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An Optimal Point Estimation Method for the Inverse Weibull Model Parameters Using the Runge-Kutta Method

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ABSTRACT

In parameter estimation techniques the maximum likelihood estimation method is the most common technique used in social sciences and psychology although it is usually biased in a situation where sample sizes are small or when the data are heavily censored. Thus, the main objective of this paper is to present an optimal technique using the Runge-Kutta method to find the point estimation for the distribution parameters to avoid the drawback of the maximum likelihood estimation method. This method has been applied to derive the estimators of the inverse Weibull model parameters and compare them with the standard maximum likelihood estimation and Bayesian estimation methods based on the generalized progressive hybrid-censoring scheme, via the Monte Carlo simulations. The simulation results indicated that the estimates are highly favorable for the Runge-Kutta method, which provides better estimates and outperforms Bayesian and maximum likelihood estimation methods for different sample sizes and several values of the true parameters. Finally, two real data analyses are presented to demonstrate the efficiency of the proposed methods.

1. Introduction

In statistical inference, the maximum likelihood estimation (MLE) method is known to be the most commonly has been used in point estimation despite its bias when the sample sizes are small or when the data are heavily censored and it is not efficient as the Bayesian estimate. Its bias can mislead subsequent inferences and in some distributions contains nonlinear equations that need numerical iteration techniques. Thus, in this paper, we present an optimal method to find point estimates for the distribution parameters that are more efficient than the standard MLE and the Bayesian estimates.

To illustrate that, we employed the proposed methods on one of the employed lifetime distributions and reliability theory, the inverse Weibull distribution

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(IWD) that has a probability density function and a cumulative distribution function, which presented respectively as follows:

$$f(x;\alpha,\beta)=\alpha\beta x^{-\alpha-1}\exp(-\beta x^{-\alpha}),\ x>0 \qquad (1.1)$$

$$F(x;\alpha,\beta)=\exp(-\beta x^{-\alpha}),\ x>0 \qquad (1.2)$$

where $\alpha,\beta>0$ are the shape and scale parameters respectively.

It includes two well-known distributions such as the inverse exponential distribution for $\alpha=1$ and the inverse Rayleigh distribution for $\alpha=2$.

This model proposed as a model in the analysis of life test data. It played an important role in many applications such as the dynamic components of diesel engines and several datasets such as the breakdown times of the insulating fluids subject to the action of constant tension. Furthermore, it has been used in the reliability engineering discipline and a variety of failure characteristics such as infant mortality, wear-out periods, and to determine the cost-effectiveness and maintenance periods of reliability-centered maintenance activities. Keller and Kamath (1982) and Keller *et al.* (1985) derived this model based on the physical considerations on some failures of mechanical components subject to degradation phenomena. Erto (1986) introduced other physical failure processes leading to this distribution. Moreover, it gives a good fit to life testing data reported in Nelson (1982), and weather phenomena (flood, drought, and rainfall, etc.) see, Maswadah (2003) who proved its usefulness in modeling extreme value data.

The estimation procedures in classical and Bayesian approaches to this distribution studied extensively in the literature. Calabria and Pulcini (1989) investigated the statistical properties of Maximum Likelihood Estimators (MLE's) of the parameters and reliability based on a complete sample. Erto (1989) used the Least-Square (LS) method to obtain the estimators of the parameters and reliability. Calabria and Pulcini (1990) derived the MLE's and the LS of the parameters. Calabria and Pulcini (1992) derived the Bayes estimator of the parameters and reliability. Calabria and Pulcini (1994) derived the prediction for some future variables. Khan *et al.* (2008 a, b) provided some theoretical analysis and derived the MLEs for the IWD parameters. Sultan (2008) derived the Bayesian estimates for some record values from the IWD. Miljenko *et al.* (2010) derived the least square estimates for the three-parameter IWD. In recent decades, some authors have been used the technical information about the real systems and converted it into a degree of belief about the model parameters that improved the accuracy of the estimators, see Calabria and Pulcini (1994). For a comprehensive study on the IWD, see Johnson *et al.* (1994) and Murthy *et al.* (2004).

In the reliability analysis, experiments are often terminate before all units on the test fail due to costs and time considerations or may be lost or removed from the test before failure. Thus, censored sampling schemes arise in such life testing experiments. The general scheme for studying such experiments is the Progressive censoring scheme, which is one of the familiar schemes useful for both industrial life testing applications and clinical trials. It allows the removal of some of the surviving experimental units at different stages before the termination of the test. Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014) presented comprehensive studies on the subject of progressive censoring and its applications. The progressive Type-II censoring scheme is the most commonly applied in life test experiments, although the experimental time may be very long due to the presence of some highly reliable units. Thus, Kundu and Joarder (2006) recently proposed a censoring scheme called Type-II progressive hybrid censoring scheme, which is a mixture of Type-II progressive and hybrid censoring schemes which can be described as follows:

Consider n identical items placed on a test with considering $R_1, R_2, ..., R_m$ are the random removal units that are fixed at the beginning of the experiment with $m(<n)$ and the time point T are fixed beforehand. Generally, at the time of the i_th failure, R_i units are randomly removed from the remaining surviving units $S_i = n - i - \sum_{j=1}^{i-1} R_j$, where $i = 1(1)m$. If the failure time of the m_th failure occurs before the time point T the experiment stops at the time point $X_{m:m:n}$ and all the remaining surviving units $S_m = n - m - \sum_{j=1}^{m-1} R_j$ are removed with $R_m = S_m$. On the other hand, if the m_th failure does not occur before the time point T and only k failures occur before the time point T, where $X_{m:m:n} > T$, then at the time point T all the remaining surviving units R_k are removed and the experiment terminates at the time $T^* = \min\{X_{m:m.n}, T\}$.

However, the drawback of the progressive hybrid-censoring scheme is that very few failures may occur, before the time point T In order to provide assurance of the number of the observed failures as well as the time to complete the test, Cho *et al.* (2014, 2015) proposed the generalized progressive hybrid- censoring scheme (GPHCS).This scheme modifies the progressive hybrid censoring scheme by allowing the experiment to continue beyond time T if the number of failures are less than m, which allows the experimenter to observe at least k failures. This scheme can be described as follows:

Consider a life-testing experiment in which n identical units $X_1, X_2, \ldots, X_n$ placed on the test. For $T \in (0, \infty)$, integers k and m are pre-fixed such that $k < m$ with $R_1, R_2, \ldots, R_m$ are the random removal units that are fixed at the beginning of the experiment where $n = m + \sum_{i=1}^{m} R_i$. Generally, at the time of the i_th failure, R_i units are randomly removed from the remaining surviving units $S_i = n - i - \sum_{j=1}^{i-1} R_j$, where $i=1(1)m$. This process continues until, immediately following the terminated time $T^* = \max\{X_{k:m}, \min\{X_{m:n}, T\}\}$, where at this time all the remaining surviving units are removed from the experiment according to the following cases. Let J denote the number of observed failures up to the time T. Thus, we have one of the following types of observations:

Case I: $X_{1:m:D} \leq \ldots \leq X_{k:m:D} < X_{k+1:m:D} < \ldots < X_{J:m:D}$

If $X_{K:m:D} < T < X_{m:m:D}$.

Case II: $X_{1:m:D} \leq \ldots \leq X_{k:m:D} < X_{k+1:m:D} < \ldots < X_{m:m:D}$

If $X_{k:m:D} < X_{m:m:D} < T$.

Case III: $X_{1:m:D} \leq \ldots \leq X_{J:m:D} < X_{J+1:m:D} < \ldots < X_{k:m:D}$

If $T < X_{k:m:D} < X_{m:m:D}$.

Note that for Case I, $X_{J:m:D} < T < X_{J+1:m:D}$ and $X_{J+1:m:D}, \ldots, X_{m:m:D}$ are not observed.

For Case III, $T < X_{k:m:D} < X_{m:m:D}$ and $X_{k+1:m:D}, \ldots, X_{m:m:D}$ are not observed.

Thus, given a generalized progressive hybrid censored sample, the likelihood function for the three different cases can be written in a unified form as follows:

$$L(\underline{x}; \theta) = C \prod_{i=1}^{D} f(x_{i:m:D}) \left[1 - F(x_{i:m:D})\right]^{R_i} \left[1 - F(T)\right]^{R^*_{T^\delta}}, \tag{1.3}$$

where, $C = \prod_{i=1}^{D} \sum_{j=1}^{m} (R_j + 1)$

$$D = \begin{cases} J, \quad \delta = 1 & \text{if} \quad X_{k:m:D} \le \ T < \ X_{m:m:D} \\ m, \quad \delta = 0 & \text{if} \quad X_{k:m:D} < \ X_{m:m:D} \le \ T \\ k, \quad \delta = 0 & \text{if} \quad T < \ X_{k:m:D} < \ X_{m:m:D} \end{cases}, \text{and}$$

$$\underline{X} = \begin{cases} (X_1, X_2, \ldots, X_J) & if \quad X_{k::m:D} \le \ T < \ X_{m:m:D} \\ (X_1, X_2, \ldots, X_m) & if \quad X_{k:m:D} < \ X_{m:m:D} \le \ T \\ (X_1, X_2, \ldots, X_J, X_{J+1} \ldots, X_k) & if \quad T < \ X_{k::m:D} < \ X_{m:m:D} \end{cases},$$

with R_T^* is the number of surviving units that are removed at the stopping time T.

$$R_i = \begin{cases} (R_1, R_2, \ldots, R_J, R_T^*), & R_T^* = D - J - \sum_{i=1}^{J} R_i & \text{if} \quad X_{k:m:D} \le \ T < \ X_{m:m:D} \\ (R_1, R_2, \ldots, R_m), & R_m = \ D - m - \sum_{i=1}^{m-1} R_i & \text{if} \quad X_{k:m:D} < \ X_{m:m:D} \le \ T \\ (R_1, R_2, \ldots, R_J, 0, 0, \ldots 0, R_k), & R_k = D - k - \sum_{i=1}^{k-1} R_i & \text{if} \quad T < \ X_{k:m:D} < \ X_{m:m:D} \end{cases}$$

The GPHCS has been applied for some distributions such as the Weibull distribution see Cho *et al.* (2015 b), the inverse Weibull distribution see Mohie El-Din and Nagy (2017), the Exponential distribution see Cho *et al.* (2015 a), the Rayleigh distribution see Cho *et al.* (2014), and the shape-scale family see Maswadah (2021).

2. Runge-Kutta Method

The likelihood function for n independent observations is the product of the probability density functions for any statistical model, which contains all the information about the unknown parameters in the sample. The log-likelihood function $H(\theta; X)$ depends on the unknown parameter $\theta = (\alpha, \beta)$ and the data $X = (X_1, X_2, \ldots, X_n)$ .Thus, the MLE $\hat{\theta} = \hat{\theta}(x)$ of θ is the solution of the stationary equations $\frac{\partial H(\theta; X)}{\partial \theta}\Big|_{\theta=\hat{\theta}} = 0$, which is a function of $\hat{\theta}(x)$ and X. Applying the implicit function theorem to the stationary equation with considering all partial derivatives, as well as the total derivatives are assumed to be evaluated at some known value of $\hat{\theta}(x) = \theta_0$, says. Taking the total derivative with respect to any $x \in X$ for the stationary equations, see Ramsay *et al.* (2007), we obtain

$$\frac{d}{dx}\left(\frac{\partial H(\theta; X)}{\partial \theta}\right)\Big|_{\theta=\hat{\theta}} = \frac{\partial^2 H(\theta; X)}{\partial \theta \partial x}\Big|_{\theta=\hat{\theta}} + \frac{\partial^2 H(\theta; X)}{\partial \theta^2}\Big|_{\theta=\hat{\theta}} \frac{d\hat{\theta}}{dx} = 0. \tag{2.1}$$

Solving (2.1) we obtain the first derivative with respect to x for $\hat{\theta}$ at $\theta = \hat{\theta}$ as:

$$\frac{d\hat{\theta}}{dx} = -\left(\frac{\partial^2 H(\theta; X)}{\partial \theta^2}\Big|_{\theta=\hat{\theta}}\right)^{-1} \frac{\partial^2 H(\theta; X)}{\partial \theta \partial x}\Big|_{\theta=\hat{\theta}}. \tag{2.2}$$

Thus, we can write (2.2) as the first-order ordinary differential equation in the maximum likelihood estimator $\hat{\theta}(x)$ of θ as

$$\frac{d\hat{\theta}}{dx} = f(x,\hat{\theta}), \text{ with the initial condition } \hat{\theta}(x) = \theta_0 \,, \tag{2.3}$$

It is clear that $f(x,\hat{\theta})$ and $\frac{df(x,\hat{\theta})}{d\tilde{\theta}}$ are defined and continuous functions in a rectangular region containing the point (x_0, θ_0), which ensures the existence of a unique solution for (2.3) in the neighborhood of the point (x_0, θ_0). Using any numerical technique such as the fourth-order Runge-Kutta, we can find the approximate solution given a trial set of parameter values and initial conditions. If the initial conditions are unavailable, they must be appended to the parameter $\hat{\theta}$ as quantities with respect to which the fit is optimized. Thus, the recurrence solution for (2.3) can be written as:

$$\hat{\theta}_{i+1} = \hat{\theta}_i + (K_1 + 2K_2 + 2K_3 + K_4)/6, \text{ for } i = 0,1,2,\ldots, \tag{2.4}$$

where

$K_1 = hf(x_i, \hat{\theta}_i)$, $K_2 = hf(x_i + h/2, \hat{\theta}_i + K_1/2)$,

$K_3 = hf(x_i + h/2, \hat{\theta}_i + K_2/2)$ and $K_4 = hf(x_i + h, \hat{\theta}_i + K_3)$.

Here h is the step height with a small value say (1E-01), and $\hat{\theta} = \theta_0$, is the initial value for $\hat{\theta}$.

For the inverse Weibull model, the likelihood function of the GPHCS (1.3) and its derivatives based on the generalized progressive hybrid censored samples can be derived as

$$L(\theta;\underline{x}) = C(\alpha\beta)^D \prod_{i=1}^{D} x_i^{-\alpha-1} e^{-\beta x_i^{-\alpha}} [1-e^{-\beta x_i^{-\alpha}}]^{R_i} [1-e^{-\beta T^{-\alpha}}]^{R_T^* \delta}$$

$$= C(\alpha\beta)^D Exp(-(\alpha+1)\sum_{i=1}^{D} In\, x_i - \beta \sum_{i=1}^{D} In\, x_i^{-\alpha}$$

$$\times Exp[\sum_{i=1}^{D} R_i\, In(1-e^{-\beta x_i^{-\alpha}})\delta R_T^*\ In(1-e^{-\beta T^{-\alpha}})] \tag{2.5}$$

Thus, the log- likelihood function can be derived as

$$H = In\, L(\alpha,\beta;x) = D\, In\,(\alpha\beta) - (\alpha+1)\sum_{i=1}^{D} In\, x_i$$

$$-\beta\sum_{i=1}^{D} x_i^{-\alpha} + \sum_{i=1}^{D} R_i \ln(1-e^{-\beta x_i^{-\alpha}}) + \delta R_T^* \ln(1-e^{-\beta T^{-\alpha}}) .$$

$$\frac{\partial H}{\partial \alpha} = D/\alpha - \sum_{i=1}^{D} \ln x_i + \beta\sum_{i=1}^{D} x_i^{-\alpha}\ln x_i$$

$$-\beta[\sum_{i=1}^{D} R_i \frac{x_i^{-\alpha}\ln x_i e^{-\beta x_i^{-\alpha}}}{1-e^{-\beta x_i^{-\alpha}}} + \delta R_T^* \frac{T^{-\alpha}\ln T e^{-\beta T^{-\alpha}}}{1-e^{-\beta T^{-\alpha}}}],$$

$$\frac{\partial H}{\partial \beta} = D/\beta - \sum_{i=1}^{D} x_i^{-\alpha} + \sum_{i=1}^{D} R_i \frac{x_i^{-\alpha} e^{-\beta x_i^{-\alpha}}}{1-e^{-\beta x_i^{-\alpha}}} + \delta R_T^* \frac{T^{-\alpha} e^{-\beta T^{-\alpha}}}{1-e^{-\beta T^{-\alpha}}},$$

$$\frac{\partial^2 H}{\partial \beta^2} = -D/\beta^2 - \sum_{i=1}^{D} R_i \frac{(1-e^{-\beta x_i^{-\alpha}})(x_i^{-2\alpha} e^{-\beta x_i^{-\alpha}}) + (x_i^{-\alpha} e^{-\beta x_i^{-\alpha}})^2}{(1-e^{-\beta x_i^{-\alpha}})^2}$$

$$-\delta R_T^* \frac{(1-e^{-\beta T^{-\alpha}})(T^{-2\alpha} e^{-\beta T^{-\alpha}}) + (T^{-\alpha} e^{-\beta T^{-\alpha}})^2}{(1-e^{-\beta T^{-\alpha}})^2},$$

$$\frac{\partial^2 H}{\partial \alpha^2} = -D/\alpha^2 - \beta\sum_{i=1}^{D} x_i^{-\alpha}(\ln x_i)^2 - \beta^2\sum_{i=1}^{D} R_i (\frac{x_i^{-\alpha}\ln x_i e^{-\beta x_i^{-\alpha}}}{1-e^{-\beta x_i^{-\alpha}}})^2 + \delta R_T^* (\frac{T^{-\alpha}\ln T e^{-\beta T^{-\alpha}}}{1-e^{-\beta T^{-\alpha}}})^2]$$

$$-\beta[\sum_{i=1}^{D} R_i \frac{(\beta x_i^{-\alpha}-1)x_i^{-\alpha}(\ln x_i)^2 e^{-\beta x_i^{-\alpha}}}{1-e^{-\beta x_i^{-\alpha}}} + \delta R_T^* [\frac{(\beta T^{-\alpha}-1)T^{-\alpha}(\ln T)^2 e^{-\beta T_i^{-\alpha}}}{1-e^{-\beta T^{-\alpha}}}],$$

$$\frac{\partial^2 H}{\partial \beta \partial x} = \alpha\sum_{i=1}^{D} x_i^{-\alpha-1} + \sum_{i=1}^{D} R_i \frac{(-\alpha x_i^{-\alpha-1} + \alpha\beta x_i^{-2\alpha})e^{-\beta x_i^{-\alpha}} + \alpha x_i^{-2\alpha-1} e^{-2\beta x_i^{-\alpha}}}{(1-e^{-\beta x_i^{-\alpha}})^2},$$

$$\frac{\partial^2 H}{\partial \alpha \partial x} = -\sum_{i=1}^{D} \frac{1}{x_i} + \beta\sum_{i=1}^{D}[-\alpha x_i^{-\alpha-1}\ln x_i + x_i^{-\alpha-1}]$$

$$-\beta[\sum_{i=1}^{D} R_i \frac{(1-e^{-\beta x_i^{-\alpha}})[x_i^{-\alpha-1} - \alpha x_i^{-\alpha-1}\ln x_i]e^{-\beta x_i^{-\alpha}} + \alpha\beta x_i^{-2\alpha-1}\ln x_i e^{-\beta x_i^{-\alpha}}}{(1-e^{-\beta x_i^{-\alpha}})^2}],$$

Thus, using (2.4) with the above corresponding derivatives we can find the point estimates for each α and β separately using the Runge-Kutta method.

3. Bayesian Estimation Based on the Informative Prior

We propose the use of piecewise independent priors for both parameters, namely each of the unknown parameters α and β has gamma distribution as given by:

$$g_1(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b\alpha}, \ a, b \geq 0 \text{ and } g_2(\beta) = \frac{d^c}{\Gamma(c)} \beta^{c-1} e^{-d\beta}, \ c, d \geq 0 .$$

Where the hyper-parameter a, b, c, and d are assumed to be known, and chosen to reflect the prior belief about the unknown parameters. Thus, the joint prior density is given by

$$g(\alpha, \beta) \propto \alpha^{a-1} \beta^{c-1} e^{-b\alpha - d\beta}. \tag{3.1}$$

Using the informative prior (3.1) and the likelihood function of the GPHCS (2.5), the posterior density for the parameters can be derived as

$$f(\alpha, \beta \mid x) = Cg(\alpha, \beta)(\alpha\beta)^D \prod_{i=1}^{D} x_i^{-\alpha-1} e^{-\beta x_i^{-\alpha}} [1 - e^{-\beta x_i^{-\alpha}}]^{R_i} [1 - e^{-\beta T^{-\alpha}}]^{R_T^* \delta}$$

$$= Cg(\alpha, \beta)(\alpha\beta)^D \exp[-(\alpha+1)\sum_{i=1}^{D} \ln x_i - \beta \sum_{i=1}^{D} x_i^{-\alpha}]$$

$$\times \prod_{i=1}^{D} \sum_{j=0}^{R_i} (-1)^j \binom{R_i}{j} e^{-\beta j x_i^{-\alpha}} \sum_{k=0}^{\delta R_T^*} (-1)^k \binom{\delta R_T^*}{k} e^{-\beta k T^{-\alpha}}$$

$$= Cg(\alpha, \beta)(\alpha\beta)^D \exp(-(\alpha+1)\sum_{i=1}^{D} \ln x_i)$$

$$\times \prod_{i=1}^{D} \sum_{j=0}^{R_i} \sum_{k=0}^{\delta R_T^*} (-1)^{j+k} \binom{\delta R_T^*}{k} \binom{R_i}{j} \exp[-\beta(d + \sum_{i=1}^{n} (1 + J_i) x_i^{-\alpha} + LT^{-\alpha})] .$$

Thus, the posterior density for the parameters can be written as

$$f(\alpha, \beta \mid x) = K\alpha^{D+a-1} \beta^{D+c-1} \exp(-\alpha(b + \sum_{i=1}^{D} \ln x_i))$$

$$\times \sum_{J_1=0}^{R_1} \sum_{J_2=0}^{R_2} \cdots\cdots \sum_{J_n}^{R_D} \sum_{L=0}^{\delta R_T^*} A(L, J_i) \exp[-\beta(d + \sum_{i=1}^{n} (1 + J_i) x_i^{-\alpha} + LT^{-\alpha})]$$

$$= K\alpha^{D+a-1} \beta^{D+c-1} \exp[-\alpha b - (\alpha+1)\sum_{i=1}^{D} \ln x_i]$$

$$\times \sum_{L,J}^{*} A(L, J_i) \exp[-\beta(d + \sum_{i=1}^{n} (1 + J_i) x_i^{-\alpha} + LT^{-\alpha})],$$

Where

$$\sum_{L,J}^{*} = \sum_{J_1=0}^{R_1} \sum_{J_2=0}^{R_2} \cdots\cdots \sum_{J_n}^{R_D} \sum_{L=0}^{\delta R_T^*} \quad \text{and} \quad A(L, J_i) = (-1)^{K+J_i} \binom{R_i}{J_i} \binom{\delta R_T^*}{L}$$

The marginal density for α can be generated as

$$f(\alpha \mid x) = K\Gamma(D+c)\alpha^{D+a-1}\exp[-\alpha b-(\alpha+1)\sum_{i=1}^{D}\ln x_i]$$

$$\times\sum_{K,J}^{*}A(L,J_i)[d+\sum_{i=1}^{D}(1+J_i)x_i^{-\alpha}+LT^{-\alpha}]^{-(n+c)}.$$

K is the normalizing constant and can be evaluated as

$$K^{-1} = \Gamma(D+c)\int_0^\infty \alpha^{D+a-1}\exp[-\alpha b-(\alpha+1)\sum_{i=1}^{D}\ln x_i)]$$

$$\times\sum_{L,J}^{*}A(L,J_i)[d+\sum_{i=1}^{D}(1+J_i)x_i^{-\alpha}+LT^{-\alpha}]^{-(D+c)}d\alpha.$$

4. Simulation Study

For studying the performance of the R-K, the standard MLE and Bayes methods, through the parameter estimates and the root mean square error (RMSE), which is given as: $RMSE(\theta^*) = \sqrt{\sum_{i=1}^{M}(\theta-\theta^*)^2 / M}$, where θ^* is the point estimate for the unknown parameter θ and M is the number of replications.

In our simulation study, we choose different combinations for the hyperparameters of α and β say: $a=(2,4,7)$, $b=(8,7,6)$, $c(4,7)$ and $d=(7,6)$. Thus, we can generate from the gamma distribution three values for the parameter $\alpha=(0.5929, \quad 1.1077, \quad 1.9737)$ and two values for the parameter= $\beta=(1.1077,1.9737)$ respectively. Using the above values of the parameters for generating different samples from the inverse Weibull distribution with sizes n = 20, 40, 60, and 100 to represent small, moderate, and large sizes. To assess the performance of these estimates, the RMSEs for each one were calculated using 1000 replications.

The generation of the generalized progressive hybrid censored order statistics can be carried out according to the following procedure:

Let $X=\{X_1,X_2,...,X_n\}$ be a random sample with size n from the parent distribution. Thus, based on the random sample the generalized progressive hybrid censored sample with size $m(<n)$: $X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}$, can be generated as follows:

i) Let $\Re=(R_1,R_2,...,R_m)$ be the predetermined number of a uniform random removal observations, which can be generated as

$$R_i = An\text{int}(2*(n-m-\sum_{j=1}^{i-1} R_j + 1)*U/(n-m)), \quad i = 1(1)(m-1),$$

where

$U \approx \text{Uniform}(0,1)$ and $R_m = n - m - \sum_{i=1}^{m-1} R_i$.

ii) For $i = 1$; choose the minimum observation of the random sample say $X_{i,m:n}$ which is the i_th observation that can be selected for the generalized progressive hybrid censored random sample.

iii) Remove randomly the uniform random observations of R_i that is $R_i = \{ r_{1,i}, r_{2,i}, \ldots r_{c,i} \}$,
$i = 1(1)m$ and c is the number of censored observations, where
$r_{j,i} = An\text{int}((n - \sum_{l=1}^{i-1} R_l + 1)*U)$,
$j = 1(1)c$ be the subscripts of the removal random observations without replacement from the
subset $X_i^* = X \setminus \{A_i \bigcup B_i\}$, where $A_i = \bigcup_{j=1}^{i} X_{j:m:n}$ and $B_i = \bigcup_{j=1}^{i-1} R_j$. With noting that
$r_{0,i} = 0$ it means that $R_i = 0$.

iv) If $X_{k:m:D} < T < X_{m:m:D}$, let $R_T^* = D - J - \sum_{j=1}^{J} R_j$, $m = J$, and stop.

v) If $X_{k:m:D} < X_{m:m:D} < T$, let $R_m^* = D - m - \sum_{j=1}^{m-1} R_j$, and stop.

vi) If $T < X_{k:m:D} < X_{m:m:D}$, let $R_K = D - k - \sum_{j=1}^{k-1} R_j$, $m = k$, and stop.

vii) If $i < m$, set $i = i + 1$ and go to step 2 or else stop.

From the simulation results in Tables 3, 4, 5, and 6, some points are quite clear based on these estimates, and the others have been summarized in the following main points:

i) It is clear that in general, the point estimates based on the R-K method have the smallest estimated RMSEs as compared to estimates based on Bayes and MLE methods.

ii) The estimated value of RMSEs increases as the value of α increases and decreases as the value of β increases.

iii) The estimated RMSEs decrease with decreasing the hyperparameters of the informative priors.

iv) The estimated RMSEs decrease with increasing the termination time of experiment T as expected.

v) It is immediate to note that, as the sample sizes increase the estimated RMSEs for the parameters decrease.

vi) In general, the estimated RMSEs of the parameters α and β based on the R-K method are often less than the estimated based on the standard MLE and the Bayes methods.

vii) The MLE method overestimates the actual value of the parameter β, especially when the sample size is small and has heavily censoring. Also, it has a higher bias and RMSEs compared with the R-K method.

As a conclusion, it appears that the point estimates based on the R-K method compete and outperform the standard MLE and Bayes methods.

5. Real Data Analysis

In this section, we studied two real data sets to demonstrate the performance of the proposed methods on the IW model in practice and to illustrate that this distribution can be considered as a good lifetime model for some new area of applications, comparing to many known distributions such as the Weibull distribution. We have fitted these datasets using some goodness of fit tests such as the Kolmogorov-Smirnov (K-S), Anderson-darling (A-D) and Chi-Square (CH2) tests for significance level test equals 0.05. Cordeiro *et al.* (2008) and Cordeiro and Lemonte (2011) provided a comprehensive study for these tests.

a) Flood Data Application

Consider the data given by Dumonceaux and Antle (1973), which represent the maximum flood levels (in millions of cubic feet per second) of the Susquehanna River at Harrisburg, Pennsylvania over 20 four-year periods (1890-1969) as:

0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265

Maswadah (2003) has fitted these data to the IWD.

We found the inverse Weibull model is a good fit for this dataset as shown in Table 1 and Figure (5.1 a).For studying the behavior of the flood levels based on this dataset we find the estimates for the parameters which represent the shape and scale of the flood levelfor 20 four-year periods. We found the R-K and Bayes estimates for α lies in the interval [2.8, 4.4] and for β lies in the interval [0.01, 0.08]. We noticed that the R-K, Bayes, and ML estimates for α are greater than one, while for β is close to zero and that indicates the graph is approximately symmetric, see Figure (5.2 a). These estimates indicate that the maximum flood levels are more stable during this period of years.

b) Reactor Pumps Data Application

A real dataset, for secondary nuclear pumps, has been analyzed to illustrate the proposed methods. An important aspect of nuclear energy is safety. One of the most severe accidents in nuclear power generation is the loss of coolant, where the re-circulating coolant of the pressurized water reactor may flash into steam. Under such conditions, the reactor cooling pumps become unable to generate the same head as that of the single-phase flow case. Thus, the secondary reactor pump is the feed water pump that takes from the desecrator storage tank feed water pressured up by the booster pump and pushes it into the steam generator through the high-pressure heater. Accordingly, the main feed pump must be a high temperature and high-pressure pump since it requires a head larger than the pressure inside the steam generator. The secondary circulation pump differs slightly in design and has been developed specifically for cooling at higher temperatures. The following dataset represents the times between the failure of the secondary reactor pumps. Singh *et al*., 2013, 2016 have been discussed the classical and Bayesian estimation methods under the Type-II censoring scheme of this data set. The times between failures of 23 secondary reactor pumps are as follows:

2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101,1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320

We found the inverse Weibull model is a good fit for this dataset as shown in Table 1 and Figure (5.2c).For studying the reliability of these reactor pumps based on this dataset we find the estimates for the parameters which represent the shape and scale of the failures between pumps using our model to determine the behavior of the failure pumps. We noticed that the R-K, Bayes, and ML estimators for α lie in the interval [0.81, 0.87] and for β lies in the interval [0.37, 0.41]. These estimates indicate that the above dataset is heavily right-skewed and that means the failure rate decreases with increasing time, see Figure (5.2d).Thus, we conclude that decreasing the reliability of safety mechanism with increasing time.

Table 1: The critical and calculated values for the K-S, A-D and CH2 tests and their powers (p-values) for the IWD based on the MLE's.

Data	**The Tests**	**Calculated value**	**Critical value**	**The p-values**	**MLES**	
					α	β
Flood N=20	K-S	0.6976	0.8482	0.2138	4.3141	0.0119
	A-D	0.3104	0.7414	0.5899		
	CH2	3.5552	31.1109	0.3294		
Reactor pumps N=23	K-S	0.4741	0.8528	0.8113	0.7832	0.4463
	A-D	0.3443	0.7472	0.4915		
	CH2	10.270	31.5744	0.1223		

Table 2: The estimate and the root mean square errors (RMSEs) for the parameter α and β based on the R-K and Bayes method for the GHPCS: for $m = n/2$, $\mathrm{k} = 3\mathrm{m}/4$.

Samples	**T**	**Parameters**	**R-K**		**Bayes**		**MLE**
			Estimate	**RMSE**	**Estimate**	**RMSE**	
Flood Data N=20	0.2	α	3.86273	0.29580	2.83818	0.72875	3.56693
		β	0.03308	0.00370	0.08185	0.05248	0.02937
	0.5	α	4.40492	0.31333	2.97248	1.11911	4.09159
		β	0.01793	0.00239	0.06327	0.04773	0.01554

Reactor Pumps Data N=23	0.5	α	0.83827	0.00877	0.81013	0.01936	0.91589
		β	0.40949	0.03295	0.41393	0.03738	0.37655
	5	α	0.87217	0.00889	0.84806	0.01522	0.86328
		β	0.38419	0.03304	0.37928	0.02813	0.35115

6. Conclusions

In this paper, it has been noticed that from our simulation study the bias of the R-K estimate is close to zero and much more efficient than the standard ML and the Bayes estimates even when using the informative prior. However, the standard ML estimation bias can be large and remains noticeable even when the sample sizes are too large. It increases rapidly with increasing the degree of censorship, based on the generalized progressive hybrid-censoring scheme.

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Table 3: The estimate and the root mean square errors (RMSEs) for the parameter α based on the improved MLE, Bayes and MLE methods with m = (n/2 and 3n/4), k=m/2 and T=0.75.

n	α	β	m	IMLE		Bayes		MLE	
				Estimate	RMSE	Estimate	RMSE	Estimate	RMSE
20	0.5929	1.1077	n/2	0.6886	0.1107	0.5894	0.1259	0.7564	0.3444
			3n/4	0.6644	0.0852	0.5837	0.1136	0.6726	0.1998
		1.9737	n/2	0.6241	0.0575	0.5265	0.1267	0.7686	0.3474
			3n/4	0.6210	0.0407	0.5375	0.1128	0.6784	0.2009
	1.1077	1.1077	n/2	1.2394	0.1632	1.0252	0.1881	1.3967	0.6125
			3n/4	1.2190	0.1338	1.0409	0.1787	1.2569	0.3611
		1.9737	n/2	1.1589	0.0976	0.9216	0.2401	1.3950	0.5930
			3n/4	1.1445	0.0647	0.9698	0.2089	1.2689	0.3982
	1.9737	1.1077	n/2	2.1994	0.3103	1.7467	0.3223	2.4941	1.1169
			3n/4	2.1559	0.2473	1.8022	0.2974	2.2710	0.6613
		1.9737	n/2	1.9845	0.1928	1.6091	0.4173	2.5423	1.1968
			3n/4	1.9913	0.1298	1.6886	0.3588	2.2766	0.6715
40	0.5929	1.1077	n/2	0.6950	0.1107	0.6034	0.1062	0.6554	0.1574
			3n/4	0.6679	0.0822	0.5973	0.0869	0.6301	0.1114
		1.9737	n/2	0.6388	0.0571	0.5633	0.0986	0.6581	0.1581
			3n/4	0.6181	0.0316	0.5694	0.0860	0.6296	0.1133
	1.1077	1.1077	n/2	1.2624	0.1681	1.0968	0.1708	1.2414	0.3071
			3n/4	1.2249	0.1298	1.0997	0.1535	1.1915	0.2227
		1.9737	n/2	1.1729	0.0941	1.0107	0.1804	1.2193	0.2969
			3n/4	1.1446	0.0524	1.0438	0.1542	1.1829	0.2173
	1.9737	1.1077	n/2	2.2172	0.2926	1.8789	0.2585	2.1883	0.5094
			3n/4	2.1500	0.2150	1.8893	0.2441	2.0879	0.3628
		1.9737	n/2	1.9829	0.1663	1.7653	0.3103	2.1831	0.5146
			3n/4	2.0009	0.0953	1.8179	0.2671	2.1002	0.3714
60	0.5929	1.1077	n/2	0.6949	0.1078	0.6007	0.0906	0.6300	0.1152
			3n/4	0.6703	0.0823	0.5945	0.0736	0.6136	0.0851
		1.9737	n/2	0.6399	0.0549	0.5744	0.0847	0.6337	0.1139
			3n/4	0.6209	0.0333	0.5755	0.0733	0.6150	0.0873
	1.1077	1.1077	n/2	1.2661	0.1675	1.0972	0.1504	1.1805	0.2153
			3n/4	1.2294	0.1299	1.0923	0.1314	1.1465	0.1619
		1.9737	n/2	1.1708	0.0836	1.0405	0.1514	1.1753	0.2063
			3n/4	1.1473	0.0513	1.0655	0.1277	1.1583	0.1626
	1.9737	1.1077	n/2	0.2352	0.2919	1.9088	0.2425	2.1043	0.3784
			3n/4	0.1652	0.2181	1.9205	0.2137	2.0485	0.2811
		1.9737	n/2	1.9869	0.1361	1.8168	0.2728	2.0959	0.3912
			3n/4	1.9958	0.0782	1.8714	0.2316	2.0677	0.3065
100	0.5929	1.1077	n/2	0.6989	0.1094	0.5991	0.0693	0.6139	0.0796
			3n/4	0.6728	0.0829	0.5979	0.0576	0.6084	0.0632
		1.9737	n/2	0.6413	0.0539	0.5815	0.0720	0.6149	0.0853
			3n/4	0.6212	0.0315	0.5839	0.5838	0.6075	0.6545
	1.1077	1.1077	n/2	1.2687	0.1665	1.1062	0.1299	1.1520	0.1613
			3n/4	1.2329	0.1308	1.1046	0.1017	1.1355	0.1165
		1.9737	n/2	1.1702	0.0765	1.0684	0.1218	1.1459	0.1467
			3n/4	1.1479	0.0479	1.0788	0.1021	1.1331	0.1160
	1.9737	1.1077	n/2	2.2371	0.2849	1.9362	0.2124	2.0459	0.2764
			3n/4	2.1773	0.2193	1.9396	0.1759	2.0137	0.2058
		1.9737	n/2	1.9917	0.1130	1.8880	0.2132	2.0573	0.2741
			3n/4	1.9967	0.0641	1.9049	0.1858	2.0202	0.2139

Table 4: The estimate and the root mean square errors (RMSEs) for the parameter α based on the improved MLE , Bayes and MLE methods with m= (n/2 and 3n/4), k=3m/4 and T=1.5.

n	α	β	M	IMLE		Bayes		MLE	
				Estimate	RMSE	Estimate	RMSE	Estimate	RMSE
20	0.5929	1.1077	n/2	0.6886	0.1107	0.5846	0.1224	0.7149	0.2695
			3n/4	0.6644	0.0852	0.5837	0.1136	0.6726	0.1998
		1.9737	n/2	0.6341	0.0575	0.5268	0.1253	0.7191	0.2789
			3n/4	0.6210	0.0407	0.5388	0.1134	0.6737	0.1972
	1.1077	1.1077	n/2	1.2394	0.1632	1.0342	0.1851	1.3576	0.5144
			3n/4	1.2190	0.1338	1.0510	0.1786	1.2739	0.3751
		1.9737	n/2	1.1589	0.0976	0.9317	0.2333	1.3389	0.4979
			3n/4	1.1445	0.0647	0.9684	0.2054	1.2621	0.3606
	1.9737	1.1077	n/2	2.1994	0.3102	1.7561	0.3141	2.3818	0.8974
			3n/4	2.1559	0.2473	1.8011	0.2856	2.2594	0.6227
		1.9737	n/2	1.9845	0.1928	1.6252	0.4071	2.4073	0.9105
			3n/4	1.9914	0.1298	1.6728	0.3646	2.2679	0.6614
40	0.5929	1.1077	n/2	0.6884	0.1052	0.5148	0.1074	0.5379	0.1073
			3n/4	0.6679	0.0822	0.5959	0.0889	0.6281	0.1129
		1.9737	n/2	0.6327	0.0530	0.5395	0.1015	0.6287	0.1398
			3n/4	0.6181	0.0316	0.5700	0.0852	0.6315	0.1144
	1.1077	1.1077	n/2	1.2392	0.1498	0.8872	0.2431	0.9347	0.2225
			3n/4	1.2249	0.1299	1.0951	0.1562	1.1861	0.2238
		1.9737	n/2	1,1456	0.0823	0.9408	0.2079	1.1169	0.2219
			3n/4	1.1446	0.0524	1.0420	0.1504	1.1805	0.2069
	1.9737	1.1077	n/2	2.1252	0.2423	1.4389	0.5507	1.4896	0.5223
			3n/4	2.1526	0.2150	1.8954	0.2538	2.0989	0.3855
		1.9737	n/2	1.8531	0.2245	1.5380	0.4642	1.8218	0.3606
			3n/4	2.0009	0.0953	1.8233	0.2700	2.1091	0.3899
60	0.5929	1.1077	n/2	0.6955	0.1083	0.6016	0.0873	0.6301	0.1109
			3n/4	0.6703	0.0823	0.5955	0.0725	0.6146	0.0844
		1.9737	n/2	0.6399	0.0549	0.5733	0.0841	0.6297	0.1099
			3n/4	0.6209	0.0333	0.5731	0.0769	0.6119	0.0903
	1.1077	1.1077	n/2	1.2661	0.1675	1.0957	0.1492	1.1751	0.2087
			3n/4	1.2294	0.1299	1.1009	0.1293	1.1565	0.1633
		1.9737	n/2	1.1709	0.0838	1.0426	0.1524	1.1713	0.2051
			3n/4	1.1473	0.0513	1.0590	0.1301	1.1500	0.1593
	1.9737	1.1077	n/2	2.2347	0.2915	1.9022	0.2345	2.0851	0.3499
			3n/4	2.1652	0.2181	1.9329	0.2184	2.0660	0.2976
		1.9737	n/2	1.9869	0.1359	1.8278	0.2663	2.1021	0.3774
			3n/4	1.9958	0.0782	1.8574	0.2368	2.0477	0.2938
100	0.5929	1.1077	n/2	0.6925	0.1035	0.5041	0.1011	0.5096	0.0983
			3n/4	0.6728	0.0829	0.5969	0.0585	0.6073	0.0640
		1.9737	n/2	0.6348	0.0485	0.5626	0.0702	0.5947	0.0744
			3n/4	0.6212	0.0315	0.5833	0.0555	0.6063	0.0618
	1.1077	1.1077	n/2	1.2440	0.1442	0.8747	0.2446	0.8875	0.2354
			3n/4	1.2329	0.1307	1.0999	0.1004	1.1304	0.1137
		1.9737	n/2	1.1399	0.0579	0.9939	0.1519	1.0633	0.1330
			3n/4	1.1479	0.0479	1.0804	0.1058	1.1348	0.1213
	1.9737	1.1077	n/2	2.1399	0.2112	1.4004	0.5814	1.4091	0.5757
			3n/4	2.1773	0.1930	1.9431	0.1766	2.0178	0.2081
		1.9737	n/2	1.8486	0.1798	1.6013	0.3956	1.7129	0.3148
			3n/4	1.9967	0.0641	1.9053	0.1864	2.0209	0.2152

Table 5: The estimate and the root mean square errors (RMSEs) for the parameter β based on the improved MLE, Bayes and MLE methods with m= (n/2 and 3n/4), k=3m/4 and T=0.75.

n	α	β	m	IMLE		Bayes		MLE	
				Estimate	RMSE	Estimate	RMSE	Estimate	RMSE
20	0.5929	1.1077	n/2	1.2243	0.1329	0.9735	0.2167	1.0655	0.3799
			3n/4	1.2173	0.1241	0.9850	0.2039	1,1446	0.3332
		1.9737	n/2	2.1057	0.1488	1.6592	0.3713	2.2149	0.8997
			3n/4	2.0871	0.1269	1.6776	0.3594	2.2444	0.8151
	1.1077	1.1077	n/2	1.2457	0.1491	0.9965	0.1933	1.0750	0.3719
			3n/4	1.2426	0.1463	0.9913	0.1974	1.1427	0.3747
		1.9737	n/2	2.1665	0.2032	1.6677	0.3569	2.2282	0.9858
			3n/4	2.1549	0.1934	1.6542	0.3748	2.1934	0.8280
	1.9737	1.1077	n/2	1.2995	0.2011	1.0072	0.1780	1.0688	0.3681
			3n/4	1.2978	0.1999	0.9976	0.1878	1.1313	0.3332
		1.9737	n/2	2.2875	0.3263	1.6528	0.3681	2.2051	1.0027
			3n/4	2.2659	0.3059	1.6587	0.3714	2.2318	0.8713
40	0.5929	1.1077	n/2	1.2235	0.1236	1.0251	0.1837	1.0883	0.2442
			3n/4	1.2178	0.1174	1.0383	0.1686	1.1246	0.2197
		1.9737	n/2	2.1009	0.1345	1.7990	0.2704	2.0804	0.4324
			3n/4	2.0908	0.1247	1.7867	0.2829	2.0563	0.3976
	1.1077	1.1077	n/2	1.2431	0.1404	1.0430	0.1684	1.0913	0.2414
			3n/4	1.2411	0.1387	1.0468	0.1572	1.1238	0.2132
		1.9737	n/2	2.1667	0.1981	1.7951	0.2739	2.0858	0.4762
			3n/4	2.1558	0.1874	1.7729	0.2820	2.0384	0.3804
	1.9737	1.1077	n/2	1.2989	0.1960	1.0548	0.1529	1.0918	0.2285
			3n/4	1.2990	0.1959	1.0488	0.1546	1.1174	0.2140
		1.9737	n/2	2.2839	0.3155	1.7776	0.2729	2.0505	0.4171
			3n/4	2.2674	0.2999	1.7782	0.2799	2.0571	0.3945
60	0.5929	1.1077	n/2	1.2213	0.1179	1.0517	0.1562	1.0979	0.1913
			3n/4	1.2184	0.1151	1.0507	0.1472	1.1076	0.1716
		1.9737	n/2	2.1039	0.1352	1.8397	0.2403	2.0197	0.3111
			3n/4	2.0899	0.1213	1.8477	0.2396	2.0353	0.3141
	1.1077	1.1077	n/2	1.2439	0.1396	1.0644	0.1388	1.0996	0.1773
			3n/4	1.2424	0.1378	1.0651	0.1388	1.1179	0.1712
		1.9737	n/2	2.1672	0.1968	1.8496	0.2289	2.0384	0.3159
			3n/4	2.1547	0.1848	1.8363	0.2379	2.0231	0.2995
	1.9737	1.1077	n/2	.3009	0.1965	1.0669	0.1419	1.0899	0.1881
			3n/4	.2992	0.1946	1.0642	0.1338	1.1088	0.1654
		1.9737	n/2	2.2831	0.3128	1.8381	0.2285	2.0236	0.3057
			3n/4	2.2643	0.2944	1.8489	0.2312	2.0494	0.3233
100	0.5929	1.1077	n/2	1.2217	0.1165	1.0686	0.1247	1,0982	0.1396
			3n/4	1.2170	0.1123	1.0761	0.1207	1.1127	0.1344
		1.9737	n/2	2.1038	0.1328	1.8849	0.1929	1.9932	0.2221
			3n/4	2.0888	0.1182	1.8943	0.1939	2.0079	0.2270
	1.1077	1.1077	n/2	1.2444	0.1387	1.0782	0.1258	1.1004	0.1469
			3n/4	1.2418	0.1362	1.0795	0.1105	1.1117	0.1243
		1.9737	n/2	2.1699	0.1984	1.8814	0.1964	1.9912	0.2281
			3n/4	2.1522	0.1808	1.8944	0.1897	2.0111	0.2245
	1.9737	1.1077	n/2	1.2994	0.1935	1.0841	0.1118	1.0989	0.1341
			3n/4	1.2974	0.1916	1.0855	0.1459	1.1141	0.1324
		1.9737	n/2	2.2822	0.3107	1.8922	0.1919	2.0078	0.2363
			3n/4	2.2649	0.2935	1.8913	0.1877	2.0102	0.2228

Table 6: The estimate and the root mean square errors (RMSEs) for the parameter β based on the improved MLE, Bayes and MLE methods with m= (n/2 and 3n/4), k=3m/4 and T=1.5.

n	α	β	m	IMLE		Bayes		MLE	
				Estimate	RMSE	Estimate	RMSE	Estimate	RMSE
20	0.5929	1.1077	n/2	1.2143	0.1228	0.9735	0.2167	1.0655	0.3799
			3n/4	1.2078	0.1142	0.9850	0.2039	1.1446	0.3332
		1.9737	n/2	2.0911	0.1331	1.6392	0.3714	2.2149	0.8997
			3n/4	2.0823	0.1240	1.6522	0.3795	2,1610	0.7413
	1.1077	1.1077	n/2	1.2068	0.1051	1.2557	0.2008	1.5534	0.5245
			3n/4	1.2267	0.1282	0.9898	0.1959	1.1333	0.3398
		1.9737	n/2	2.1437	0.1794	1.6677	0.3569	2.2282	0.9838
			3n/4	2.1309	0.1673	1.6617	0.3681	2.1984	0.7252
	1.9737	1.1077	n/2	1.2445	0.1412	1.3292	0.2538	1.7061	0.6459
			3n/4	1.2771	0.1779	0.9929	0.1939	1.1253	0.3512
		1.9737	n/2	2.2488	0.2859	1.6554	0.3659	2.2157	1.0158
			3n/4	2.2299	0.2674	1.6603	0.3698	2.2426	0.9419
40	0.5929	1.1077	n/2	1.2135	0.1134	1.0251	0.1837	1.0883	0.2446
			3n/4	1.2082	0.1076	1.0383	0.1686	1.1246	0.2197
		1.9737	n/2	2.0867	0.1199	1.7990	0.2704	2.0803	0.4324
			3n/4	2.0771	0.1099	1.7863	0.2816	2.0554	0.3956
	1.1077	1.1077	n/2	1.2286	0.1255	1.0430	0.1689	1.0916	0.2418
			3n/4	1.2284	0.1259	1.0414	0.1669	1.1655	0.2226
		1.9737	n/2	2.1439	0.1748	1.7951	0.2739	2.0858	0.4762
			3n/4	2.1315	0.1625	1,7869	0.2783	2.0684	0.4237
	1.9737	1.1077	n/2	1.2779	0.1745	1.0532	0.1537	1.0896	0.2293
			3n/4	1.2772	0.1744	1.0467	0.1557	1.1125	0.2138
		1.9737	n/2	2.2475	0.2784	1.7759	0.2729	2.0463	0.4121
			3n/4	2.2331	0.2649	1.7784	0.2816	2.0579	0.3999
60	0.5929	1.1077	n/2	1.2113	0.1078	1.0517	0.1562	1.0979	0.1913
			3n/4	1.2087	0.1053	1.0507	0.1472	1.1076	0.1716
		1.9737	n/2	2.0895	0.1205	1.8397	0.2403	2.0197	0.3111
			3n/4	2.0784	0.1098	1.8466	0.2357	2.0328	0.3044
	1.1077	1.1077	n/2	1.2036	0.0977	1.4599	0.3681	1.5874	0.4879
			3n/4	1.2276	0.1231	1.0647	0.1435	1.1173	0.1767
		1.9737	n/2	2.1444	0.1737	1.8496	0.2289	2.0384	0.3159
			3n/4	2.1323	0.1619	1.8435	0.2335	2.0342	0.3049
	1.9737	1.1077	n/2	1.2559	0.1503	1.3628	0.2920	1.4789	0.4139
			3n/4	1.2752	0.1705	1.0751	0.1315	1.1224	0.1672
		1.9737	n/2	2.2458	0.2751	1.8394	0.2275	2.0259	0.3067
			3n/4	2.2330	0.2625	1.8343	0.2311	2.0239	0.2930
100	0.5929	1.1077	n/2	1.1884	0.0823	1.4047	0.3122	1.5729	0.3814
			3n/4	1.2083	0.1034	1.0730	0.1149	1.1091	0.1262
		1.9737	n/2	2.0894	0.1182	1.8849	0.1929	1.9932	0.2221
			3n/4	2.0775	0.1066	1.8864	0.1925	1.9977	0.2189
	1.1077	1.1077	n/2	1.2297	0.1238	1.0782	0.1258	1.1004	0.1469
			3n/4	1.2278	0.1220	1.0802	0.1152	1.1128	0.1303
		1.9737	n/2	2.1469	0.1751	1.8814	0.1964	1.9912	0.2281
			3n/4	2.1309	0.1591	1.8966	0.1841	2.0136	0.2189
	1.9737	1.1077	n/2	1.2536	0.1471	1.4335	0.3631	1.5101	0.4439
			3n/4	1.2769	0.1709	1.0821	0.1106	1.1099	0.1265
		1.9737	n/2	2.2455	0.2738	1.8922	0.1909	2.0077	0.2348
			3n/4	2.2311	0.2595	1.8895	0.1877	2.0079	0.2211

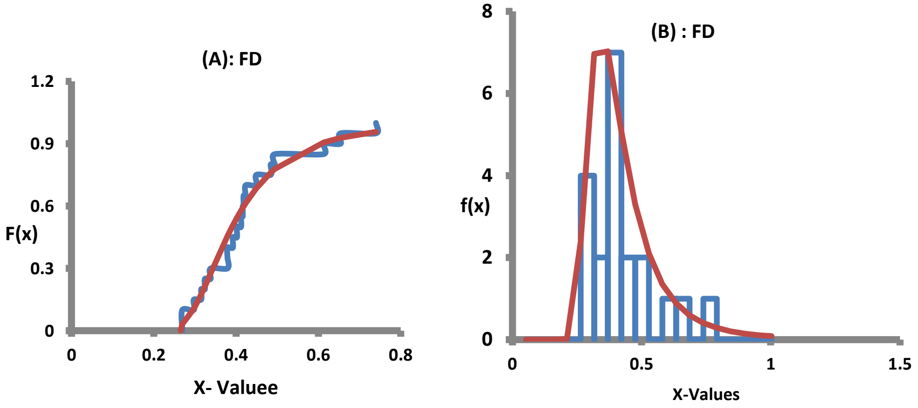

Figure 5(a): a) The Empirical CDF and the fitted CDF for the Flood Data.
b) The Histogram and the fitted PDF for the Flood Data.

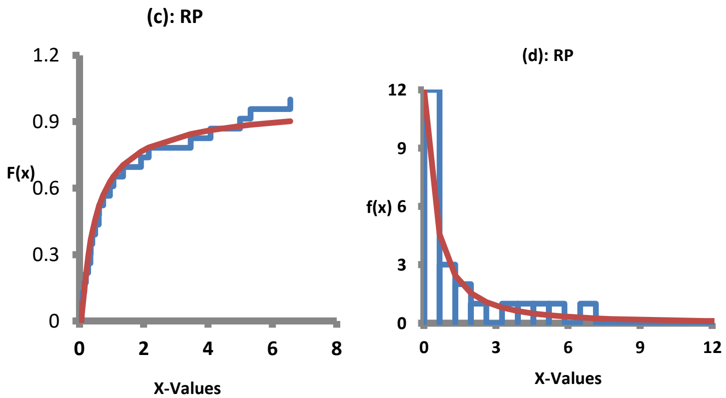

Figure 5(b): c) The Empirical CDF and the fitted CDF for the Reactor Pumps Data.
d) The Histogram and the fitted PDF for the Reactor Pumps Data.