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Circular Geiger-Counter Type Pólya-Eggenberger Distributions

Sonali Bhattacharya*
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ABSTRACT

Motivated by the paper of Dandekar (1955), a one-urn model with Polya-Eggenberger sampling scheme is developed based on circular arrangement. The model is further modified to obtain some new generalized circular non-overlapping distributions of order k, circular l-overlapping distributions of order k and some waiting time distributions.

1. Introduction

Dandekar (1955) gave a new form of modified binomial distribution in which probability of success p does not remain constant but whenever at any given trial success results with probability p, then in the next (m-1) trials the probability of success will be zero. Under this modified binomial setup, if in n Bernoulli trials there are x successes, probability of not more than x successes, is denoted by $F(x,n)$, and its limiting form were obtained. This limiting form of this distribution can be used for accident analysis. In industrial accidents, it is possible that some of the workers might have met with an accident. Not long after, we have started our observational studies about susceptibility of the workers towards accidents. So it is expected that the workers will be immune to accidents for some time at the beginning of the observations, as they will be more alert and cautious. The observational record will be such that the experimenter is given a number of trials for a certain event under given probability conditions, while an investigator stepping in observing a sequence of n trials and noting down the number of occurrences of the event. Such a sequence was called an abrupt sequence and probability of x successes in such an abrupt sequence of size n was obtained. This model was originally used for obtaining frequency distributions of fertility

*Corresponding author**: Sonali Bhattacharya, Professor, Symbiosis Centre for Management and Human Resource Development, Symbiosis International University, Pune, INDIA-411057.
E-mail: Sonali_bhattacharya@scmhrd.edu

enquiry, wherein cost towards payment of pregnancy allowance was estimated based on two physiological factors that:

- A pregnant woman has little chance of getting exposed to the risk of getting pregnant during the pregnancy period.
- The chance of pregnancy for a mother of a neo-natal infant is also very low.

In this paper we consider an urn containing '*w*' white balls and '*b*' black balls. Balls are drawn from the urn, one by one, by the Polya-Eggenberger sampling scheme. A ball is drawn randomly from the urn, its colour is noted and it is returned to the urn with 's' additional balls of the same colour. Balls (trials) are drawn (made) at uniform rate. In a given trial, if a white ball is drawn, drawings are stopped for subsequent (m-1) trials and after these (m-1) trials, trials are again made.

The probability distribution of the number of white balls in n trials when trials are arranged in a **circular sequence gives a** new **Circular Geiger- Counter Polya –Eggenberger Model (CGCPEM).**

Waiting time distributions of similar kind, in which distribution of number of trials leading to first nth white ball drawn is obtained, gives the new **Inverse Circular Geiger- Counter Polya-Eggenberger Model (ICGCPEM**, **Circular Geiger- Counter Polya-Eggenberger Model of Order 'k' (both for Non-overlapping and l-overlapping) and Inverse Circular Geiger- Counter Polya-Eggenberger Model of Order 'k' (non-overlapping and l-overlapping)**are developed which yield CGCPEM and ICGCPEM respectively for *k*=1.

Lemma 1.1 Let $C(\alpha;i,\ r\text{-}i;m\text{-}1,n\text{-}1)$ be the number of allocations of α indistinguishable balls in r distinguishable cells, *i* specified of which have capacity m-1 and each of the rest *r-i* has capacity *n*-1. Then,

$$C\left(\alpha;i,\ r-i;m-1,n-1\right)=\sum_{j_1=0}^{[\alpha/m]}\sum_{j_2=0}^{\left[(\alpha-mj_1)/n\right]}\left(-1\right)^{j_1+j_2}\binom{i}{j_1}\binom{r-i}{j_2}\binom{\alpha-mj_1-nj_2+r-1}{r-1} \tag{1.1}$$

(Makri *et al*., 2007a)

where [x] denotes the greatest integer less than or equal to x.

Corollary 1.1: Let $C(\alpha;i,\ r\text{-}i;m\text{-}1,n\text{-}1)$ be as in Lemma 2.1. Then,

$$C(\alpha; i,\ r-i; m-1, m-1) \equiv C(\alpha; r; m-1) = \sum_{j=0}^{[\alpha/m]} (-1)^j \binom{r}{j} \binom{\alpha - mj + r - 1}{r-1} \tag{1.2}$$

(see e.g., Freund, 1956; Riordan, 1964, p. 104).

2. Circular Geiger-Counter Polya-Eggenberger Model (GCPEM)

Under the above sampling scheme, let

$X_{n,m}^{C}$ be the number of white balls in n trials arranged in a circular arrangement.

Let $P\left(X_{n,m}^{C} = x\right)$ be the probability that there are x white balls drawn in n trials arranged in circular arrangement where $0 \le x \le 1 + \left[\frac{n-1}{m}\right]$, where $[a_1]$ is the greatest integer in a_1.

Theorem 1: For $n(\ge 1)$, w(>0), $b(>0)$ and $s(\ge -1)$ we have,

(i) $$P\left(X_{n,m}^{C} = 0\right) = \frac{b^{(n,s)}}{(w+b)^{(n,s)}} \tag{2.1}$$

(ii) $$P\left(X_{n,m}^{C} = x\right) = \sum_{j=0}^{\min(m-1, n-m(x-1)-1)} \binom{n-(m-1)(x-1)-j-1}{x-1} \frac{w^{(x,s)} b^{(n-m(x-1)-j-1,s)}}{(w+s)^{(n-(m-1)(x-1)-j,s)}} +$$

$$\left(\frac{n-mx+x}{x}\right)\binom{n-mx+x-1}{x-1} \frac{w^{(x,s)} b^{(n-mx,s)}}{(w+b)^{((n-(m-1)x,s)}}$$

$$I\left(x \le \left[\frac{n-1}{m}\right]\right), \quad x = 1,2,\ldots,1+\left[\frac{\text{n-1}}{m}\right] \tag{2.2}$$

where, $x^{(n,s)} = x(x+s)(x+2s)\ldots(x+(n-1)s), I(.)$ is the indicator function.

Proof: (i) For x = 0, i.e., when no white ball is drawn or all the balls drawn by Polya-Eggenberger sampling scheme from the urn are of black colour then probability of drawing n black balls can be given as:

$$P_{n,m;0}(w,b) = \frac{b^{(n,s)}}{(w+b)^{(n,s)}}. \tag{2.3}$$

(ii) For $1 \le x \le 1 + \left[\frac{n-1}{m}\right]$, let the sequence of *n* trials be

$$\underbrace{BB\ldots B}_{r_1}W_1 \underset{(m-1)}{\_\_\_} \underbrace{BB\ldots B}_{r_2}W_2 \underset{(m-1)}{\_\_\_} \underbrace{BB\ldots B}_{r_3}W_3 \underset{(m-1)}{\_\_\_} \ldots \underbrace{BB\ldots B}_{r_x}W_x \underbrace{\underset{(m-1)}{\_\_\_} BB\ldots B}_{j}$$

where, r_t = number of black balls preceding the t^{th} white ball drawn, $t = 1,2,\ldots,x$; $r_t \ge 0$,and j = the number of remaining trials after x^{th} white ball has been drawn.

$$\therefore \quad \sum_{t=1}^{x} r_t + m(x-1) + 1 + j = n.$$

For j≤ m-1,

$$\Rightarrow \sum_{t=1}^{x} r_t = n - j - m(x-1) - 1 = \text{ number of black balls,}$$

$\left(\sum_{t=1}^{x} r_t = n - j - m(x-1) - 1\right)$ black balls will be distributed in x cells preceding the x white balls by 'balls-into-cells' technique in

$$\binom{n-m(x-1)-1-j+x-1}{x-1} = \binom{n-(m-1)(x-1)-j-1}{x-1} \quad \text{ways,}$$

$$j \le n - m(x-1) - 1, \; x \le 1 + \left[\frac{n-1}{m}\right]. \tag{2.4}$$

Probability of drawing x white balls and $\sum_{t=1}^{x} r_t = n - j - m(x-1) - 1$ black balls by Polya-Eggenberger sampling scheme is given as:

$$\frac{w^{(x,s)} b^{(n-m(x-1)-j-1,s)}}{(w+b)^{((n-(m-1)(x-1)-j,s)}} \tag{2.5}$$

For j≥ m, the number of black balls would be *n-mx* which will be distributed into x cells created by the x white balls such that there is at least one black ball in the last cell and number of such distributions is given as:

$$\binom{n-mx+x-1}{x-1} \tag{2.6}$$

Total number of such cyclic arrangements can be x+n-mx and each of these arrangements are partitioned into x parts.

So, total number of such arrangements $\left(\frac{n-mx+x}{x}\right)$ (2.7)

Probability of drawing x white balls and *n - mx* black balls by Polya-Eggenberger sampling scheme is given as:

$$\frac{w^{(x,s)}b^{(n-mx,s)}}{(w+b)^{((n-(m-1)x,s)}} \tag{2.8}$$

From (2.3), (2.4, (2.5), (2.7) and (2.8) we get (2.2).

3. Circular Geiger- Counter Polya-Eggenberger Model of Order 'k'

In this section, we have derived the Circular Geiger Counter Polya-Eggenberger Distribution of order k, i.e., the distribution of $N^{C}_{n,k,m,s}$ by Polya-Eggenberger urn model, where $N^{C}_{n,k,m,s}$ is the number of sequences of

$W_1 \underset{(m-1)}{\_\_\_} W_2 \underset{(m-1)}{\_\_\_} W_3 \underset{(m-1)}{\_\_\_} \ldots W_{k-1} \underset{(m-1)}{\_\_\_\_} W_k .$ in a circular arrangement.

Consider an urn that contain w white and b black balls. A ball is drawn at random by Polya-Eggenberger sampling scheme, if however a white ball is drawn then there is no drawings is the next (m-1) trials. The procedure is repeated n times. Let $N^{C}_{n,k,m,s}$ representing the number of non-overlapping success runs of length k (k≥1) in the n drawings arranged in a circular sequence with support $\left\{0,1,\ldots,1+\left[\frac{n-m(k-1)-1}{mk}\right]\right\}$ for s≥0; and

$$\left\{\max\left(0,\left[\frac{n-((k-1)m+1)b-1}{mk}\right]\right),\ldots,1+\left[\frac{n-m(k-1)-1}{mk}\right]\right\} \text{when } n \le (w-1)m+1 \text{ or}$$

$$\left\{\max\left(0,\left[\frac{n-((k-1)m+1)b-1}{mk}\right]\right),\ldots,1+\left[\frac{m(w-k)-1}{mk}\right]\right\} \text{when } n > (w-1)m+1 \text{ for } s=-1$$

Theorem 2 : For $n(\geq 1)$, $a(>0)$, $b(>0)$ and $s(\geq -1)$ we have,

$$P\left(N^{C}_{n,k,m,s}=y\right)=\binom{n-mx+y-1}{y}C\left(x-yk,n-mx;k-1\right)\frac{(n-mx+x)}{\left(n-mx\right)}\frac{w^{(x,s)}b^{(n-mx,s)}}{\left(w+b\right)^{(n-(m-1)x,s)}}\ \backslash$$

$$+\sum_{j=0}^{m-2}\left[\binom{n-j-m(x-1)-1+y-1}{y-1}C\left(x-yk,n-j-m(x-1)-1;k-1\right)+\right.$$

$$\left.\binom{n-j-m(x-1)-1+y}{y}\sum_{l=0}^{k-2}C\left(x-yk-1-l,n-j-m(x-1)-1;k-1\right)\right]$$

$$\frac{a^{(x,s)}b^{(n-m(x-1)-j-1,s)}}{\left(a+b\right)^{(n-(m-1)(x-1)-j,s)}} \tag{3.1}$$

D = Maximum value of $x = yk + (k\text{-}1) + (k\text{-}1)\,[a_1] + [a_2]$,

$$a_1=\frac{n-ymk-m(k-1)}{m(k-1)+1},\ a_2=\frac{1}{m}\left\{n-\left(ymk+m(k-1)\right)-\left(m(k-1)+1\right)\left[a_1\right]\right\}.$$

Proof: Let the sequence of n trials with y white ball runs of length k be:

$$...BB\ldots W\underset{(m-1)}{\_\_\_}BB\ldots\underbrace{W_1\underset{(m-1)}{\_\_\_}W_2\underset{(m-1)}{\_\_\_}W_3\underset{(m-1)}{\_\_\_}\ldots W_{k-1}\underset{(m-1)}{\_\_\_\_}W_k\underset{(m-1)}{\_\_\_\_}}_{1^{st}\ run}BB\ldots W\underset{(m-1)}{\_\_\_}BB...W\underset{(m-1)}{\_\_\_}$$

$$\ldots W\underset{(m-1)}{\_\_\_}BB...\underbrace{W_1\underset{(m-1)}{\_\_\_}W_2\underset{(m-1)}{\_\_\_}W_3\underset{(m-1)}{\_\_\_}\ldots W_{k-1}\underset{(m-1)}{\_\_\_\_}W_k\underset{(m-1)}{\_\_\_\_}}_{2nd\ \text{run}}BB\ldots$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\underbrace{\ldots W_1\underset{(m-1)}{\_\_\_}W_2\underset{(m-1)}{\_\_\_}W_3\underset{(m-1)}{\_\_\_}\ldots W_{k-1}\underset{(m-1)}{\_\_\_\_}W_k\underset{(m-1)}{\_\_\_\_}}_{(y-1)^{th}\ run}BB\ldots W\underset{(m-1)}{\_\_\_}BB...$$

$$...W\underset{(m-1)}{\_\_\_}BB...\underbrace{W_1\underset{(m-1)}{\_\_\_}W_2\underset{(m-1)}{\_\_\_}W_3\underset{(m-1)}{\_\_\_}\ldots W_{k-1}\underset{(m-1)}{\_\_\_\_}W_k\underset{(m-1)}{\_\_\_\_}}_{y^{th}\ run}W\underset{(m-1)}{\_\_\_}BB...$$

$$\ldots W\underset{(m-1)}{\_\_\_}W\underset{(m-1)}{\_\_\_}BB\ldots W_x\underbrace{\underset{(m-1)}{\_\_\_}BB\ldots}_{j}$$

x is j is the number of remaining trials after the x^{th} white ball.

(i) Suppose j > m-1, then the total number of black balls are equal to n-mx, which create (n-mx) cells in which the y white ball runs of length k can be arranged by balls-into-cells technique in $\binom{n-mx+y-1}{y}$ ways. The remaining x-yk white balls can be arranged in (n-mx) cells such that no cell has more than (k-1) white balls in $C(x\text{-}yk, n\text{-}mx; k\text{-}1)$ ways. Each such arrangements gives *x+n-mx*cyclic arrangements by rotations and each such rotations can be partitioned into (n-mx) arrangements which are alike.

Hence, the required probability of drawing x white balls and n-mx black balls such that there are y white ball runs of length k each and ends with a run of black balls is given as

$$\binom{n-mx+y-1}{y} C(x-yk, n-mx; k-1) \frac{(n-mx+x)}{(n-mx)} \frac{w^{(x,s)} b^{(n-mx,s)}}{(w+b)^{(n-(m-1)x,s)}} \tag{3.2}$$

(ii) Suppose j ≤ m-1 and the last white ball drawn has contributed to y^{th} run of white balls of length k then,

$$\sum_{t=1}^{x} r_t + m(x-1) + 1 + j = n$$

$$\Rightarrow \sum_{t=1}^{x} r_t = n - j - m(x-1) - 1,$$

the number of ways of distributions of (y-1) white ball runs of length k and n-j-m(x-1)-1 black balls is given as:

$$\binom{n-j-m(x-1)-1+y-1}{y-1}.$$

The remaining x-yk white balls can be distributed in n-j-m(x-1)-1 cells such that no cell has more than (k-1) white balls in $C(x-yk, n-j-m(x-1)-1; k-1)$ ways.

Hence, the required probability of drawing x white balls and n-m(x-1)-j-1 black balls such that there are y runs of white balls of length k each and ends with a white ball run followed by j blank trials (0 ≤ j ≤ m-1) is given by

$$\binom{n-j-m(x-1)-1+y-1}{y-1} C(x-yk, n-j-m(x-1)-1; k-1) \frac{w^{(x,s)} b^{(n-m(x-1)-j-1,s)}}{(w+b)^{(n-(m-1)(x-1)-j,s)}},$$

(see, Godbole, 1990). (3.3)

Further, suppose j≤ m-1 and the last white ball drawn does not contribute to runs of white balls of length k then, number of ways of distributions of y white ball runs of length k each and n-j-m(x-1)-1 black balls is given as

$$\binom{n-j-m(x-1)-1+y}{y}.$$

The remaining x-yk-1 white balls can be arranged in n-j-m(x-1)cells such that no cell has more than (k-1) white balls and the last cell has at most (k-2) white balls in:

$$\sum_{l=0}^{k-2} C\left(x-yk-1-l, n-j-m(x-1)-1; k-1\right)$$

ways (see also Godbole, 1990). It could also be obtained by using lemma 2.1 of Makri *et al*. (2007b).

Thus, the required probability of drawing x white balls and n-j-m(x-1)-1 black balls such that there are y runs of white balls of length k each and ends with a white ball followed by j blank trials (0 ≤ j≤ m-1) is given as

$$\binom{n-j-m(x-1)-1+y}{y}\sum_{l=0}^{k-2} C\left(x-yk-1-l, n-j-m(x-1)-1; k-1\right)\frac{a^{(x,s)}b^{(n-m(x-1)-j-1,s)}}{(a+b)^{(n-(m-1)(x-1)-j,s)}} \tag{3.4}$$

From (3.2), (3.3) and (3.4) we get the required probability (3.1).

4. Circular Geiger- Counter Polya-Eggenberger Model of Order 'k' for l Overlapping Distribution

Aki and Hirano (2000) introduced the l-overlapping counting Let k≥ 2 and 0≤*l*≤*k* be two non-negative integers. A success run of length k which has overlapping part of length at most length *l* with the previously counted runs is termed as *l* overlapping runs. For example in a sequence of outcomes

SFSSSSFSSSSSSSSFSFF

There are four 1-overlapping success runs of length 3, three 1-overlapping runs of length 4, and four 2-overlapping success runs of length 4.

The special cases of l-overlapping success runs of k are non-overlapping success run of length k for l=0 and overlapping success run of length k for l=k-1. Number of (l-1) overlapping occurrences of success runs of length k until the nth overlapping occurrence of success run of length l is distributed as $B_{k-l}(n, p, 2)$.

In this section, we have derived the Circular Geiger Counter Polya-Eggenberger Distribution of order k for l-overlapping success run of length k, i.e., the distribution of $N^{C}_{n,k,l,m,s}$, $(0 \le l \le k-1, k \ge 1, m \ge 1)$ by Polya-Eggenberger urn model in the n drawings arranged in a circular sequence with support

$$\left\{0,1,\ldots,1+\left[\frac{n-m(k-l-1)-1}{m(k-l)}\right]\right\} \text{ for } s \ge 0; \text{ a}$$

$$\left\{\max\left(0,\left[\frac{n-\left((k-1)m+1\right)b-ml}{m(k-l)}\right]\right),\ldots,1+\left[\frac{n-m(k-l-1)-1}{m(k-l)}\right]\right\} \text{ when } n \le (w\text{-}1)m\text{+}1 \text{ or}$$

$$\left\{\max\left(0,\left[\frac{n-\left((k-1)m+1\right)b-1}{mk}\right]\right),\ldots,1+\left[\frac{m(w-k)-1}{m(k-l)}\right]\right\} \text{ when } n > (w\text{-}1)m\text{+}1 \quad \text{for } s = -1$$

Theorem 3 : For n (≥1), a (>0), b (>0) and s (≥ -1) we have,

$$\begin{aligned} P\left(N^{C}_{n,k,l,m,s}=y\right) = & \sum_{i=\min(1,y)}^{n-mx}\binom{n-mx}{i}\binom{y-1}{i-1} C\left(x-ik-(y-i)(k-l);i,n-mx-i;k-l-1,k-1\right) \\ & \frac{(n-mx+x)}{\left(n-mx\right)}\frac{w^{(x,s)}b^{(n-mx,s)}}{\left(w+b\right)^{(n-(m-1)x,s)}} + \\ & \sum_{j=0}^{m-1}\left[\sum_{i=\min(1,y)}^{n-m(x-1)-1-j}\sum_{t=0}^{k-2}\binom{n-m(x-1)-1-j}{i}\binom{y-1}{i-1}\right. \\ & C\left(x-ik-(y-i)(k-l)-t;i,n-m(x-1)-j-i;k-l-1,k-1\right)+ \\ & \left.\sum_{t=0}^{k-2}\binom{n-m(x-1)-1-j}{i}\binom{y-1}{i-1}C\left(x-ik-(y-i)(k-l)-t;i,n-m(x-1)-j-i;k-l-1,k-1\right)\right] \\ & \frac{a^{(x,s)}b^{(n-m(x-1)-j-1,s)}}{\left(a+b\right)^{(n-(m-1)(x-1)-j,s)}} \end{aligned} \tag{4.1}$$

for s ≥ 0

$$\left((y-1)(k-l)+1\right) \le x \le \left((y-1)(k-l)+k\right) + \left[n - \frac{n - \left((y-1)(k-l)m-1\right) - (k-l-1)m-1}{(k-1)m+1} \right]$$

and for s = -1,

$$\max\left(\left((y-1)(k-l)+k\right), b\right) \le x$$
$$\le \min\left(w, \left((y-1)(k-l)+k\right) + \left[n - \frac{n - \left((y-1)(k-l)m-1\right) - (k-l-1)m-1}{(k-1)m+1} \right] \right)$$

Proof :

Let the sequence of n trials with y white ball runs of length k be:

$$...BB\ldots W \underset{(m-1)}{\_\_\_} \underbrace{BB\ldots W_1 \underset{(m-1)}{\_\_\_} W_2 \underset{(m-1)}{\_\_\_} W_3 \underset{(m-1)}{\_\_\_} \ldots W_{k-1} \underset{(m-1)}{\_\_\_\_} W_k \underset{(m-1)}{\_\_\_\_}}_{1^{st}\ run} BB\ldots W \underset{(m-1)}{\_\_\_} BB...W \underset{(m-1)}{\_\_\_}$$

$$\ldots W \underset{(m-1)}{\_\_\_} BB...\underbrace{W_1 \underset{(m-1)}{\_\_\_} W_2 \underset{(m-1)}{\_\_\_} W_3 \underset{(m-1)}{\_\_\_} \ldots W_{k-1} \underset{(m-1)}{\_\_\_\_} W_k \underset{(m-1)}{\_\_\_\_}}_{2nd\ \text{run}} BB\ldots$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\underbrace{\ldots W_1 \underset{(m-1)}{\_\_\_} W_2 \underset{(m-1)}{\_\_\_} W_3 \underset{(m-1)}{\_\_\_} \ldots W_{k-1} \underset{(m-1)}{\_\_\_\_} W_k \underset{(m-1)}{\_\_\_\_}}_{(y-1)^{th}\ run} BB\ldots W \underset{(m-1)}{\_\_\_} BB...$$

$$...W \underset{(m-1)}{\_\_\_} BB...\underbrace{W_1 \underset{(m-1)}{\_\_\_} W_2 \underset{(m-1)}{\_\_\_} W_3 \underset{(m-1)}{\_\_\_} \ldots W_{k-1} \underset{(m-1)}{\_\_\_\_} W_k \underset{(m-1)}{\_\_\_\_}}_{y^{th}\ run} W \underset{(m-1)}{\_\_\_} BB...$$

$$\ldots W \underset{(m-1)}{\_\_\_} W \underset{(m-1)}{\_\_\_} BB\ldots W_x \underbrace{\underset{(m-1)}{\_\_\_} BB\ldots}_{j}$$

x is the number of white balls drawn and j is the number of remaining trails after the x^{th} white ball.

(i) Suppose j> m-1, then the total number of black balls are equal to n-mx, which create (n-mx) cells. Suppose that out of (n-mx) cells there are i cells which contribute to the y “l-overlapping white ball runs of k” is given as $\binom{y-1}{i-1}$ ways and $i$cells can be selected out of n-mx cells in $\binom{\text{n-mx}}{\text{i}}$ ways. The remaining x-i k-(y-i)(k-l) white balls can be arranged in (n-mx) cells such that none of the “i" cells have more than (k-l-1) white balls, and rest of the n-mx-i cells do not have more than (k-1) white balls in $C\left(x\text{-}ik\text{-}(y\text{-}i)(k\text{-}l);i,n\text{-}mx\text{-}i;k\text{-}l\text{-}1,k\text{-}1\right)$ ways. Each such arrangements, gives x+n-mx cyclic arrangements by rotations and each such rotations can be partitioned into (n-mx) arrangements which are alike.

Hence the required probability of drawing x white balls and n-mx black balls such that there are y white ball runs of length k each and ends with a run of black balls is given as

$$\sum_{i=\min(1,y)}^{n-mx} \binom{n-mx}{i}\binom{y-1}{i-1} C\left(x-ik-(y-i)(k-l);i,n-mx-i;k-l-1,k-1\right) \frac{(n-mx+x)}{(n-mx)} \frac{w^{(x,s)}b^{(n-mx,s)}}{(w+b)^{(n-(m-1)x,s)}} \tag{4.2}$$

(ii) Suppose j ≤ m-1 and the last white ball drawn has contributed to y^{th} *l*-overlapping run of white balls of length k then,

$$\sum_{t=1}^{x} r_t + m(x-1) + 1 + j = n$$

$$\Rightarrow \sum_{t=1}^{x} r_t = n - j - m(x-1) - 1,$$

the total number of black balls are equal to n-m(x-1)-1-j, which create (n-m(x-1)-1-j) cells. Suppose that out of (n-m(x-1)-1-j) cells there are *i*-1 cells which contribute to the y -1 " l-overlapping white ball runs of k" is given as $\binom{y-2}{i-2}$ ways and these *i*cells can be selected out of n-m(x-1)-1-j cells in $\binom{n-m(x-1)-1-j}{i-1}$ ways (the last cell having a white ball run of length k). The remaining *x*-*ik*-(*y*-*i*)(*k*-*l*) white balls can be arranged in (n-m(x-1)-j) cells such that none of the "*i*" cells have more than (*k*-*l*-1) white balls, and rest of the n-m(x-1)-j-icells do not have more than (k-1) white balls in $C\left(x\text{-}ik\text{-}(y\text{-}i)(k\text{-}l);i,n\text{-}m(x\text{-}1)\text{-}j\text{-}i;k\text{-}l\text{-}1,k\text{-}1\right)$ ways.

Hence the required probability of drawing x white balls and n-m(x-1)-1-j black balls such that there are y white ball runs of length k each and ends with a run of black balls is given as

$$\sum_{i=\min(1,y)}^{n-m(x-1)-1-j} \binom{n-m(x-1)-1-j}{i}\binom{y-2}{i-2} C\big(x-ik-(y-i)(k-l); i, n-m(x-1)-j-i; k-l-1, k-1\big) \frac{w^{(x,s)} b^{(n-m(x-1)-1-j,s)}}{(w+b)^{(n-(m-1)(x-1)-j,s)}}$$

(4.3)

Further, suppose j≤ m-1 and the last white ball drawn does not contribute to runs of white balls of length k then, Suppose that out of (n-m(x-1)-1-j) cells there are *i* cells which contribute to the y -1 "l-overlapping white ball runs of k" and is given as $\binom{y-1}{i-1}$ ways and *i*cells can be selected out of n-m(x-1)-1-j cells in $\binom{n\text{-}m(x\text{-}1)\text{-}1\text{-}j}{i}$ ways.

The remaining *x*-*ik*-(*y*-*i*)(*k*-*l*) white balls can be arranged in n-j-m(x-1)-1cells such that none of the "*i*" cells have more than (*k*-*l*-1) white balls, and rest of the n-m(x-1)-j-1-i cells do not have more than (k-1) white balls in C(x-ik-(y-i)(k-l)-t;i,n-m(x-1)-j-i-1;k-l-1,k-1) ways.

$$\sum_{t=0}^{k-2} C\big(x-ik-(y-i)(k-l)-t; i, n-m(x-1)-j-i-1; k-l-1, k-1\big)$$

ways (see also Godbole, 1990). It could also be obtained by using lemma 2.1 of Makri *et al.* (2007b).

Hence the required probability of drawing x white balls and n-j-m(x-1)-1 black balls such that there are y runs of white balls of length k each and ends with a white ball followed by j blank trials (0 ≤ j≤ m-1) is given as

$$\sum_{i=\min(1,y)}^{n-m(x-1)-1-j} \sum_{t=0}^{k-2} \binom{n-m(x-1)-1-j}{i}\binom{y-1}{i-1}$$

$$C\big(x-ik-(y-i)(k-l)-t; i, n-m(x-1)-j-i; k-l-1, k-1\big) \frac{w^{(x,s)} b^{(n-m(x-1)-1-j,s)}}{(w+b)^{(n-(m-1)(x-1)-j,s)}}$$

(4.4)

From (4.2), (4.3) and (4.4) we get the required probability (4.1).

5. Circular Inverse Geiger-Counter *l*-Overlapping Polya-Eggenberger Model of Order k

Under the Polya-Eggenberger sampling scheme, if $W_{k,n,l,m,s}$ represents the waiting time (number of trials) till the n^{th} white ball run of length *k* occurs, then let $P\left(W^{(C)}_{k,n,l,m,s} = u\right)$ = the probability that n^{th} white ball run of length k occurs at the u^{th} trial.

Let the sequence be with the n^{th} white ball run of length k at the u^{th} trial be:

$$\ldots BBW\underset{(m-1)}{\_\_}\underbrace{\ldots W_1\underset{(m-1)}{\_\_}W_2\underset{(m-1)}{\_\_}W_3\underset{(m-1)}{\_\_}\ldots W_{k-1}\underset{(m-1)}{\_\_\_}W_k\underset{(m-1)}{\_\_\_}}_{1^{st}\,run}BB....W\underset{(m-1)}{\_\_\_}W\underset{(m-1)}{\_\_\_}BB\ldots W\underset{(m-1)}{\_\_\_}BB\ldots$$

$$\underbrace{\ldots W_1\underset{(m-1)}{\_\_}W_2\underset{(m-1)}{\_\_}W_3\underset{(m-1)}{\_\_}\ldots W_{k-1}\underset{(m-1)}{\_\_\_}W_k\underset{(m-1)}{\_\_\_}}_{2nd\text{ run}}BB\ldots W\underset{(m-1)}{\_\_\_}BB\ldots W\underset{(m-1)}{\_\_\_}BB\ldots$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\underbrace{\ldots W_1\underset{(m-1)}{\_\_}W_2\underset{(m-1)}{\_\_}W_3\underset{(m-1)}{\_\_}\ldots W_{k-1}\underset{(m-1)}{\_\_\_}W_k\underset{(m-1)}{\_\_\_}}_{(n-1)^{th}\,run}BB\ldots W\underset{(m-1)}{\_\_\_}BB\ldots W\underset{(m-1)}{\_\_\_}BB\ldots$$

$$\underbrace{\ldots W_1\underset{(m-1)}{\_\_}W_2\underset{(m-1)}{\_\_}W_3\underset{(m-1)}{\_\_}\ldots W_{k-1}\underset{(m-1)}{\_\_\_}W_k}_{n^{th}\text{ run}}$$

Theorem 3: For $n(\geq 1)$, $a(>0)$, $b(>0)$ and $s(\geq 0)$ we have,

$$P\left(W^{(C)}_{k,n,l,m,s} = u\right) = \sum_{i=\min(1,u)}^{u-xm+m-1} \binom{u-xm+m-1}{i}\binom{n-1}{i} C\left(x\text{-}(i+1)k+\left(n-(i+1)\right)\left(k-l\right); i+1, u\text{-}xm\text{+}m\text{-}1; k-l-1, k-1\right)$$

$$\frac{w^{(x,s)}b^{(u-xm+m-1,s)}}{\left(w+b\right)^{(u-(m-1)(x-1),s)}}, \quad u = lm+n(k-l)m+1, lm+n(k-l)m+2, \ldots \tag{5.1}$$

Proof : The total number of black balls are equal to u -(x-1)m-1, where x is the number of white balls drawn. Suppose there are i cells out of the u-xm+m-1which has exactly k white balls, then these *i* cells can be selected in

$$\binom{u-xm+m-1}{i} \text{ ways} \tag{5.2}$$

The remaining n-(i+1) runs of white balls of length (k-l) can be distributed in the (i+1) cells in $\binom{n-1}{i}$ ways. (5.3)

The remaining $x\text{-}(i+1)k+(n-(i+1))(k-l)$ white balls can be arranged in u-xm+m-1 cells such that none of the (i+1) cells have more than (k-l-1) white balls and the rest of the u-xm+m-i in

$$C\left(x\text{-}(i+1)k+(n-(i+1))(k-l)\ ; i+1, u\text{-}xm+m\text{-}1; k-l-1, k-1\right) \text{ways.} \tag{5.4}$$

Hence the required probability is given as

$$\sum_{i=\min(1,u)}^{u-xm+m-1}\binom{u-xm+m-1}{i}\binom{n-1}{i}C\left(x\text{-}(i+1)k+(n-(i+1))(k-l)\ ; i+1, u\text{-}xm+m\text{-}1; k-l-1, k-1\right)\frac{w^{(x,s)}b^{(u-xm+m-1,s)}}{(w+b)^{(u-(m-1)(x-1),s)}} \tag{5.5}$$

As in Makri *et al.* (2007b), the case s = -1 requires special attention. This is so because the Polya-Eggenberger sampling procedure might terminate without observing nl-overlapping white ball runs of length k each, when sampling is done without replacement (s = -1). Thus, the random variable $W_{k,n,l,m,s}$ might take on the value ∞ with positive probability. We have found in this case, the probability $P(W_{k,n,l,m,s}=u)$ is given by (5.1) with

$$\max\left(nk, \left[\frac{u-b+m-1}{m}\right]+\delta\right) \le x \le \min\left(a, \left[\frac{nk+(u+m-1)(k-1)}{1+m(k-1)}\right]\right), \tag{5.6}$$

$$\max\left(n(k-l)+l, \left[\frac{u-b+m-1}{m}\right]+\delta\right) \le x$$

$$\le \min\left(w, n(k-l)+l+\left[\frac{u-(n(k-l)-1)m-1}{1+m(k-1)}\right]\bullet(k-1)m\right)$$

where $\delta = \begin{cases} 0, & \text{if } [c] = c \\ 1, & \text{otherwise.} \end{cases}$

Further, $P\left(W^{(C)}_{k,n,l,m,-1}=\infty\right)=\sum_{y=0}^{n-1}P\left(N^{C}_{wm+b,k,l,m,-1}=y\right)$

where $P\left(N^{C}_{wm+b,k,l,m,-1}=y\right)$ are as given in (4.1)

6. Conclusion

In this research paper a simple combinatorial approach using ball into cell techniques with Polya-Eggenberger sampling scheme has been used to develop several Generalized Circular Geiger Counter Type Distributions. Geiger counter of these types is used in problems in which a counter registering only successes gets locked abruptly for few trials, say (r-1) trials after which it starts re-registering. Several use of these kinds of counters are mentioned in the literature (Feller, 1968) for example for detecting radioactive rocks and minerals during mineral prospecting, checking environmental level of radioactivity in nuclear power plants, testing level of danger in nuclear accidents and for detecting radioactive contamination of food and uniforms of workers of nuclear power plants. In general, in the insurance sector to decide premium for covering incidents of burglary it is presumed that after occurrence of each incidence of burglary the chance of occurrence of the event would be minimal for a specific period of time. During this period, the chance of moral or morale hazard would be very less. Similarly, buildings, monuments, and places of religious worships which are insured from terrorist attack will have less chances of such attacks for some time, once incidents like 9/11 happens.

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