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Efficient Circular Strongly Partially Balanced Repeated Measurements Designs in Periods of Two Different Sizes

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ABSTRACT

With the use of repeated measurements designs (RMDs), residual effects may often arise. Minimal strongly balanced RMDs are useful to balance out these effects economically but unfortunately, these designs cannot be constructed for all cases. In such situations, strongly partially balanced RMDs are preferred. In this article, efficient circular strongly partially balanced RMDs are presented in periods of two different sizes.

1. Introduction

Repeated measurements designs (RMDs) are called strongly balanced if each treatment follows exactly λ' times by every other treatment, including itself, otherwise called strongly partially-balanced. If λ'_i takes values only 0 and 1, or 1 and 2 then design is called minimal strongly partially balanced. Williams (1949, 1950) first introduced RMDs. Magda (1980) introduced the idea of circular balanced RMDs. Cheng and Wu (1980) constructed balanced and strongly balanced uniform RMDs. Afsarinejad (1994) constructed minimal balanced and strongly balanced RMDs with unequal period sizes. Using method of cyclic shifts, (a) Iqbal and Jones (1994) constructed (i) efficient RMDs and strongly balanced RMDs for two unequal period sizes, (b) Iqbal and Tahir (2009) constructed circular strongly balanced RMDs for some cases, (c) Iqbal *et al.* (2010) constructed some first and second order circular balanced and strongly balanced RMDs, (d) Rajab *et al*. (2018) constructed some infinite series to generate circular balanced RMDs for equal period sizes, (e) Rasheed *et al.* (2018) developed some infinite series to obtain the minimal circular strongly balanced

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RMDs in periods of three different sizes, (f) Khan *et al.* (2019) developed some infinite series to obtain circular partially balanced RMDs in periods of equal sizes. Minimal strongly partially balanced RMDs are recommended for the situations where minimal strongly balanced RMDs cannot be constructed. Jabeen *et al*. (2019) introduced circular strongly partially balanced RMDs (CSPBRMDs) in periods of equal sizes. Nazeer *et al*. (2019) constructed minimal CSPBRMDs in periods of two different sizes for a few cases. Using method of cyclic shifts (Rule II), Jabeen *et al*. (2019) developed some generators to obtain minimal CSPBRMDs in periods of two different sizes. Some more references of the designs constructed through method of cyclic shifts can be found in Ahmed *et al*. (2020). All these constructions are for p_1 even. These designs are no more useful if the setup of experiments is to run in two different period sizes with p_1 odd. To overcome this problem, efficient CSPBRMDs are constructed here in periods of two different sizes with p_1 odd using method of cyclic shifts (Rule I).

Rest of the paper is organized as follows: Method of cyclic shifts (Rule I) is explained in Section 2 to obtain CSPBRMDs in periods of two different sizes. Procedures for calculating efficiency of separability and of residual effects are described in Section 3. In Section 4, efficient CSPBRMDs are constructed in periods of two different sizes along with their efficiencies for (i) $11 \le p_1$ (odd) ≤ 15, $3 \le p_2 \le 7$, $v \le 50$, and (ii) $9 \le p_1$ (odd) ≤ 15, $8 \le p_2 \le 14$, $v \le 50$.

2. Method of Cyclic Shifts

- **How to obtain a CSPBRMD in Periods of Two Different Sizes**

Let $S_1 = [q_{11,}\ q_{12},\ldots,\ q_{1(r\ -1)}]$ and $S_2 = [q_{21,}\ q_{22},\ldots,\ q_{2(s-1)}]$ be sets of shifts, where $0 \le q_{ij} \le v\text{-}1$, if each element 0, 1, 2, …, v-1 appears an equal number of times, say λ' in S^*, where $S^* = [q_{11}\ ,\ q_{12}\ ,\ldots,\ \ q_{1(r\ -1),}\ v\text{-}(q_{11} + q_{12} +\ldots+ q_{1(r-1)}) \bmod v,\ q_{21}\ ,\ q_{22}\ ,\ldots,\ q_{2(s\ -1),}\ v\text{-}(q_{21} + q_{22} +\ldots+q_{2(s-1)}) \bmod v]$ then it will be CSBRMD in periods of two different sizes ($p_1 = r$ and $p_2 =$ s). It will be minimal CSPBRMD, if S^* contains some of 0, 1, 2… v-1 either (i) no time and others once, or (ii) two times and others once. If sum of any two, three, …, (p-2) elements of consecutive shifts of a set is 0 (mod) v then any permutation within any shift is accepted as long as 0 (mod) v is not satisfied. It is described here with the help of following example.

Example 2.1: Sets of shifts, S_1 = [1, 3, 2, 11, 10] and S_2 = [4, 5, 8, 7] provide minimal CSPBRMD for $v = 12$, $p_1 = 6$ and $p_2 = 5$.

Check for minimal CSPBRMD: Here, S* = [1,3,2,11,10,9,4,5,8,7,0] it means each of 0, 1, …, 5, 7, 8, …, 11 appears once while 6 does not appear. So it is minimal CSPBRMD, using 24 subjects.

Subjects											
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0_3	1_4	2_5	3_6	4_7	5_8	6_9	7_{10}	8_{11}	9_0	10_1	11_2
1_0	2_1	3_2	4_3	5_4	6_5	7_6	8_7	9_8	10_9	11_{10}	0_{11}
4_1	5_2	6_3	7_4	8_5	9_6	10_7	11_8	0_9	1_{10}	2_{11}	3_0
6_4	7_5	8_6	9_7	10_8	11_9	0_{10}	1_{11}	2_0	3_1	4_2	5_3
5_6	6_7	7_8	8_9	9_{10}	10_{11}	11_0	0_1	1_2	2_3	3_4	4_5
3_5	4_6	5_7	6_8	7_9	8_{10}	9_{11}	10_0	11_1	0_2	1_3	2_4
Subjects											
13	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
0_0	1_1	2_2	3_3	4_4	5_5	6_6	7_7	8_8	9_9	10_{10}	11_{11}
4_0	5_1	6_2	7_3	8_4	9_5	10_6	11_7	0_8	1_9	2_{10}	3_{11}
9_4	10_5	11_6	0_7	1_8	2_9	3_{10}	4_{11}	5_0	6_1	7_2	8_3
5_9	6_{10}	7_{11}	8_0	9_1	10_2	11_3	0_4	1_5	2_6	3_7	4_8
0_5	1_6	2_7	3_8	4_9	5_{10}	6_{11}	7_0	8_1	9_2	10_3	11_4

Here 9_4 means 9^{th} treatment is applied in current period while 4^{th} treatment was applied in preceding period.

3. Efficiency of the Proposed Designs

3.1 Efficiency of Residual Effects

In order to consider the efficiency of the constructed designs, the model for circular RMDs proposed by Davis and Hall (1969) is used.

$$\mathbf{Y} = \mu\mathbf{E} + \mathbf{D}\boldsymbol{\delta} + \mathbf{R}\boldsymbol{\rho} + \mathbf{U}\boldsymbol{\nu} + \mathbf{P}\boldsymbol{\pi} + \boldsymbol{\varepsilon} \tag{3.1}$$

Here $\mathbf{Y}$ is the $np\times 1$ column vector of the np observations, μ is the overall mean, $\boldsymbol{\delta}$ is the vector of direct effects of order $v\times 1$, $\boldsymbol{\rho}$ is residual effect vector of order $v\times 1$, $\boldsymbol{\nu}$ is the unit vector of order $n\times 1$, $\boldsymbol{\pi}$ is the vector of period effects having order $p\times 1$ and $\boldsymbol{\varepsilon}$ is random error vector of order $np\times 1$ with mean zero and constant variance σ^2. $\mathbf{E}$ is the matrix of 1's with order $p\times q$. **D, R, U, P** are design matrices of observations versus direct effects, residual effects, unit effects and period effects of treatments with order $np\times v$,$np\times v$, $np\times bv$ and $np\times p$ respectively.

Using the identities, $\mathbf{D'D} = \mathbf{R'D} = bp\mathbf{I}_v$, $\mathbf{D'R} = \mathbf{L}$, $\mathbf{D'U} = \mathbf{N}$, $\mathbf{D'P} = b\mathbf{E}_{v,p}$, $\mathbf{R'U} = \mathbf{N}$, $\mathbf{U'U} = p\mathbf{I}_n$, $\mathbf{U'U} = \mathbf{E}_{k,q}$,$\mathbf{P'P} = n\mathbf{I}_n$. The reduced normal equations for $\widehat{\boldsymbol{\delta}}$ and $\widehat{\boldsymbol{\rho}}$ will be:

$$\mathbf{C}\begin{bmatrix}\hat{\boldsymbol{\delta}}\\ \hat{\boldsymbol{\rho}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{\pi}\\ \boldsymbol{\pi}' & \boldsymbol{\Theta}\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{\delta}}\\ \hat{\boldsymbol{\rho}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{T}\\ \boldsymbol{S}\end{bmatrix}$$

$$\begin{bmatrix} bp\mathbf{I}_v - p^{-1}\mathbf{NN}' & \mathbf{L}' - p^{-1}\mathbf{NN}' \\ \mathbf{L}' - p^{-1}\mathbf{NN}' & bp\mathbf{I}_v - p^{-1}\mathbf{NN}' \end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{\delta}}\\ \hat{\boldsymbol{\rho}}\end{bmatrix} = \begin{bmatrix}\mathbf{D}'\mathbf{Y} - p^{-1}\mathbf{NU}'\mathbf{Y}\\ \mathbf{R}'\mathbf{Y} - p^{-1}\mathbf{NU}'\mathbf{Y}\end{bmatrix}$$

$\boldsymbol{\theta} = bp\mathbf{I}_v - p^{-1}\mathbf{NN}'$, $\boldsymbol{\Theta} = bp\mathbf{I}_v - p^{-1}\mathbf{NN}'$, $\boldsymbol{\pi} = \mathbf{L}' - p^{-1}\mathbf{NN}'$, $\mathbf{T} = \mathbf{D}'\mathbf{Y} - p^{-1}\mathbf{NU}'\mathbf{Y}$, $\boldsymbol{S} = \mathbf{R}'\mathbf{Y} - p^{-1}\mathbf{NU}'\mathbf{Y}$,

For the period of two different sizes information matrix can be presented as:

$$\mathbf{C}* = \begin{bmatrix} bp\mathbf{I}_v - {p_1}^{-1}\mathbf{N_1N'_1} - {p_2}^{-1}\mathbf{N_2N'_2} & \mathbf{L}' - {p_1}^{-1}\mathbf{N_1N'_1} - {p_2}^{-1}\mathbf{N_2N'_2} \\ \mathbf{L}' - {p_1}^{-1}\mathbf{N_1N'_1} - {p_2}^{-1}\mathbf{N_2N'_2} & bp\mathbf{I}_v - {p_1}^{-1}\mathbf{N_1N'_1} - {p_2}^{-1}\mathbf{N_2N'_2} \end{bmatrix}$$

The information matrix for direct and residual effects denoted by $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$ respectively can be specified by their initial rows:

$$\boldsymbol{\theta} = [\theta_0, \theta_1, \ldots, \theta_{t-1}] \qquad \text{and} \qquad \boldsymbol{\Theta} = [\Theta_0, \Theta_1, \ldots, \Theta_{t-1}]$$

According to the duality presented in the model (3.1), both direct and residual effects share the same information matrix. The non-zero Eigen values of information matrix C^* are called the canonical efficiency factors, see James and Wilkinson (1971) and Pearce *et al*. (1974). The canonical efficiency factor is calculated by working out harmonic mean of non-zero Eigen values of their respective information matrix relative to that of an orthogonal with the same number of treatments having same number of replications. It is further assume that σ^2 is the same for the proposed design and the orthogonal design to which it is compared. The high value of E_r shows that design is suitable for the estimation of residual effects.

3.2 Efficiency of Separability

RMD must be characterized for its ability of separating the direct and first order residual effects. Divecha and Gondaliya (2014) used the following measure of separability called efficiency of separabiliy (Es) for balanced RMDs.

$$\text{Es} = \left[1 - \left\{\frac{(\lambda_3-\lambda_2)^2}{\lambda_3+(v-1)\lambda_2(\lambda_1+\lambda_3+(v-1)\lambda_2)}\right\}^{1/2}\right] \times 100\ \%$$

where

- λ_1 is number of units allocated for each treatment in two successive periods.
- Each ordered pair of distinct treatments is given to λ_2 units.
- Each pair of treatments with itself is given to λ_3 units.

4. Construction of Efficient CSPBRMDs

In this Section, efficient CSPBRMDs are constructed using method of cyclic shifts (Rule I). If sum of any two, three, …, (p-2) elements of consecutive shifts of a set is 0 (mod) v then any permutation within any shift is accepted as long as 0 (mod) v is not satisfied.

Construction 4.1. Efficient CSPBRMDs can be constructed for $v = ri+2p_2+1$ in $p_1 = r$ (odd) and $p_2 = s$, using following i sets of shifts for p_1 and two sets for p_2. In these designs, some ordered pairs of treatments do not appear together while all other appear once.

$$S_{j+1} = [q_{j1},q_{j2}, \ldots,q_{j(r-1)}]; \qquad j = 0, 1\ldots i\text{-}1.$$
$$S_{i+1} = [q_{(i+1)1},q_{(i+1)2}, \ldots, q_{(i+1)(s-1)}] \qquad S_{i+2} = [q_{(i+2)1},q_{(i+2)2}, \ldots, q_{(i+2)(s-1)}]$$

Where

- $0 \le q_{j1}, q_{j2}, \ldots,q_{j(r-1)}, q_{(i+1)1},q_{(i+1)2}, \ldots, q_{(i+1)(s-1)}, q_{(i+2)1}, q_{(i+2)2}, \ldots, q_{(i+2)(s-1)} \le v-1$ but $\ne v/2$.
- S* contains each of 0, 1, 2… v-1 exactly once except $v/2$ which do not appear.
- S* = $[q_{j1}, q_{j2}, \ldots,q_{j(r-1)}, v-(q_{j1}+\ldots+ q_{j(r-1)}) \bmod v, q_{(i+1)1},q_{(i+1)2},\ldots,q_{(i+1)(s-1)}, v-(q_{(i+1)1}+\ldots+ q_{(i+1)(s-1)}) \bmod v, q_{(i+2)1},q_{(i+2)2},\ldots, q_{(i+2)(s-1)}, v-(q_{(i+2)1}+\ldots+q_{(i+2)(s-1)}) \bmod v]$

Designs constructed through this procedure for $v \le 50$ are presented with (i) $11 \le p_1$ (odd) ≤ 15, $3 \le p_2 \le 7$, and (ii) $9 \le p_1$ (odd) ≤ 15, $8 \le p_2 \le 14$ in Table 1 and 2 respectively.

Table 1. Efficient CSPBRMDs for $v \le 50$, $11 \le p_1$ (odd) ≤ 15, $3 \le p_2 \le 7$.

v	p_1	p_2	**Sets of Shifts**	**Er**	**Es**
26	11	3	[23,1,3,2,5,4,6,7,8,9]+ [12,25,14,24,16,17,18,19,20,22]+[11,15]	0.82	0.96
48	11	3	[41,1,2,3,4,5,6,7,8,9]+ [12,13,14,15,27,16,17,18,19,20]+ [22,23,25,26,28,29,30,32,31,42]+ [34,35,37,36,38,39,46,47,33,43]+[45,40]	0.76	0.98
30	13	3	[1,2,3,4,5,6,7,8,9,10,11,24]+ [14,12,17,16,18,19,21,22,23,25,26,28]+[27,20]	0.67	0.97
34	15	3	[1,3,2,5,4,7,6,8,10,9,11,12,13,14]+ [18,33,16,32,22,20,21,23,25,24,26,28,27,29]+ [15,19]	0.86	0.97
16	11	4	[1,2,3,4,5,6,7,15,10,11]+[12,13,14]	0.70	0.94
38	11	4	[1,2,3,4,5,6,7,8,9,31]+ [11,23,13,14,16,15,17,18,22,21]+ [24,37,25,27,26,28,29,30,10,32]+[34,35,33]	0.86	0.97

18	13	4	[1,2,3,4,5,6,7,16,10,11,12,13]+[14,15,8]	0.55	0.94
44	13	4	[1,2,3,4,5,6,7,8,9,10,12,21]+ [13,37,15,17,16,18,19,20,11,26,24,24]+ [27,43,41,30,31,32,33,34,42,36,39,14]+ [40,29,35]	0.84	0.94
20	15	4	[1,2,3,4,5,6,7,8,11,12,13,14,15,19]+[16,17,18]	0.59	0.95
50	15	4	[45,1,2,4,3,5,6,8,7,9,11,10,12,13]+ [15,1,17,18,19,35,22,21,24,23,26,28,27,30]+ [31,32,33,34,49,47,37,38,39,40,41,42,43,44]+ [46,36,48]	0.77	0.95
28	11	5	[1,2,3,4,5,6,7,8,9,11]+ [10,13,25,15,17,16,18,19,20,21]+[23,24,12,26]	0.74	0.96
50	11	5	[45,1,2,3,4,5,6,7,8,9]+ [35,12,13,14,15,16,17,18,19,20]+ [48,23,24,49,26,27,28,29,33,31]+ [30,34,11,36,37,47,39,40,41,42]+[22,38,44,46]	0.87	0.98
32	13	5	[18,1,2,3,4,5,6,7,8,9,10,11]+ [30,14,15,17,19,20,21,22,23,24,25,26]+ [27,28,29,13]	0.77	0.97
36	15	5	[1,2,3,4,5,6,7,8,9,10,11,12,13,17]+ [16,15,19,20,35,21,23,22,24,25,26,27,28,29]+ [31,32,33,34]	0.71	0.97
16	11	6	[1,2,3,4,5,15,7,8,9,10]+[11,12,13,14,6]	0.74	0.94
38	11	6	[21,1,2,3,4,5,6,7,8,9]+ [11,12,13,14,15,32,18,17,19,20]+ [22,24,23,25,27,26,28,29,30,33]+ [31,34,35,36,16]	0.87	0.97
18	13	6	[13,2,3,4,5,6,7,8,9,10,11,12]+[1,14,17,15,16]	0.54	0.95
44	13	6	[1,2,3,4,5,6,7,8,9,10,11,22]+ [40,14,15,16,17,18,19,20,21,12,23,24]+ [43,27,28,29,30,31,34,32,33,42,36,37]+ [29,13,41,35,26]	0.85	0.96
20	15	6	[13,11,10,9,12,7,6,8,5,4,2,3,1,15]+ [16,17,18,19,10]	0.85	0.95
50	15	6	[46,2,3,4,5,6,7,8,9,10,11,12,13,14]+ [35,16,18,17,19,20,21,22,23,24,25,26,27,29]+ [31,45,32,33,34,36,37,15,38,39,40,41,43,42]+ [30,1,47,48,49]	0.80	0.98
30	11	7	[1,2,3,4,5,6,7,8,14,10]+ [11,12,9,16,13,18,19,17,20,21]+ [22,25,26,27,28,29]	0.80	0.97
34	13	7	[1,2,3,5,4,7,6,8,10,9,11,12]+ [13,14,15,16,18,19,20,21,22,23,25,32]+ [27,28,29,30,31,26]	0.66	0.97
38	15	7	[1,2,3,4,5,6,7,8,9,10,11,12,13,23]+ [17,14,16,18,21,20,22,25,24,26,28,29,30,37]+ [31,32,33,34,35,36]	0.80	0.97

Table 2. Efficient CSPBRMDs for $v \le 50$, $9 \le p_1$ (odd) ≤ 15, $8 \le p_2 \le 14$.

v	p_1	p_2	Sets of Shifts	Er	Es
18	9	8	[1,3,2,5,4,6,7,8]+[12,11,14,13,15,16,17]	0.82	0.94
36	9	8	[29,1,3,4,5,6,7,8]+[11,10,12,13,14,15,16,17]+ [20,21,23,22,24,25,26,27]+[2,30,31,32,33,34,35]	0.85	0.97
20	11	8	[1,2,3,4,5,6,7,12,9,11]+[14,13,15,16,17,18,19]	0.63	0.95
42	11	8	[1,3,2,4,5,6,7,8,9,10]+ [13,12,14,15,16,17,18,19,22,23]+ [38,24,25,26,27,28,30,31,32,33]+ [35,36,37,20,39,40,11]	0.79	0.98
22	13	8	[1,2,3,4,5,14,7,8,10,9,12,13]+ [15,16,17,19,18,20,21]	0.66	0.95
48	13	8	[1,2,3,4,5,6,7,8,9,10,11,12]+ [13,14,15,16,17,28,20,21,22,23,25,26]+ [19,41,31,30,32,33,34,35,36,37,38,39]+ [29,42,43,44,45,46,47]	0.69	0.98
24	15	8	[1,2,3,16,5,6,7,8,9,10,11,13,14,15]+ [17,18,19,20,21,22,23]	0.49	0.96
32	11	9	[1,11,3,4,5,6,7,8,9,10]+ [13,15,31,14,17,18,19,20,21,22]+ [24,25,26,27,28,29,30,12]	0.87	0.97
36	13	9	[2,1,4,3,5,6,7,9,8,10,11,12]+ [14,15,27,20,17,21,19,22,23,24,26,25]+ [16,28,29,31,32,33,34,13]	0.88	0.97
40	15	9	[1,2,3,4,6,5,7,8,9,10,11,12,13,14]+ [16,17,19,18,23,21,24,25,22,26,27,28,29,30]+ [31,32,33,34,36,37,38,39]	0.89	0.97
22	11	10	[12,2,3,4,5,6,7,8,9,10]+ [13,15,14,16,17,18,19,21,20]	0.71	0.95
44	11	10	[2,1,3,4,5,6,7,8,9,10]+ [13,12,14,15,16,17,18,19,20,21]+ [43,37,25,26,27,28,29,30,31,32]+ [35,36,24,38,39,40,41,42,23]	0.84	0.98
24	13	10	[1,2,3,4,5,7,6,8,9,11,10,13]+ [14,15,16,18,19,20,21,22,23]	0.87	0.96
50	13	10	[1,2,3,4,5,6,8,7,9,10,11,12]+ [37,14,15,16,17,18,19,20,21,23,24,26]+ [48,29,30,31,32,33,34,35,36,13,41,39]+ [38,42,43,44,45,46,47,28,27]	0.83	0.98
26	15	10	[15,1,14,3,2,5,4,6,7,8,9,10,11,12]+ [16,17,18,19,20,21,22,24,25]	0.81	0.96
38	13	11	[1,2,3,4,5,6,7,8,9,10,12,11]+ [18,14,17,16,20,21,22,23,24,25,26,27]+ [15,28,29,30,31,32,33,34,35,37]	0.89	0.97

42	15	11	[22,2,3,4,5,6,7,8,9,10,11,12,13,14]+ [1,16,18,17,19,20,23,24,25,26,27,28,30,29]+ [31,32,34,35,37,36,38,39,40,41]	0.75	0.98
26	13	12	[1,2,3,4,5,6,7,8,9,10,11,12]+ [16,15,18,17,19,20,21,22,23,24,25]	0.76	0.96
28	15	12	[1,2,3,4,5,6,7,8,9,16,11,12,13,15]+ [17,18,19,21,20,22,24,23,25,27,26]	0.64	0.96
44	15	13	[2,1,3,4,5,6,7,8,9,10,11,12,14,13]+ [16,18,20,17,21,19,23,24,25,26,41,28,29,30]+ [31,32,33,34,35,36,37,38,39,40,42,43]	0.91	0.98
30	15	14	[16,2,3,4,5,6,7,8,9,10,11,12,13,14]+ [17,19,18,29,20,21,22,23,25,24,27,26,28]	0.54	0.97

Construction 4.2. Efficient CSPBRMDs can be constructed for $v = ri+2p_2+1$ in $p_1 = r$ (odd) and $p_2 = s$, using following i sets of shifts for p_1 and two sets for p_2. In these designs, some ordered pairs of treatments appear twice together while all other appear once.

$S_{j+1} = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $j = 0, 1\ldots i-1$.
$S_{i+1} = [q_{(i+1)1}, q_{(i+1)2}, \ldots, q_{(i+1)(s-1)}]$ $\qquad$ $S_{i+2} = [q_{(i+2)1}, q_{(i+2)2}, \ldots, q_{(i+2)(s-1)}]$

Where

- $0 \le q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, q_{(i+1)1}, q_{(i+1)2}, \ldots, q_{(i+1)(s-1)}, q_{(i+2)1}, q_{(i+2)2}, \ldots, q_{(i+2)(s-1)} \le v-1$.
- S* contains each of 0, 1, 2… v-1 exactly once except $v/2$ which appears twice.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, v-(q_{j1}+\ldots+q_{j(r-1)}) \bmod v, q_{(i+1)1}, q_{(i+1)2}, \ldots, q_{(i+1)(s-1)}, v-(q_{(i+1)1}+\ldots+q_{(i+1)(s-1)}) \bmod v, q_{(i+2)1}, q_{(i+2)2}, \ldots, q_{(i+2)(s-1)}, v-(q_{(i+2)1}+\ldots+q_{(i+2)(s-1)}) \bmod v]$

Designs constructed through this procedure for $v \le 50$ are presented with (i) $11 \le p_1$ (odd) ≤ 15, $3 \le p_2 \le 7$, and (ii) $9 \le p_1$ (odd) ≤ 15, $8 \le p_2 \le 14$ in Table 3 and 4 respectively.

Table 3. Efficient CSPBRMDs for $v \le 50$, $11 \le p_1$ (odd) ≤ 15, $3 \le p_2 \le 7$.

v	p_1	p_2	**Sets of Shifts**	**Er**	**Es**
24	11	3	[1,2,3,4,5,6,7,8,10,9]+ [13,15,14,11,16,18,19,20,21,22]+[12,12]	0.82	0.96
46	11	3	[37,1,2,3,4,5,6,7,8,9]+ [45,13,14,15,16,17,19,18,20,21]+ [24,26,25,27,28,29,31,22,33,34]+ [35,36,38,11,39,40,41,42,30,44]+[23,23]	0.86	0.98

28	13	3	[22,2,3,4,5,6,7,8,9,10,12,11]+ [15,16,17,18,20,19,21,23,24,27,25,26]+[14,14]	0.84	0.97
32	15	3	[30,26,2,3,4,5,6,7,8,9,10,11,12,13]+ [15,31,17,18,20,19,21,23,22,24,25,1,27,28]+ [16,16]	0.86	0.97
14	11	4	[1,2,3,4,5,6,7,8,9,11]+[10,12,13]	0.52	0.94
36	11	4	[1,2,3,4,5,6,7,8,9,27]+ [10,11,12,14,15,16,18,19,20,32]+ [22,23,24,26,17,28,29,30,33,21]+[31,34,25]	0.77	0.97
16	13	4	[1,2,3,4,5,6,7,8,9,10,11,14]+[13,12,15]	0.83	0.94
42	13	4	[1,2,3,4,5,6,13,8,9,10,11,12]+ [15,29,16,18,17,20,19,21,22,21,24,23]+ [41,40,27,30,28,31,32,33,34,35,38,37]+ [39,26,25]	0.80	0.98
18	15	4	[1,2,3,4,5,6,7,8,9,10,11,12,13,17]+[15,16,14]	0.45	0.95
48	15	4	[40,2,3,4,5,6,7,8,9,10,11,12,13,14]+ [15,17,18,19,20,16,21,22,23,24,25,26,27,28]+ [32,31,34,36,33,37,39,38,1,41,43,42,45,47]+ [44,46,30]	0.89	0.98
26	11	5	[24,2,3,4,5,6,7,8,9,10]+ [12,11,13,14,15,22,18,17,19,20]+[16,23,25,13]	0.69	0.96
48	11	5	[42,2,3,4,5,6,7,8,9,10]+ [11,12,14,13,15,32,17,18,19,20]+ [22,23,24,25,26,27,28,29,30,38]+ [47,34,35,36,37,31,39,40,41,1]+[44,45,46,33]	0.88	0.98
30	13	5	[1,2,3,4,5,6,7,8,9,10,11,24]+ [14,13,16,15,17,18,19,20,22,21,28,12]+ [26,27,23,29]	0.68	0.97
34	15	5	[32,2,3,4,5,6,7,8,9,10,11,12,13,14]+ [15,16,17,18,19,20,22,31,24,23,25,26,27,29]+ [30,21,33,17]	0.72	0.97
16	11	6	[1,2,3,4,5,15,7,8,9,10]+[11,12,13,14,6]	0.74	0.94
38	11	6	[21,1,2,3,4,5,6,7,8,9]+ [11,12,13,14,15,32,18,17,19,20]+ [22,24,23,25,27,26,28,29,30,33]+[31,34,35,36,16]	0.87	0.97
18	13	6	[13,2,3,4,5,6,7,8,9,10,11,12]+[1,14,17,15,16]	0.54	0.95
44	13	6	[1,2,3,4,5,6,7,8,9,10,11,22]+ [40,14,15,16,17,18,19,20,21,12,23,24]+ [43,28,27,29,31,34,30,33,42,36,37,38]+ [39,13,41,35,26]	0.85	0.98
20	15	6	[15,13,11,10,12,2,1,3,4,5,6,8,7,9]+ [16,17,18,19,10]	0.85	0.95
50	15	6	[46,2,3,4,5,6,7,8,9,10,11,12,13,14]+ [35,16,18,17,19,20,21,22,24,23,26,25,27,29]+ [31,45,32,33,34,36,37,15,38,39,40,41,43,42]+ [30,1,47,48,49]	0.80	0.98
28	11	7	[1,2,3,4,5,6,7,8,9,11]+ [27,13,12,14,15,16,17,18,19,20]+ [22,23,24,21,26,10]	0.75	0.97

50	11	7	[46,2,3,4,5,6,7,8,9,10]+ [35,12,13,14,15,16,17,18,19,20]+ [22,23,24,25,26,27,28,29,30,34]+ [33,31,11,36,37,38,49,39,41,42]+ [44,45,1,47,48,40]	0.88	0.98
32	13	7	[1,2,3,4,5,6,7,8,9,10,11,30]+ [14,13,15,16,17,18,19,29,21,22,23,24]+ [26,27,20,28,12,31]	0.88	0.97
36	15	7	[1,2,3,4,5,6,7,8,9,10,11,12,13,17]+ [35,16,14,18,19,20,32,23,22,24,25,26,27,28]+ [29,30,33,34,21,18]	0.73	0.97

Table 4. Efficient CSPBRMDs for $v \leq 50$, $9 \leq p_1$ (odd) ≤ 15, $8 \leq p_2 \leq 14$.

v	p_1	p_2	**Set of Shifts**	**Er**	**Es**
16	9	8	[1,2,3,4,5,6,7,8]+[8,9,10,11,13,14,15]	0.73	0.94
34	9	8	[1,2,3,4,5,6,7,8]+[10,11,12,13,14,15,16,20]+ [19,18,17,21,22,23,24,9]+[26,27,28,29,30,31,33]	0.85	0.97
18	11	8	[1,2,3,4,5,6,7,8,9,9]+[12,11,14,13,15,16,17]	0.58	0.95
40	11	8	[1,2,3,4,6,5,7,8,9,10]+ [12,14,13,15,16,17,18,20,19,20]+ [39,22,23,24,26,27,28,29,30,32]+ [34,33,35,11,37,38,21]	0.88	0.98
20	13	8	[1,2,3,4,5,6,7,12,9,10,10,11]+ [14,13,15,16,17,18,19]	0.62	0.95
46	13	8	[1,2,3,4,5,6,7,8,9,10,11,12]+ [45,15,16,17,18,19,20,22,21,23,24,23]+ [27,38,28,29,30,31,32,33,34,35,37,36]+ [27,39,40,41,42,43,44]	0.89	0.95
22	15	8	[1,2,3,4,5,14,7,8,9,10,11,11,12,13]+ [15,16,17,19,18,20,21]	0.64	0.96
30	11	9	[1,2,3,4,5,11,7,8,9,10]+ [15,12,14,13,16,15,17,19,18,20]+ [22,23,24,25,27,26,28,29]	0.81	0.97
34	13	9	[1,2,3,5,4,7,6,8,10,9,11,12]+ [33,14,15,16,17,17,18,19,20,21,25,23]+ [26,27,28,29,30,31,32,13]	0.71	0.97
38	15	9	[1,2,3,4,5,15,7,8,9,10,11,12,13,14]+ [16,17,18,19,20,21,22,23,24,25,26,27,28,19]+ [30,31,32,33,34,35,36,6]	0.85	0.97
20	11	10	[1,2,3,4,10,6,7,8,9,10]+ [11,12,14,13,15,16,17,18,19]	0.57	0.95
42	11	10	[1,3,2,4,5,6,7,8,9,10]+ [12,14,13,16,15,21,18,19,20,21]+ [40,39,23,24,25,26,27,28,30,32]+ [33,35,34,36,37,38,22,11,17]	0.80	0.98

22	13	10	[12,2,3,4,5,6,7,8,9,10,11,11]+ [13,15,14,16,17,18,19,21,20]	0.64	0.96
48	13	10	[1,2,3,4,5,6,7,8,9,10,11,12]+ [25,14,15,16,17,19,20,21,22,23,24,24]+ [47,28,46,29,31,30,32,33,34,35,36,38]+ [39,40,41,42,43,44,45,27,26]	0.76	0.98
24	15	10	[1,12,2,12,3,4,7,5,6,8,9,11,10,13]+ [14,15,16,18,19,20,21,22,23]	0.88	0.96
36	13	11	[2,1,4,3,5,6,7,9,8,10,11,12]+ [15,17,14,18,16,18,19,20,21,23,22,24]+ [13,26,27,28,29,31,32,33,34,35]	0.84	0.97
40	15	11	[1,2,3,4,6,5,7,8,9,10,11,12,13,14]+ [30,17,18,19,20,20,21,22,23,24,25,26,27,28]+ [16,31,32,33,34,35,36,37,38,39]	0.69	0.98
24	13	12	[1,2,4,3,5,6,7,8,10,11,9,12]+ [12,13,14,15,16,17,19,20,21,22,23]	0.69	0.96
50	13	12	[1,2,3,4,5,6,8,7,9,10,11,12]+ [38,14,15,16,17,18,19,20,21,23,24,25]+ [48,47,46,42,30,31,32,33,34,35,36,37]+ [39,40,41,29,43,44,45,28,26,27,25]	0.83	0.98
26	15	12	[1,2,3,4,5,6,7,8,9,10,11,12,13,13]+ [16,15,18,17,19,20,21,22,23,24,25]	0.68	0.96
42	15	13	[1,3,2,4,5,7,6,8,9,10,11,13,12,14]+ [16,17,20,18,19,22,23,21,24,25,26,28,27,35]+ [29,30,31,32,33,34,36,37,38,39,40,41]	0.83	0,98
28	15	14	[1,2,3,4,5,6,14,8,9,10,11,12,13,14]+ [15,17,16,18,19,21,20,22,23,24,25,27,26]	0.74	0.97

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