XII . Zimmer *et al*. (1998) presented probabilistic and statistical properties of the Burr type XII distribution.

The Kullback-Leibler divergence given by Kullback and Leibler (1951) has many uses in science and engineering for the distance between two probability distributions. Although it is closely related to statistical issues such as model selection and parameter estimation, it has been used to a wide range of other analytical and experimental ideas. The reader is recommended to see (Dixit *et al*. 2010 and Olver *et al*. 2010) for a list of applications.

For two continuous random variables X and Y with probability density functions f and g respectively, the K-L divergence of f relative to g is defined by:

$$D\left(f \parallel g\right)=\int_R f\left(x\right)*\ln\frac{f\left(x\right)}{g\left(x\right)}dx \tag{1.4}$$

for x such that $g\left(x\right)\neq 0$. The function $D\left(f \parallel g\right)$ is always non-negative and it is zero if and only if $f=g$. This definition is based on the density of two random variables which in general may or may not exist Gholamhossein *et al*. (2013). Even if the densities exist, it is difficult to determine them from sample data. No assurance exists that the estimated density will match its real value, even with a larger sample size. To solve the aforesaid difficulties, scholars have suggested several K-L estimate approaches. e.g. methods defined in (Gholamhossein *et al*. (2013) and Kullback & Leibler (1951)). Also, several alternative measures have been defined in the literature, e.g. in Perez-Cruz (2008).

The aforesaid difficulties still persist when one tries to quantify the distance between a collection of data sample and a probability distribution using K-L divergence. Liu (2007) has defined a new divergence measure between sample data and a probability distribution which is based on the survival function of the random variable X , namely $f\left(x\right)=P\left(X>x\right)$, instead of its density function $f\left(x\right)$. The survival function is more conventional than the probability density because it exists continuously, can be accurately computed from data set, and its estimation is convergence by the law of large numbers. Reliability theory is based on the survival function, which is of interest and/or measurably important in the field. It quantifies the difference between an empirical survival function and a predefined survival function. Liu (2007) has used it to estimate the parameters of exponential and uniform distributions. Gholamhossein *et al*. (2013) have studied the estimation of the Weibull parameters by Kullback-Leibler divergence of Survival functions. Ramirez *et al*. (2009) have estimated the entropy and Kullback-Leibler divergence based on Szego's theorem.

The paper is organized as follows: In sections 2 and 3, we have derived MLE and UMVUE of Reliability function respectively. R^{th} order raw moments and Mean

Square Errors are presented in the form of theorems. Kullback-Leibler divergence for survival function for the shape parameter of Lomax distribution has been discussed in section 4. Simulation study and Conclusion are given in sections 5 and 6 respectively to show the advantages of KLS over UMVUE and MLE of reliability function.

2. MLE of Reliability Function of Lomax Distribution

Let $X_1, X_2, ..., X_n$ be a random sample of size n from the Lomax distribution. Using Maximum Likelihood method we can obtain the MLE of α represented a $\tilde{\alpha}$, where

$$\tilde{\alpha} = n\left[\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right)\right]^{-1}$$

Using the Invariance property of MLE which states that “MLE's are invariant under functional transformation i.e., if $\hat{\alpha}$ is a MLE of α then any function of $\hat{\alpha}$ say $g(\hat{\alpha})$ will be MLE for $g(\alpha)$”, therefore we can obtain the estimator of reliability function with replacement of $\hat{\alpha}$ instead of α in the reliability function. Then,

$$\tilde{R}(x) = \left(\frac{\lambda}{x+\lambda}\right)^{n*\left[\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\lambda}\right)\right]^{-1}} ,\alpha,\lambda > 0 \tag{2.1}$$

As a notational convenience, let us use,

$$z = z(x) = \ln\left(1+\frac{x}{\lambda}\right),\ z_x = z_x(x) = \frac{dz}{dx} = \left(\frac{1}{x+\lambda}\right),$$

through rest of the paper. Therefore, we have,

$$\tilde{R}(x) = e^{-z\tilde{\alpha}},\ \tilde{\alpha}>0 \tag{2. 2}$$

We know that, pdf of $t = \sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\lambda}\right)$ follows Gamma distribution with parameters n and α, thus,

$$h(t) = \frac{\alpha^n * t^{n-1}}{\Gamma(n)} * \exp(-\alpha t),\ t > 0, \alpha > 0 \tag{2.3}$$

Next, we have obtained the R^{th} order raw moments:
In mathematical statistics, moments involve basic calculations through which some basic properties of probability distribution like mean, variance, skewness, kurtosis etc. can be derived. The moments of a distribution are a set of

parameters that outline it. Given an arbitrary variable X, its first moment about the origin, is defined to be $E\left[X\right]$. Its second moment about the origin, is defined as the expected estimation of the arbitrary variable X_2 or $E\left[X^2\right]$. In general, the R^th^ moment of X about the origin, is characterized as $E\left[X^r\right]$.

We can similarly characterize the R^th^ moment about the mean, by $E\left[\left(X-\mu\right)r\right]$. Note that the variance of the distribution, denoted by σ^2, or $V\left[X\right]$, is the same as second order moment. The third moment about the mean is utilized to build a measure of skewness, which describes whether the probability mass is more to left side or the right side of the mean, compared with a normal distribution. The fourth moment about the mean is utilized to build a measure of peakedness, or kurtosis, which measures the "width" of a distribution.

Theorem 1: For $n > r > 0$, the R^th^ raw moments of $\tilde{R}(x)$ is given by

$$E\left(\tilde{R}(x)\right)^r = \frac{2}{\Gamma(n)}\left(\sqrt{nr\alpha z}\right)^n * K_n\left(2\sqrt{nr\alpha z}\right) \tag{2.4}$$

Where, $K_v(x)$ is the modified Bessel function Olver *et al*. (2010).

Proof: We first need to note the well-known integral representation Olver *et al*. (2010),

$$k_v(x) = \frac{1}{2} * \left(\frac{x}{2}\right)^v * \int_0^\infty \exp\left(-t - \frac{x^2}{4t}\right) * \frac{dt}{t^{v+1}} \tag{2.5}$$

The R^th^ order moment of MLE of Reliability function of Lomax distribution can be obtained as:

$$E\left[\tilde{R}(x)\right]^r = \int_0^\infty \left(\tilde{R}(x)\right)^r * h(t)dt$$

$$= \frac{1}{\Gamma(n)}\int_0^\infty (\alpha t)^{n-1} * \exp\left(-\alpha t - \frac{\left(2\sqrt{n\alpha zr}\right)^2}{4\alpha t}\right) d(\alpha t) \tag{2.6}$$

Taking, $\alpha t = h$, we get,

$$E\left[\tilde{R}(x)\right]^r = \frac{1}{\Gamma(n)}\int_0^\infty h^{n-1} * \exp\left(-h - \frac{\left(2\sqrt{n\alpha zr}\right)^2}{4h}\right)$$

By using Bessel function, we get,

$$2\left(\sqrt{n\alpha zr}\right)^2 k_{-n}\left(2\sqrt{n\alpha zr}\right)=\int_0^\infty \exp\left(-h-\frac{\left(2\sqrt{n\alpha zr}\right)^r}{4h}\right)\frac{dh}{h^{-n+1}} \tag{2.7}$$

Using Connection formula Olver *et al*. (2010), $k_{-n}\left(2\sqrt{n\alpha zr}\right)=k_n\left(2\sqrt{n\alpha zr}\right)$ and after simplications we get R[th] order moment of MLE of Reliability function in the form of following expression: $E\left(\tilde{R}(x)\right)^r=\frac{2}{\Gamma(n)}\left(\sqrt{nr\alpha z}\right)^n * K_n\left(2\sqrt{nr\alpha z}\right)$ (2.8)

Theorem 2: The mean square error of $\tilde{R}(x)$ is given by

$$MSE\left[\tilde{R}(x)\right]=\frac{2}{\Gamma(n)}\left(\sqrt{2n\alpha z}\right)^2 * K_n\left(2\sqrt{2n\alpha z}\right)-\frac{4}{\Gamma(n)} * R(x) * \left(\sqrt{n\alpha z}\right)^n$$
$$*K_n\left(2\sqrt{n\alpha z}\right)+R^2(x)$$

where, $K_v(x)$ is modified Bessel function of second kind and $R(x)$ is Reliability function of Lomax distribution.

Proof: To obtain the *MSE* of *MLE* of Reliability function, we need to find second order moment that can be obtained by putting $r=2$ in (3.8). After substituting the required terms in $E\left[\hat{R}(x)\right]^2+R^2(x)-2*R(x)*E\left[\hat{R}(x)\right]$, we can get the final expression for MSE of MLE of Reliability function.

3. UMVUE of Reliability Function

In statistics, Uniformly Minimum-Variance Unbiased Estimator (UMVUE) is an unbiased estimator that has minimum variance than any other unbiased estimator for all possible values of the parameter. Let $\underline{X}=X_1,X_2,...,X_n$ be a random sample from $f(x,\theta);\theta\in\Theta$ (say) and ν is a real-valued parameter related to θ. An unbiased estimator T(X) of ν is called the Uniformly Minimum Variance Unbiased Estimator (UMVUE) if and only if V(T(X)) $\leq$ V(U(X)) for any $\theta\in\Theta$ and any other unbiased estimator U(X) of ν. T is complete and sufficient statistic for the family $f(x,\theta);\theta\in\Theta$. The conditional density of X_1 given $T=T_1$ is denoted by $g(x|t)$ which is unbiased for $f(x,\theta)$ since,

$$E[g(x|t)]=\int_t g(x|t)*h(t,\theta) \tag{3.1}$$

$$= \int_t k(x,t,\theta)dt \tag{3.2}$$

$$= f(x,\theta) \tag{3.3}$$

where, $k(x,t,\theta)$ and $f(x,\theta)$ denote the joint pdf of X_1 given $T = T_1$. Therefore, UMVUE of Reliability function for Lomax distribution is defined by

$$\hat{R}(x) = \int_x^{\infty} g(x|\theta)dx \; = \int_x^{\infty} \frac{n-1}{t} * z_x * \left(1-\frac{z}{t}\right)^{n-2} dx$$

$$= \frac{n-1}{t} * \int_x^{\infty} \frac{dz}{dx} * \left(1-\frac{z}{t}\right)^{n-2} dx$$

$$= \frac{n-1}{t^{n-1}} * \frac{(t-z)^{n-1}}{n-1}$$

$$= \left(1-\frac{z}{t}\right)^{n-1}$$

Therefore, the final expression for UMVUE of the reliability function for the Lomax distribution is

$$\hat{R}(x) = \left(1-\frac{z}{t}\right)^{n-1} \tag{3.4}$$

where, $z = \ln\left(1+\frac{x}{\lambda}\right)$ and $t = \sum_{i=1}^{n}\left(\frac{1}{x+\lambda}\right)$.

Theorem 3: For $n > r > 1$, the Rth raw moment of $\hat{R}(x)$ is given by

$$E\left(\hat{R}(x)\right)^r = \frac{\Gamma(nr-r+1)}{\Gamma(n)} * R(x) * U(nr-r-n+1, 1-n, \alpha z) \tag{3.5}$$

where, $U(a,b,c)$ is the Kummer Confluent Hypergeometric function Gholamhossein *et al*. (2013).

Proof: Note that the Kummer confluent hypergeometric function has an integral representation Gholamhossein *et al*. (2013),

$$U(a,b,c) = \frac{1}{\Gamma(a)} \int_0^{\infty} t^{\alpha-1} * (1+t)^{b-a-1} * e^{-ct} dt, \alpha > 0, c > 0, b > a. \tag{3.6}$$

and the proof is completed by using the Kummer transformation Gholamhossein *et al.* (2013),

$$U(a,b,c) = e^{1-b} * U(1+a-b, 2-b, c) \tag{3.7}$$

The Rth order moment of UMVUE of reliability function of Lomax distribution is obtained by

$$E[\hat{R}(x)]^r = \int_0^\infty \hat{R}(x)^r * h(t)dt$$

$$= \int_0^\infty \left(1-\frac{z}{t}\right)^{(n-1)r} * \frac{\alpha^n t^{n-1}}{\Gamma(n)} * e^{-\alpha t} dt$$

Put $\alpha t = h,$ we have,

$$E[\hat{R}(x)]^r = \frac{1}{\Gamma(n)} \int_0^\infty (h-\alpha z)^{nr-r} * h^{-nr+r+n-1} * e^{-\alpha zh} dh \tag{3.8}$$

By using Kummer Confluent Hypergeometric function, we get,

$$U(nr-r+1, n+1, \alpha z) = \frac{1}{\Gamma(nr-r+1)} \int_0^\infty h^{(nr-r+1)-1} * (1+h)^{(n+1)-(nr-r+1)-1} * e^{-\alpha zh} dh$$

$$\Gamma(nr-r+1).U(nr-r+1, n+1, \alpha z) = \int_0^\infty h^{(nr-r+1)-1} * (1+h)^{(n+1)-(nr-r+1)-1} * e^{-\alpha zh} dh \tag{3.9}$$

After substituting (3.9) in (3.8), we have,

$$E(\hat{R}(x))^r = \frac{1}{e^{\alpha z}\Gamma n} * (\alpha z)^n * \Gamma(nr-r+1, n+1, \alpha z) \tag{3.10}$$

Now, Kummer transformation is of the following form:

$$U(nr-r+1, n+1, \alpha z) = (\alpha z)^{-n} * U(1+nr-r-n, 1-n, \alpha z)$$

On substituting the above expression in (3.10), we get,

$$E[\hat{R}(x)]^r = \frac{1}{e^{\alpha z}\Gamma n} * \Gamma(nr-r+1) * U(1+nr-r-n, 1-n, \alpha z)$$

Therefore, we obtain the final expression in the following form:

$$E(\hat{R}(x))^r = \frac{\Gamma(nr-r+1)}{\Gamma n} * R(x) * U(nr-r-n+1, 1-n, \alpha z); \left[\because e^{-\alpha z} = e^{-\alpha \log\left(1+\frac{x}{\lambda}\right)^{-\alpha}}\right] \tag{3.11}$$

Theorem 4: The mean square error of $\hat{R}(x)$ is given by

$$MSE(\hat{R}(x)) = \frac{\Gamma(2n-1)}{\Gamma n} * R(x) * U(n-1, 1-n, \alpha z) - R^2(X) \tag{3.12}$$

where, $U(a,b,c)$ is the Kummer Confluent Hypergeometric function.

Proof: Similarly, second order raw moment for $\hat{R}(x)$ can be obtained by putting *r=2* in (3.11), i.e.

$$E(\hat{R}(x))^2 = \frac{\Gamma(2n-1)}{\Gamma n} * R(x) * U(n-1, 1-n, \alpha z)$$

Therefore, after simplifications the mean square error of $\hat{R}(x)$ is obtained as (3.12).

4. Estimation of the Lomax Shape Parameter α by Kullback-Leibler Divergence of Survival Function

In the Kullback-Leibler divergence for survival function, we use survival function in place of density functions and add a new term to make sure that the new measure is always a positive. Also, this definition fits with the principle that the logarithm of the probability of an event should represent the information content in the event.

Next, first we recall the following definitions:

Definition 1: Let $X_1, X_2, \ldots$ be a sequence of positive, independent and identically distributed random variables from a non-increasing survival function $F(x,\alpha) = P_\alpha(X > x)$ with support S_x and vector of parameters α. Define the empirical survival function of a random sample of size *n* by

$$G_n(x) = \sum_{i=1}^{n-1} \left(1 - \frac{i}{n}\right) * I_{[x_i, x_{i=1}]}(x) \tag{4.1}$$

Where, *I* is the indicator function and $0 = X_{(0)} \le X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$ are the ordered sample.

Definition 2: Let $F(x,\theta)$ be the true survival function with unknown parameters θ and $G_n(x)$ be the empirical survival function of a random sample of size n from $F(x,\theta)$. Define the Kullback-Leibler divergence of Survival functions G_n and F by :

$$KLS(G_n \| F) = \int_0^\infty G_n(x) * \ln \frac{G_n(x)}{F(x)} - [G_n(x) - F(x)] dx \tag{4.2}$$

Gholamhossein *et al.* (2013) that the *KLS* is a divergence measure which converges to zero with increasing sample size.

The simplified form of (4.2) and is given as:

$$KLS(G_n \| F) = \int_0^\infty G_n(x) * \ln \frac{G_n(x)}{F(x)} - \left[\bar{x}_n - E(X_1)\right] dx \qquad (4.3)$$

where, $E(X_1)$ is the mean of distribution under consideration.

In order to use the *KLS* to estimate the parameters of a Lomax distribution, we just have to put the Lomax survival function given in (1.3) instead of *F* in (4.3), we get,

$$KLS(G_n \| F) = \int_0^\infty \sum_{i=0}^{n-1} \left(1 - \frac{i}{n}\right) * I_{[x_i, x_{i+1}]}(x) * \ln \left\{ \frac{\sum_{i=1}^{n-1} \left(1 - \frac{i}{n}\right) I_{[x_i, x_{i+1}]}(x)}{\left(1 + \frac{x}{\lambda}\right)^{-\alpha}} \right\} - \left\{\bar{x}_n - \frac{\lambda}{\alpha - 1}\right\} dx$$

$$= \sum_{i=0}^{n-1} \left(1 - \frac{i}{n}\right) * \ln\left(1 - \frac{i}{n}\right) * \Delta x_{i+1} - \int_0^\infty \sum_{i=0}^{n-1} \left(1 - \frac{i}{n}\right) * I_{[x_i, x_{i+1}]}(x) * \ln\left(1 + \frac{x}{\lambda}\right)^{-\alpha} dx - \left\{\bar{x}_n - \frac{\lambda}{\alpha - 1}\right\}$$

$$= \sum_{i=0}^{n-1} \left(1 - \frac{i}{n}\right) * \ln\left(1 - \frac{i}{n}\right) * \Delta x_{i+1} - \sum_{i=1}^{n-1} \left(1 - \frac{i}{n}\right) \int_{x_{(i)}}^{x_{(i+1)}} \ln\left(1 + \frac{x}{\lambda}\right)^{-\alpha} dx - \left\{\bar{x}_n - \frac{\lambda}{\alpha - 1}\right\} \qquad (4.4)$$

Consider the integral,

$$\sum_{i=0}^{n-1} \left(1 - \frac{i}{\lambda}\right) \int_{x_{(i)}}^{x_{(i+1)}} \ln\left(1 + \frac{x}{\lambda}\right)^{-\alpha} dx \qquad (4.5)$$

Define, $h(x) = \int_0^x \ln\left(1 + \frac{t}{\lambda}\right)^{-\alpha} dt$ for $x \in S_x$. Therefore, (4.5) becomes

$$\sum_{i=0}^{n-1} \left(1 - \frac{i}{n}\right) \left[h(x_{(i+1)}) - h(x_{(i)})\right] \qquad (4.6)$$

$$= \sum_{i=0}^{n-1}\left[h(x_{(i+1)}) - h(x_{(i)})\right] - \frac{1}{n}\sum_{i=0}^{n-1} i * \left[h(x_{(i+1)}) - h(x_{(i)})\right]$$

$$= \frac{1}{n} * \sum_{i=1}^{n}\int_{0}^{x} \ln\left(1+\frac{t}{\lambda}\right)^{-\alpha}$$

$$= \frac{-\alpha}{n} * \sum_{i=1}^{n}\left\{(x_i+\lambda) * \ln\left(1+\frac{x_i}{\lambda}\right) - x_i\right\} \qquad (4.7)$$

Now, using (4.7) in (4.4), we get,

$$KLS(G_n \| F) =$$

$$\sum_{I=0}^{n-1}\left(1-\frac{i}{n}\right) * \ln\left(1-\frac{i}{n}\right) * \Delta x_{i+1} - \frac{\alpha}{n}\sum_{i=1}^{n}\left\{(x_i+\lambda)\ln\left(1+\frac{x_i}{\lambda}\right) - x_i\right\} - \left\{\bar{x}_n - \frac{\lambda}{\alpha-1}\right\}$$

(4.8)

where, $\Delta x_{i+1} = x_{i+1} - x_i$ and $x_0 = 0$.

Equation (4.8) should be minimized for the values α to get the *KLS* estimation. Therefore, differentiating (4.4) w.r.t. α and equating to zero, we get,

$$\frac{\partial KLS(G_n \| F)}{\partial \alpha} = 0$$

$$\Rightarrow -\frac{1}{n}\sum_{i=1}^{n}\left\{(x_i+\lambda) * \ln\left(1+\frac{x}{\lambda}\right) - x_i\right\} - \frac{\lambda}{(\alpha-1)^2} = 0$$

On simplifications, we get,

$$\alpha = 1 + \left\{\frac{n\lambda}{\sum_{i=1}^{n}\{x_i+\lambda\} * \ln\left(1+\frac{x}{\lambda}\right) - x_i}\right\}^{\frac{1}{2}} \qquad (4.9)$$

Substituting (4.9) in (4.8), we obtain,

$$KLS(G_n \| F) =$$

$$\sum_{I=0}^{n-1}\left(1-\frac{i}{n}\right)*\ln\left(1-\frac{i}{n}\right)*\Delta x_{i+1}-\frac{1}{n}\left\{1+\left(\frac{n\lambda}{\sum_{i=1}^{n}(x_i+\lambda)*\ln\left(1+\frac{x_i}{\lambda}\right)-x_i}\right)^{\frac{1}{2}}\right\}$$

$$\sum_{i=1}^{n}\left\{(x_i+\lambda)*\ln\left(1+\frac{x_i}{\lambda}\right)-x_i\right\}-\bar{x}_n+\frac{\lambda}{1+\left(\frac{n\lambda}{\sum_{i=1}^{n}\{x_i+\lambda\}-*\ln\left(1+\frac{x}{\lambda}\right)-x_i}\right)^{\frac{1}{2}}} \quad (4.10)$$

which is the required expression for Kullback-Leibler Divergence for survival function for estimation of the shape parameter α.

5. Simulation Study and Results

We performed simulation in MATLAB where a series of codes was written to generate values $X_1, X_2, ..., X_n$ from Lomax distribution with α = 0.5 and λ = 1. For convenience we assumed α true = 0.5. To exemplify the effect of the sample size, random samples of different sizes n = 10, 20, 30, 50, 100, 200 and 500 were used. This set of simulated values was used to estimate the MLE and UMVUE of α using (2.1) and (3.4) and the parameter α using (4.9) to find the KLS for Lomax distribution. In this section, we introduce the simulation study which has been conducted to assess the statistical performance of the reliability function that we have obtained in the previous sections. To examine the performance of MLE, UMVUE and KLS some simple simulation studies have been conducted by considering values of the Reliability function of the Lomax distribution. Sample size is varied to observe the effect of small and large samples on the estimators. The results are listed in Table 1 and Table 2 for comparison purposes. Table (1) shows the estimates of the MLE, UMVUE and KLS of the Reliability function for λ = 1.0 and Table (2) shows the estimates of the MLE, UMVUE and KLS of the Reliability function for λ = 1.5 and the graphs for the values of α and λ are plotted respectively.

Table 1: UMVU, MLE and KLS estimates of R(t) for n = 10, 20, 30, 50, 100, 200 & 500; α = 0.5, λ = 1.0.

n↓	t↓	UMVUE	BIAS	MSE	MLE	BIAS	MSE	KLS	BIAS	MSE
10	0.2	0.4398	-0.325	0.1453	0.3955	-0.2945	0.1521	0.4621	-0.4281	0.1234
	0.4	0.3219	-0.2888	0.2562	0.2781	-0.2496	0.2787	0.457	-0.3855	0.1125
	0.6	0.2883	-0.2027	0.2807	0.224	-0.1784	0.3128	0.427	-0.352	0.1011
	0.8	0.2039	-0.3721	0.3292	0.1983	-0.3456	0.3483	0.369	-0.396	0.0984
20	0.2	0.4996	-0.2913	0.128	0.4944	-0.271	0.1217	0.553	0.1054	0.112
	0.4	0.3951	-0.27	0.2491	0.3678	-0.248	0.2902	0.4792	-0.358	0.1082
	0.6	0.3138	-0.2332	0.2671	0.3062	-0.2014	0.2699	0.438	-0.3114	0.1236
	0.8	0.2871	-0.2011	0.2723	0.2646	-0.1928	0.3208	0.3958	-0.273	0.13
30	0.2	0.5751	0.1921	0.1502	0.5738	0.1938	0.1343	0.5480	0.1201	0.1095
	0.4	0.4750	-0.2692	0.1520	0.4453	-0.2230	0.1868	0.4980	-0.2654	0.0822
	0.6	0.4070	-0.2721	0.2298	0.3670	-0.2400	0.2640	0.4368	-0.3260	0.1175
	0.8	0.3660	-0.3127	0.2410	0.3357	-0.2832	0.2921	0.3970	-0.3550	0.1284
50	0.2	0.6488	0.2795	0.1677	0.6648	0.3058	0.1188	0.5550	0.1399	0.0871
	0.4	0.5792	0.1965	0.1230	0.5069	0.1101	0.1436	0.5471	0.1165	0.0736
	0.6	0.5214	0.1033	0.1001	0.4297	-0.0803	0.0920	0.5017	0.0985	0.0326
	0.8	0.4095	-0.2701	0.2267	0.3534	-0.0887	0.2806	0.4562	-0.3204	0.0501
100	0.2	0.6488	0.2795	0.1677	0.6648	0.3058	0.1188	0.5510	0.1399	0.0722
	0.4	0.5792	0.1965	0.1230	0.5069	0.1101	0.1436	0.5370	0.1165	0.0681
	0.6	0.5214	0.1033	0.1001	0.4297	-0.0803	0.0920	0.5180	0.0985	0.0387
	0.8	0.4095	-0.2701	0.2267	0.3534	-0.0887	0.2806	0.4858	-0.3204	0.0402
200	0.2	0.6488	0.2795	0.1677	0.6648	0.3058	0.1188	0.5510	0.1399	0.0722
	0.4	0.5792	0.1965	0.1230	0.5069	0.1101	0.1436	0.5370	0.1165	0.0681
	0.6	0.5214	0.1033	0.1001	0.4297	-0.0803	0.0920	0.5180	0.0985	0.0387
	0.8	0.4095	-0.2701	0.2267	0.3534	-0.0887	0.2806	0.4958	-0.3204	0.0402
500	0.2	0.6488	0.2795	0.1677	0.6648	0.3058	0.1188	0.5510	0.1399	0.0722
	0.4	0.5792	0.1965	0.1230	0.5069	0.1101	0.1436	0.5370	0.1165	0.0681
	0.6	0.5214	0.1033	0.1001	0.4297	-0.0803	0.0920	0.5180	0.0985	0.0387
	0.8	0.4095	-0.2701	0.2267	0.3534	-0.0887	0.2806	0.4958	-0.3204	0.0402

From Table 1 and Table 2, we observed that on comparing the estimates performance of KLS of R(t) is better than that of UMVUE of R(t) and MLE of R(t). Also observed with increasing sample size, the estimators values approaches to true value and MSE values are decreasing. Also, we observed that for large sample sizes, the estimator provide better estimation. Again, we observed that for λ = 1.5, the estimator provide better estimation than for λ = 1.0.

Table 2: UMVU, MLE and KLS estimates of R(t) for n = 10, 20, 30, 50, 100, 200 & 500; α = 0.5, λ = 1.5.

n↓	**t↓**	**UMVUE**	**BIAS**	**MSE**	**MLE**	**BIAS**	**MSE**	**KLS**	**BIAS**	**MSE**
10	0.2	0.4483	-0.3150	0.1217	0.4125	-0.3240	0.1435	0.4831	-0.4384	0.1113
	0.4	0.3299	-0.2519	0.2410	0.2999	-0.2663	0.2533	0.4625	-0.3921	0.1038
	0.6	0.2925	-0.2201	0.2791	0.2396	-0.1834	0.2866	0.4308	-0.3580	0.1007
	0.8	0.2150	-0.3725	0.3041	0.2123	-0.3466	0.3498	0.3854	-0.3981	0.0865
20	0.2	0.5125	-0.3023	0.1080	0.4992	-0.2918	0.1189	0.5419	0.1010	0.1099
	0.4	0.4158	-0.2925	0.2054	0.3812	-0.2681	0.2215	0.4940	-0.3889	0.1002
	0.6	0.3521	-0.2587	0.2271	0.3469	-0.2685	0.2528	0.4490	-0.3328	0.1185
	0.8	0.3056	-0.2253	0.2501	0.2836	-0.2144	0.2988	0.4122	-0.3181	0.1216
30	0.2	0.5621	0.1856	0.1425	0.5632	0.1825	0.1210	0.5314	0.1187	0.1022
	0.4	0.4958	-0.2755	0.1481	0.4628	-0.2436	0.1740	0.5102	-0.2763	0.0724
	0.6	0.4344	-0.2966	0.2001	0.3837	-0.2643	0.2441	0.4560	-0.3399	0.1102
	0.8	0.3927	-0.3233	0.2280	0.3605	-0.3021	0.2711	0.4288	-0.3620	0.1178
50	0.2	0.6145	0.2418	0.1503	0.6428	0.2996	0.1126	0.5400	0.1302	0.0811
	0.4	0.5533	0.1824	0.1164	0.5145	0.1112	0.1487	0.5365	0.1102	0.0711
	0.6	0.5204	0.1010	0.0997	0.4573	-0.0961	0.0876	0.5154	0.0968	0.0388
	0.8	0.4368	-0.2978	0.2104	0.3887	-0.1004	0.0453	0.4842	-0.3287	0.0498
100	0.2	0.6102	0.2521	0.1568	0.6466	0.2846	0.1107	0.5361	0.1302	0.0688
	0.4	0.5563	0.1855	0.1190	0.5012	0.1098	0.1340	0.5286	0.1106	0.0620
	0.6	0.5198	0.1010	0.0997	0.4423	-0.0912	0.0864	0.5120	0.0936	0.0314
	0.8	0.4226	-0.2782	0.2189	0.3760	-0.0942	0.2631	0.5014	-0.3244	0.0379
200	0.2	0.6102	0.2521	0.1568	0.6466	0.2846	0.1107	0.5361	0.1302	0.0688
	0.4	0.5563	0.1855	0.1190	0.5012	0.1098	0.1340	0.5286	0.1106	0.0620
	0.6	0.5198	0.1010	0.0997	0.4423	-0.0912	0.0864	0.5120	0.0936	0.0314
	0.8	0.4226	-0.2782	0.2189	0.3760	-0.0942	0.2631	0.5014	-0.3244	0.0379
500	0.2	0.6102	0.2521	0.1568	0.6466	0.2846	0.1107	0.5361	0.1302	0.0688
	0.4	0.5563	0.1855	0.1190	0.5012	0.1098	0.1340	0.5286	0.1106	0.0620
	0.6	0.5198	0.1010	0.0997	0.4423	-0.0912	0.0864	0.5120	0.0936	0.0314
	0.8	0.4226	-0.2782	0.2189	0.3760	-0.0942	0.2631	0.5014	-0.3244	0.0379

6. Conclusion

In this paper, we have obtained UMVUE, MLE and KLS for the Reliability function (RF) of Lomax Distribution. The R^{th} order raw moments are obtained for the MLE and the UMVUE of the Reliability function. From the results of simulation study, conclusions are drawn regarding the behaviour of the estimators. Estimates for KLS, MLE and UMVUE of Reliability function have been estimated. From Table 1 and Table 2, it can be easily observe that the KLS estimates work more efficiently than the MLE and UMVUE for the RF for different sample sizes and different values of the parameters λ. Therefore, we conclude that KLS is more efficient than MLE and UMVUE for Reliability function of Lomax distribution.

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