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The Poisson-Prakaamy Distribution and its Applications

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ABSTRACT

In this paper, a study on Poisson-Prakaamy distribution, a Poisson mixture of Prakaamy distribution, has been carried out. The expression for r[th] factorial moment about origin has been derived. The expressions for its coefficient of variation, skewness, kurtosis and index of dispersion have been obtained and their behaviors illustrated graphically. The method of moment and the method of maximum likelihood estimation for estimating the parameter have been discussed. The simulation study has also been presented in order to illustrate the performance of maximum likelihood estimator. The goodness of fit of the proposed distribution has been explained with two count datasets and its fit was found quite satisfactory over Poisson distribution, Poisson-Lindley distribution, Poisson-Akash distribution and Poisson-Ishita distribution.

1. Introduction

The probability density function (pdf) and the cumulative distribution function (cdf) of Prakaamy distribution proposed by Shukla (2018) for modeling lifetime data from engineering and biomedical science are given by

$$f_1(x,\theta)=\frac{\theta^6}{\theta^5+120}\left(1+x^5\right)e^{-\theta x}\,;x>0,\theta>0 \tag{1.1}$$

$$F_1\left(x;\theta\right)=1-\left[1+\frac{\theta x\left(\theta^4x^4+5\theta^3x^3+20\theta^2x^2+60\theta x+120\right)}{\theta^5+120}\right]e^{-\theta x}\;; \tag{1.2}$$

$x>0,\theta>0$

Prakaamy distribution is a mixture of exponential distribution having scale parameter θ and gamma distribution having shape parameter 4 and scale parameter θ with their mixing proportions $\frac{\theta^5}{\theta^5+120}$ and $\frac{120}{\theta^5+120}$ respectively.

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Important statistical properties of Prakaamy distribution including its shapes, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, Renyi entropy measure and stress-strength reliability have been discussed by Shukla (2018) . It was observed and shown by Shukla (2018) that Prakaamy distribution provides much closer fit than exponential distribution, Lindley distribution by Lindley (1958), Akash distribution by Shanker (2015) and Ishita distribution by Shanker and Shukla (2017). The pdf and the cdf of Lindley, Akash and Ishita distributions are presented in the following table 1.

Table 1: The pdf and the cdf of Lindley, Akash and Ishita distributions.

Name of distributions	pdf	Cdf
Lindley	$f\left(x,\theta\right)=\frac{\theta^2}{\theta+1}\left(1+x\right)e^{-\theta x}$; $x>0,\theta>0$	$F\left(x;\theta\right)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}$; $x>0,\theta>0$
Akash	$f\left(x;\theta\right)=\frac{\theta^3}{\theta^2+2}\left(1+x^2\right)e^{-\theta x}$; $x>0,\ \theta>0$	$F\left(x;\theta\right)=1-\left[1+\frac{\theta x\left(\theta x+2\right)}{\theta^2+2}\right]e^{-\theta x}$; $x>0,\theta>0$
Ishita	$f\left(x;\theta\right)=\frac{\theta^3}{\theta^3+2}\left(\theta+x^2\right)e^{-\theta x}$; $x>0,\ \theta>0$	$F\left(x,\theta\right)=1-\left[1+\frac{\theta x\left(\theta x+2\right)}{\theta^3+2}\right]e^{-\theta x}$; $x>0,\theta>0$

The statistical properties, estimation of parameter using both the method of moment and the method of maximum likelihood and various applications of Lindley, Akash, and Ishita distributions are available in Ghitany *et al.* (2008), Shanker (2015) and Shanker and Shukla (2017). Shanker *et al.* (2015) have detailed comparative study on modeling of various lifetime data from engineering and medical science using exponential and Lindley distributions and observed that in some lifetime data exponential is better than the Lindley distribution while in some lifetime data Lindley distribution is better than the exponential distribution.

The Probability mass function (pmf) of Poisson mixture of these one parameter lifetime distributions along with their introducer(s) and year are presented in table 2.

Table 2 : The probability mass function (pmf) of Poisson mixture of one parameter lifetime distributions

Name of distributions	pmf	Introducer and years
Poisson-Lindley distribution (PLD)	$P(x,\theta)=\dfrac{\theta^2(x+\theta+2)}{(\theta+1)^{x+3}}$; $x=0,1,2,...,\theta>0$	Sankaran (1970)
Poisson-Akash distribution (PAD)	$P(x,\theta)=\dfrac{\theta^3}{\theta^2+2}\cdot\dfrac{x^2+3x+(\theta^2+2\theta+3)}{(\theta+1)^{x+3}}$; $x=0,1,2,...,\theta>0$	Shanker (2017)
Poisson-Ishita distribution (PID)	$P(x,\theta)=\dfrac{\theta^3\left[x^2+3x+(\theta^3+2\theta^2+\theta+2)\right]}{(\theta^3+2)(\theta+1)^{x+3}}$; $x=0,1,2,...,\theta>0$	Shukla and Shanker (2019)

The statistical and mathematical properties, estimation of parameter using method of moment and maximum likelihood estimation and applications to model count data from various fields of knowledge of PLD, PAD, and PID are available in Sankaran (1970), Ghitany and Al-Mutairi (2009), Shanker (2017) and Shukla and Shanker (2019), respectively. Shanker and Hagos (2015) have detailed study on applications of PLD for modeling count datasets from biological sciences and observed that PLD is better than Poisson distribution in biological data.

The main motivation of proposing a Poisson mixture of Prakaamy distribution are (i) It is observed from previous study that Prakaamy distribution is better model than exponential, Lindley, Akash and Ishita distributions for modeling lifetime data (ii) it is expected that Poisson mixture of Prakaamy distribution would provide a better model for count data than PLD, PAD and PID.

In the present paper, Poisson Prakaamy distribution (PPD) which is a Poisson mixture of Prakaamy distribution has been proposed and studied. The paper has been divided into seven sections. Second section deals with the derivation of the pmf of Poisson-Prakaamy distribution (PPD), nature of shapes of its pmf for varying values of parameter, and increasing hazard rate property and unimodality. The moments and moments based measures have been studied in section three. The estimation of its parameter has been discussed using method of moment and maximum likelihood estimation section four. The simulation study has been presented in section five. In the sixth section the goodness of fit of the PPD along with the goodness of fit given by PLD, PAD and PID has been discussed. Finally, section seven deals with the concluding remarks of the whole paper.

2. Poisson-Prakaamy Distribution

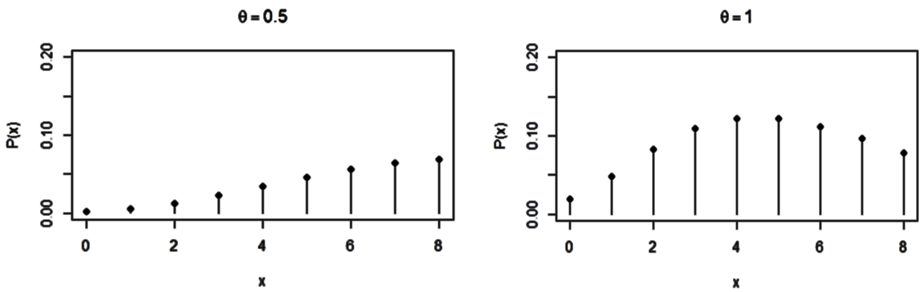
Suppose the parameter λ of the Poisson distribution follows Prakaamy distribution (1.1). Then the Poisson mixture of Prakaamy distribution (1.1) can be obtained as

$$P(x;\theta)=P(X=x)=\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^6}{\theta^5+120}\left(1+\lambda^5\right)e^{-\theta\lambda}d\lambda \tag{2.1}$$

$$=\frac{\theta^6}{\left(\theta^5+120\right)x!}\int_0^\infty e^{-(\theta+1)\lambda}\left(\lambda^x+\lambda^{x+5}\right)d\lambda$$

$$=\frac{\theta^6}{\theta^5+120}\left[\frac{(\theta+1)^5+\left(x^5+15x^4+85x^3+225x^2+274x+120\right)}{(\theta+1)^{x+6}}\right];x=0,1,2,...,\theta>0 \tag{2.2}$$

We name this distribution "Poisson-Prakaamy distribution (PPD)". The behavior of the pmf of PPD for different values of parameter is shown in the figure 1.



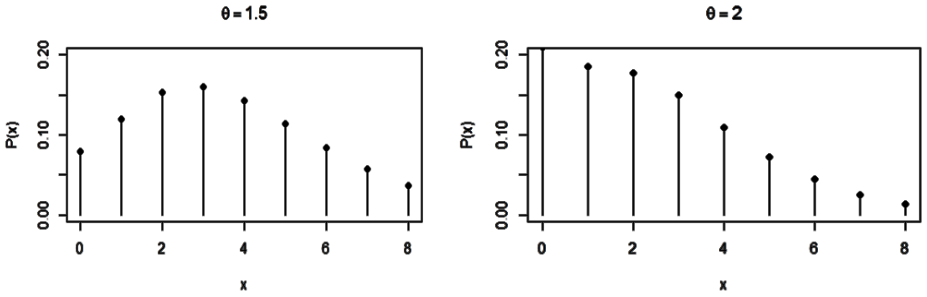


Fig. 1: Behavior of the PPD for different values of the parameter θ.

3. Moments

The r[th] factorial moment about origin of PPD (2.2), denoted by $\mu_{(r)}'$, can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)}\,|\,\lambda\right)\right], \quad \text{where} \quad X^{(r)} = X\left(X-1\right)\left(X-2\right)...\left(X-r+1\right)$$

Using (2.1), $\mu_{(r)}'$ can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)}\,|\,\lambda\right)\right] = \frac{\theta^6}{\theta^5+120}\int_0^\infty \left[\sum_{x=0}^{\infty} x^{(r)}\frac{e^{-\lambda}\lambda^x}{x!}\right]\left(1+\lambda^5\right)e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^6}{(\theta^5+120)}\int_0^\infty \lambda^r\left[\sum_{x=r}^{\infty}\frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right]\left(1+\lambda^5\right)e^{-\theta\lambda}d\lambda$$

Taking x+r in place of x within the bracket, we get

$$\mu_{(r)}' = \frac{\theta^6}{\theta^5+120}\int_0^\infty \lambda^r\left[\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^{x}}{x!}\right]\left(1+\lambda^5\right)e^{-\theta\lambda}d\lambda$$

The expression within the bracket is clearly unity and hence we have

$$\mu_{(r)}' = \frac{\theta^6}{\theta^5+120}\int_0^\infty \lambda^r\left(1+\lambda^5\right)e^{-\theta\lambda}d\lambda$$

Using gamma integral and a little algebraic simplification, we get finally, a general expression for the r th factorial moment about origin of PPD (2.2) as

$$\mu_{(r)}{}' = \frac{r!\left[\theta^5 + (r+1)(r+2)(r+3)(r+4)(r+5)\right]}{\theta^r\left(\theta^5+120\right)} \; ; r = 1,2,3,.... \tag{3.1}$$

Substituting $r = 1,2,3, \text{and } 4$ in (3.1), the first four factorial moments about origin can be obtained and using the relationship between factorial moments about origin and moments about origin, the first four moment about origin of PPD (2.2) are obtained as

$$\mu_1{}' = \frac{\theta^5 + 720}{\theta\left(\theta^5+120\right)} \qquad \mu_2{}' = \frac{\theta^6 + 2\theta^5 + 720\theta + 5040}{\theta^2\left(\theta^5+120\right)}$$

$$\mu_3{}' = \frac{\theta^7 + 6\theta^6 + 6\theta^5 + 720\theta^2 + 15120\theta + 40320}{\theta^3\left(\theta^5+120\right)}$$

$$\mu_4{}' = \frac{\theta^8 + 14\theta^7 + 36\theta^6 + 24\theta^5 + 720\theta^3 + 35280\theta^2 + 2419920\theta + 362880}{\theta^4\left(\theta^5+120\right)}$$

Using the relationship between moments about mean and the moments about origin, the moments about the mean of PPD (2.2) can be obtained as

$$\mu_2 = \sigma^2 = \frac{\theta^{11} + \theta^{10} + 840\theta^6 + 3840\theta^5 + 86400\theta + 86400}{\theta^2\left(\theta^5+120\right)^2}$$

$$\mu_3 = \frac{\begin{pmatrix}\theta^{17} + 3\theta^{16} + 2\theta^{15} + 960\theta^{12} + 11880\theta^{11} + 25920\theta^{10} + 1872000\theta^7 \\ +1641600\theta^6 + 3456000\theta^5 + 103680000\theta^2 + 311040000\theta + 20736000\end{pmatrix}}{\theta^3\left(\theta^5+120\right)^3}$$

$$\mu_4 = \frac{\begin{pmatrix}\theta^{23} + 10\theta^{22} + 18\theta^{21} + 9\theta^{20} + 1080\theta^{18} + 33600\theta^{17} + 1850400\theta^{16} + 227520\theta^{15} \\ +302400\theta^{13} + 9792000\theta^{12} + 369792000\theta^{11} + 184896000\theta^{10} + 328320000\theta^8 \\ +9676800000\theta^7 + 23016960000\theta^6 + 24468480000\theta^5 \\ +12441600000\theta^3 + 311040000000\theta^2 + 597196800000\theta + 29859840000\end{pmatrix}}{\theta^4\left(\theta^5+120\right)^4}$$

The coefficient of variation $(C.V.)$, coefficient of Skewness $\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis (β_2), and index of dispersion (γ) of the PPD (2.2) can be obtained using following formula:

$C.V = \dfrac{\sigma}{\mu_1'}$, $\sqrt{\beta_1} = \dfrac{\mu_3}{\mu_2^{3/2}}$, $\beta_2 = \dfrac{\mu_4}{\mu_2^{2}}$ and $\gamma = \dfrac{\sigma^2}{\mu_1'}$, respectively. The behaviors of mean and variance of PPD for varying values of parameter θ are shown in figure 2. The mean and variance of PPD are presented graphically in figure 2.

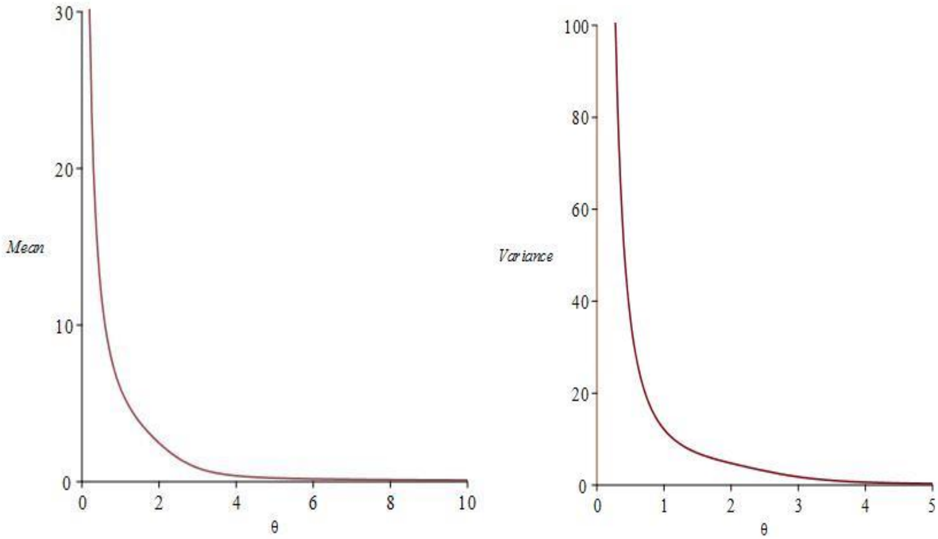
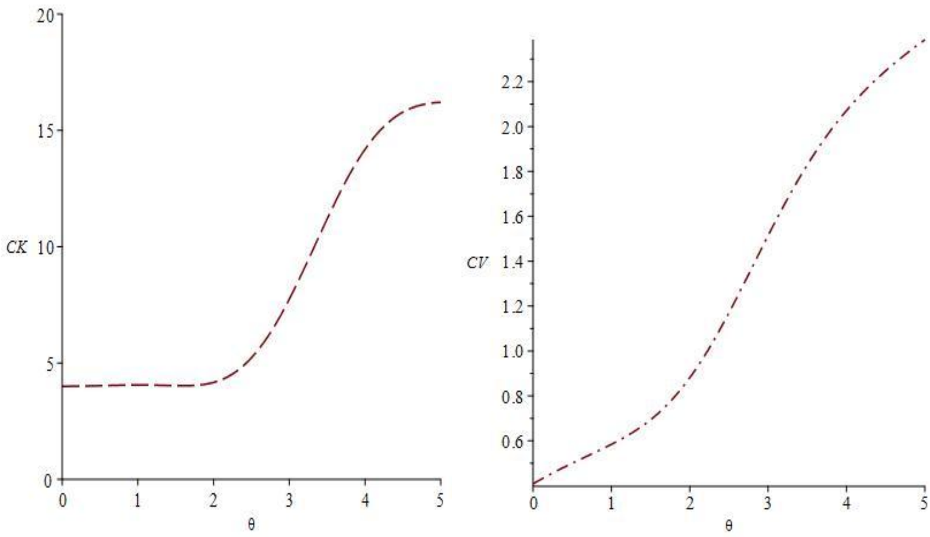


Fig. 2: Plots of Mean and Variance for varying values of θ.

The behaviors of C.V, $\sqrt{\beta_1}$, β_2 and γ of the PPD have been shown graphically for different values of parameter θ in figure-3.



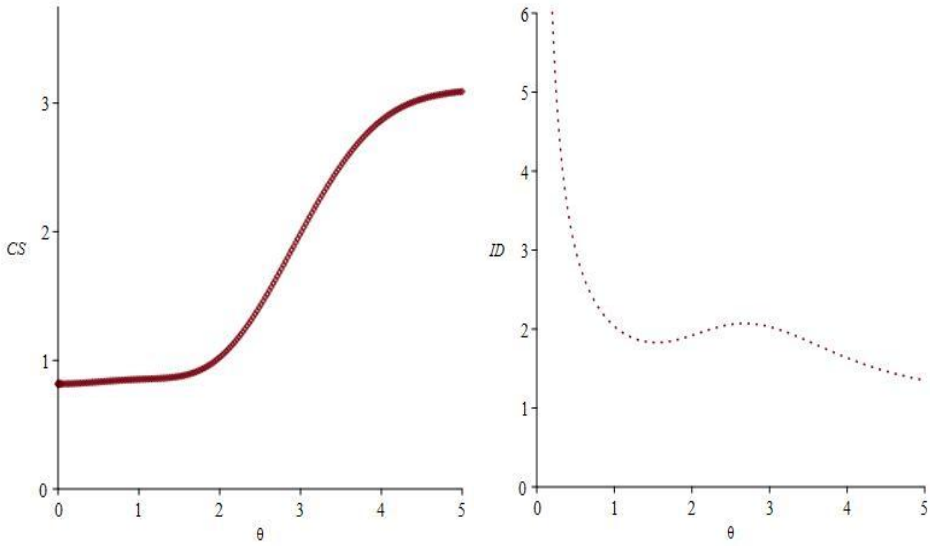


Fig.3: Behaviors of coefficient of variation(CV), coefficient of skewness (CS), coefficient of kurtosis(CK) and Index of dispersion(ID) of PPD for different values of the parameter θ.

4. Estimation of Parameter

In this section the estimation of parameter of PPD has been discussed using both the method of moment and the method of maximum likelihood.

a) Method of Moment Estimate (MOME)

Let $\left(x_1,x_2,...,x_n\right)$ be a random sample of size n from the PPD (2.2). Equating the population mean to the corresponding sample mean, the MOME $\tilde{\theta}$ of the parameter θ of PPD (2.2) is the solution of the following sixth degree polynomial equation

$\bar{x}\theta^6-\theta^5+120\bar{x}\theta-720=0$,

where $\bar{x}$ is the sample mean.

b) Maximum Likelihood Estimate (MLE)

Let $\left(x_1,x_2,...,x_n\right)$ be a random sample of size n from the PPD (2.2) and let f_x be the corresponding observed frequency. The likelihood function L of the PPD (2.2) is given by

$$L=\left(\frac{\theta^6}{\theta^5+120}\right)^n\frac{1}{\left(\theta+1\right)^{\sum_{x=1}^{k}(x+6)f_x}}\prod_{x=1}^{k}\left[\left(\theta+1\right)^5+\left(x^5+15x^4+85x^3+225x^2+274x+120\right)\right]^{f_x}$$

The log likelihood function is thus obtained as

$$\log L=n\log\left(\frac{\theta^6}{\theta^5+120}\right)-\sum_{x=1}^{k}\left(x+6\right)f_x\log\left(\theta+1\right)+\sum_{x=1}^{k}f_x\log\left[\left(\theta+1\right)^5+\left(\begin{array}{l}x^5+15x^4+85x^3+225x^2\\+274x+120\end{array}\right)\right]$$

The first derivative of the log likelihood function is given by

$$\frac{d\log L}{d\theta}=\frac{n\left(\theta^5+720\right)}{\theta\left(\theta^5+120\right)}-\frac{n\left(\bar{x}+6\right)}{\theta+1}+\sum_{x=1}^{k}\frac{5\left(\theta+1\right)^4f_x}{\left[\left(\theta+1\right)^5+\left(x^5+15x^4+85x^3+225x^2+274x+120\right)\right]},$$

where $\bar{x}$ is the sample mean.

The maximum likelihood estimate (MLE), $\hat{\theta}$ of the parameter θ of PPD (2.2) is the solution of the equation $\frac{d\log L}{d\theta}=0$

This non-linear equation can be solved by any numerical iteration methods such as Newton- Raphson method, Bisection method, Regula –Falsi method etc. R-software is used to solve the log likelihood equation to get the value of parameter. Note that initial value of the parameter θ can be taken as the value given by MOME.

5. Simulation Study

The simulation process consists in generating N=10,000 pseudo-random sample of sizes n =50, 100, 150, and 200 of a variable X having PPD (2.2). The procedure is based on the Monte Carlo simulation method to estimate the average bias and the mean squared error (MSE) of the MLEs of the parameter θ are calculated using valued of θ =1.5, 2, 2.5 and 3.0. The following formulas are used for finding bias and MSE of the parameter.

$$B(\hat{\theta})=\frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j-\theta),\qquad \mathrm{MSE}(\hat{\theta})=\frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j-\theta)^2$$

The following algorithm can be used to generate a single random variable from PPD

Algorithm

Generate, $u \sim U(0,1)$

$x \to 0$

$$p_x \Rightarrow \frac{\theta^4\left(\theta(\theta+1)^3+6\right)}{(\theta+1)^4(\theta^4+6)}$$

while($p_x < u$)do

$x \to x+1$

$p_{x1} = p_x * p_{x-1}$

$p_x \Rightarrow p_x + p_{x1}$

while

return(x)

end

The ML estimate, biases and the mean squares error (MSE) of the simulation results are presented in table 3.

Table 3: Estimated Bias and MSE of MLEs $(\hat{\theta})$.

Sample size (n)	θ	Bias	MSE
50	1.5	- 0.00146	0.00010
	2.0	0.00545	0.00148
	2.5	0.00237	0.00282
	3.0	0.000516	0.000013
100	1.5	- 0.000343	0.000011
	2.0	- 0.000257	0.000006
	2.5	0.001433	0.000205
	3.0	- 0.00105	0.000111
150	1.5	- 0.000433	0.000028
	2.0	- 0.00002	0.0000006
	2.5	0.000123	0.0000022
	3.0	- 0.00308	0.001431
200	1.5	0.00518	0.00537
	2.0	0.00118	0.00028
	2.5	0.00039	0.000030
	3.0	0.00088	0.000156

The table 3 shows the bias and MSE of the MLEs of the parameter θ for different sample sizes.

The result indicates that bias and mean square error tends to zero when the sample size and parameter increases. It is noteworthy that the MLE of θ is negative bias in some cases. The mean squared error of $\hat{\theta}$ remains small even for small n.

Table 4: Distribution of mistakes in copying groups of random digits.

6. Goodness of Fit

No. of errors per group	Observed Frequency	Expected Frequency				
		PD	**PLD**	**PAD**	**PID**	**PPD**
0	35	27.4	33.0	33.5	33.7	35.0
1	11	21.5	15.3	14.7	14.5	12.6
2	8	8.4 }	6.8 }	6.6	6.5	6.1
3	4	2.2 }	2.9 }	2.9 }	2.9 }	3.2 }
4	2	0.5 }	2.0 }	2.3 }	2.4 }	3.1 }
Total	60	60.0	60.0	60.0	60	60
ML estimate		$\hat{\theta} = 0.7833$	$\hat{\theta} = 1.7434$	$\hat{\theta} = 2.07797$	$\hat{\theta} = 1.8643$	$\hat{\theta} = 3.0831$
χ^2		7.98	2.20	1.40	1.33	0.80
d.f.		1	1	2	2	2
p-value		0.0047	0.1380	0.4966	0.514	0.6703

The PPD has been fitted to a number of datasets to test its goodness of fit over PD, PLD, PAD and PID. Note that PLD, PAD and PID are always over-dispersed and hence their comparison regarding goodness of fit is justifiable. The maximum likelihood estimate has been used to fit the PD, PLD, PAD, PID and PPD on two examples of observed count datasets. The first dataset in table 4 is due to Kemp and kemp (1965) regarding the distribution of mistakes in copying groups of random digits, the second dataset in table 5 is the distribution of number of Chromatid aberrations, available in Loeschke and Kohler (1976) and Janardan and Schaeffer (1977).

Table 5: Distribution of number of Chromatid aberrations (0.2 g chinon 1, 24 hours).

No. of Chromatid aberrations	Observed Frequency	Expected Frequency				
		PD	PLD	PAD	PID	PPD
0	268	231.3	257.0	260.4	260.8	272.0
1	87	126.7	93.4	89.7	89.3	77.7
2	26	34.7	32.8	32.1	31.8	28.8
3	9	6.3 }	11.2	11.5	11.5	12.4
4	4	0.8 }	3.8 }	4.1 }	4.2 }	5.4 }
5	2	0.1 }	1.2 }	1.4 }	1.5 }	2.2 }
6	1	0.1 }	0.4 }	0.5 }	0.6 }	0.9 }
7+	3	0.1 }	0.2 }	0.3 }	0.3 }	0.6 }
Total	400	400.0	400.0	400.0	400.0	400.0
ML estimate		$\hat{\theta}=0.5475$	$\hat{\theta}=2.380442$	$\hat{\theta}=2.659408$	$\hat{\theta}=2.3362$	$\hat{\theta}=3.5328$
χ^2		38.21	6.21	4.17	3.61	2.46
d.f.		2	3	3	3	3
p-value		0.0000	0.1018	0.2437	0.3067	0.4816

7. Conclusion

In this paper Poisson-Prakaamy distribution (PPD) has been proposed. The expression for the r th factorial moment has been derived and hence its moments about origin and the moments about mean have been given. The expressions for coefficient of variation, skewness, kurtosis and Index of dispersion have been obtained. The maximum likelihood estimation and the method of moment for estimating its parameter have been discussed. The goodness of fit of PPD over PD, PLD, PAD and PID has been discussed with two examples of observed real datasets. It is observed that PPD gives much better fit than PD, PLD and PAD and PID on all the datasets and hence it can be considered an important discrete distribution for modeling count data over these distributions.

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(Appendex-I)

The fitted plots of the distributions for dataset in tables 4 and 5 have been presented in figures 4 and 5 respectively.

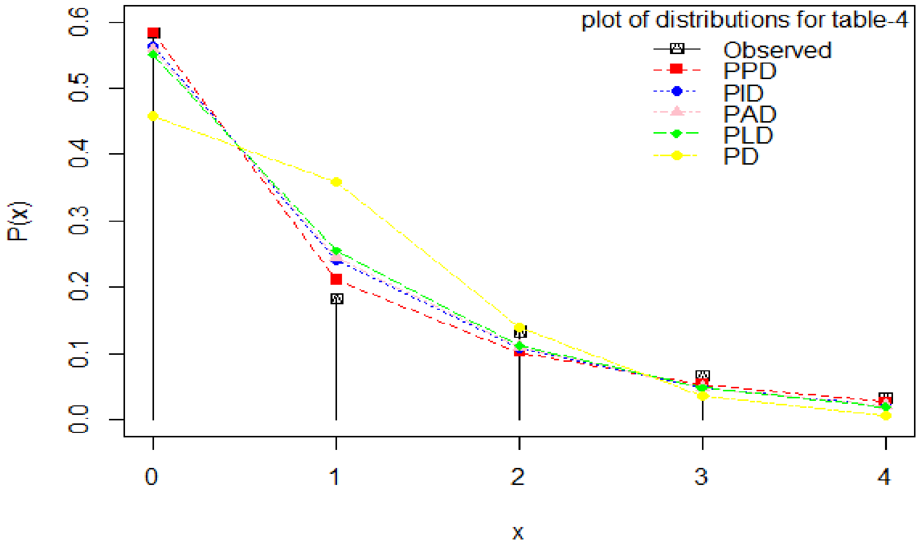

Fig 4: Fitted plot of distributions for datasets in table 4.

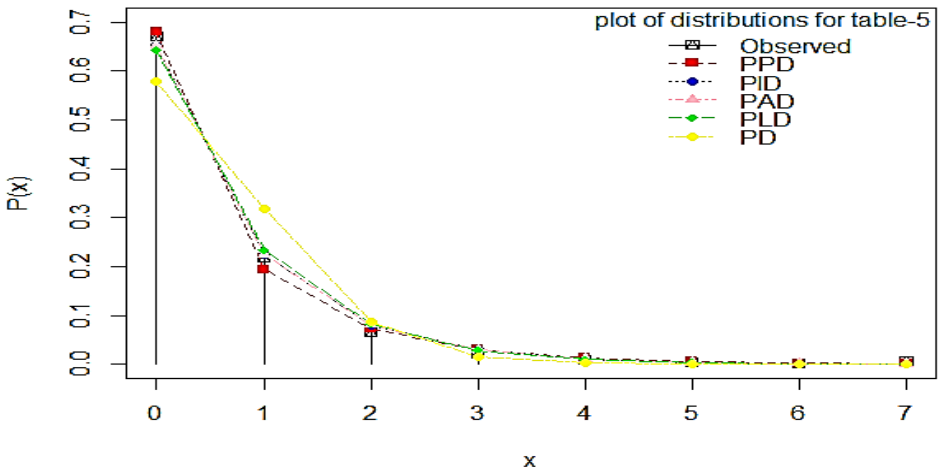

Fig. 5: Fitted plot of distributions for datasets in table 5.