**ISSN 0971-0388**

*Aligarh Journal of Statistics,*
Vol. 39 (2019), 31-40

# Characterization of Sine-Skewed von Mises Distribution

Mohammad Ahsanullah and M. Z. Anis*
[Received on May, 2019 and Accepted on May, 2020]

## ABSTRACT

The von Mises distribution is one of the most important distributions in statistics dealing with circular data. However, since circular data is very often asymmetric, we propose the sine-skewed von Mises distribution for modelling such data. In this paper we shall consider some basic properties and characterizations of the sine-skewed von Mises distribution.

## 1. Introduction

The von Mises distribution, also known as the circular normal or the Tikhonov distribution, is one of the principal symmetric distributions on the circle. However, most of the classical models such as von Mises, cardioid and wrapped Cauchy are symmetric-unimodal distributions and rarely applied in practice, since circular data is very often asymmetric and multimodal. Therefore, several new unimodal/multimodal circular distributions capable of modeling symmetry as well as asymmetry have been proposed in the literature. Mention may be made of asymmetric Laplace distribution proposed by Jammalamadaka and Kozubowski (2003), the nonnegative trigonometric sums distribution considered by Fernandez-Duran (2004), the asymmetric version of the von Mises distribution studied by Umbach and Jammalamadaka (2009), and the stereographic extreme–value distribution considered by Phani et al. (2012). Recently Hatami and Alamatsaz (2019) proposed a new transformation in order to construct a large class of new skew-symmetric circular models.

Earlier skew-symmetric distributions were introduced by Azzalini (1985). It was presented as a skewing normal distribution. This technique is used to skew any continuous symmetric distribution. See - for example - in the univariate case, Jammalamadaka (2009). Pewsey (2000) proposed the wrapped skew-normal Arnold and Beaver (2000), Azzalini (2014), Gupta et al. (2002) and Umbach and

*Corresponding Author**: M. Z. Anis, SQC and OR Unit, Indian Statistical Institute, 203, B T Road, Calcutta 700 108, India, Email: zafar@isical.ac.in. Mohammad Ahsanullah, Department of Management Sciences, Rider University, Lawrenceville, New Jersey, USA. Email: ahsan@rider.edu.

distribution as a model for circular data. Abe and Pewsey (2011) considered sine-skewed circular distributions. Yilmaz (2018) proposed the wrapped flexible skew Laplace distribution and studied its properties. In a similar spirit we propose and characterize the sine-skewed von Mises distribution. It should be noted that the von Mises distribution is one of the most important distributions in statistics to deal with the circular data or directional data. It is the circular analog of the normal distribution on a line.

Suppose $f_\theta(\theta)$ is asymmetric circular density of a circular random variable $\theta$ which is symmetric about zero. Then the sine-skewed circular probability density $f_\theta(\theta_1)$is defined as

$$f_\theta(\theta_1) = f_0(\theta_1)(1 + \lambda \sin \theta_1) \tag{1.1}$$

where $-\pi \leq \theta_1 \leq \pi$ and $-1 \leq \lambda \leq 1$. It should be noted that our construction of the sine-skewed circular probability density is slightly different from that of Abe and Pewsey (2011). In this paper$f_0(\theta_1)$will be taken as von Mises distribution with $f_0(\theta_1)$ as

$$f_0(\theta_1) = \frac{e^{k \cos \theta_1}}{2\pi I_0(k)}, \quad -\pi \leq \theta_1 \leq \pi, \quad k > 0 \tag{1.2}$$

where $\theta_1$ is measured in radian, $k$ is a scale factor and $I_0(k)$ is the modified Bessel function of the first kind and order 0. Recall that

$$I_0(k) = \sum_{j=0}^{\infty} \left(\frac{k}{2}\right)^2 \left(\frac{1}{j!}\right)^2;$$

see Abramowitz and Stegan (1970) for details.

Combining (1.1) and (1.2), the sine-skewed von Mises circular distribution with probability density function (pdf) $f_{svM}(\theta_1)$ is given by

$$f_{svM}(\theta_1) = \frac{e^{k \cos \theta_1}}{2\pi I_0(k)}(1 + \lambda \sin \theta_1), \quad -\pi \leq \theta_1 \leq \pi; -1 \leq \lambda \leq 1 \text{ and } k \geq 0. \tag{1.3}$$

We will denote the random variable $\theta$ with pdf as given in (1.3) above as $svM(\theta, \lambda, k)$. It should be noted that the distribution in (1.3) is different from the asymmetric generalized von Mises (AGvM) distribution proposed by Kim and SenGupta (2013). While inferential issues with respect to this distribution has been discussed in the literature, see for example, Ley and Verdebout (2014) and Abe and Ley (2017); to the best of our knowledge no characterization results are available. We aim to fill this gap. The paper is organized as follows. Section 2 gives the main results; while Section 3 presents two characterizations of the sine-skewed von Mises distribution. Finally Section 4 concludes the paper.

## 2. Main Results

It is known that

$$e^{k\cos\theta} = I_0(k) + 2\sum_{j=1}^{\infty} I_j(k)\cos j\theta, \tag{2.1}$$

where $I_j(k)$ is the Bessel function of order $j$.

Using (2.1) the pdf of the sine-skewed von Mises distribution can be written as

$$f_{svM}(\theta_1,\lambda,k) = \frac{1}{2\pi I_0(k)}\left(I_0(k) + 2\sum_{j=1}^{\infty} I_j(k)\cos j\theta_1\right)(1+\lambda\sin\theta_1), \tag{2.2}$$

where$-\pi \le \theta_1 \le \pi; -1 \le \lambda \le 1$ and $k \ge 0$.

The cumulative distribution function (cdf) of sine-skewed von Mises distribution with the pdf as given in (1.3) can be written as

$$F_{svM}(\theta_1) = \frac{1}{2\pi}\left\{(\pi+\theta_1) + 2\sum_{j=1}^{\infty}\frac{I_j(k)}{j}\sin(j\theta_1)\right\} + \frac{\lambda}{2\pi I_0(k)}\left(e^{-k} - e^{k\cos\theta_1}\right) \tag{2.3}$$

where

$$I_j(k) = \left(\frac{k}{2}\right)^j \sum_{i=0}^{\infty}\left\{\left(\frac{k}{2}\right)^{2i}\left(\frac{1}{i!\,(j+1)!}\right)\right\}, \quad -\pi \le \theta_1 \le \pi,$$
$$-1 \le \lambda \le 1, \text{and } k \ge 0.$$

We have $k = \frac{1}{2}\ln\frac{f(0)}{f(\pi)}$.

The mode is obtained by solving the equation $\frac{d}{d\theta} f_{svM}(\theta,\lambda,k) = 0$.

Now,

$$\frac{d}{d\theta}\left(\frac{e^{k\cos\theta}}{2\pi I_0(k)}(1+\lambda\sin\theta)\right) = 0,$$

implies

$$k\lambda(\sin\theta)^2 - \lambda\cos\theta + k\sin\theta = 0. \tag{2.4}$$

Thus, the mode is a solution to the equation in (2.4) above. The distribution is unimodal. Furthermore, if for any fixed $k$ the mode of $F_{svM}(\theta, \lambda, k)$ is at $\theta_0$, then the mode of $F_{svM}(\theta, -\lambda, k)$will be at $-\theta_0$. The following Table gives the mode of $F_{svM}(\theta, \lambda, k)$ for $k = 1, 2$ and $10$, and $\lambda = 0.1$ to $0.9$.

**Table 2.1:** Mode of $svM(\theta, \lambda, k)$.

| $k$ ↓ λ→ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.0987 | 0.1904 | 0.2709 | 0.3393 | 0.3968 | 0.4450 | 0.4855 | 0.5199 | 0.5494 |
| 2 | 0.0497 | 0.0978 | 0.1429 | 0.1842 | 0.2216 | 0.2550 | 0.2840 | 0.3109 | 0.3314 |
| 10 | 0.0010 | 0.0199 | 0.0297 | 0.0394 | 0.0488 | 0.0479 | 0.0668 | 0.0753 | 0.0835 |

From the table, we find that for a fixed $k$, the mode increases with $\lambda$ and for a fixed λ, the mode decreases with $k$.
If the random variable $\theta$ has the pdf as given in (1.3), then

$$
\begin{aligned}
E(\theta) &= \int_{-\pi}^{\pi} \frac{\theta_1 e^{k\cos\theta_1}}{2\pi I_0(k)}(1 + \lambda\sin\theta_1)d\theta_1 \\
&= \int_{-\pi}^{\pi} \frac{\theta_1 e^{k\cos\theta_1}}{2\pi I_0(k)} d\theta_1 + \lambda \int_{-\pi}^{\pi} \frac{\theta_1 e^{k\cos\theta_1}}{2\pi I_0(k)} \sin\theta_1 \, d\theta_1 \\
&= \int_{-\pi}^{\pi} \frac{\theta_1}{2\pi I_0(k)}\left(I_0(k) + 2\sum_{j=1}^{\infty} I_j(k)\cos j\theta_1\right)d\theta_1 + \frac{\lambda}{k}\left(-\theta_1 \frac{e^{k\cos\theta_1}}{2\pi I_0(k)}\right)\Big|_{-\pi}^{\pi} + \frac{\lambda}{k}\int_{-\pi}^{\pi}\frac{e^{k\cos\theta_1}}{2\pi I_0(k)}d\theta_1 \\
&= \frac{\lambda}{2\pi I_0(k)}\left(-\frac{2\pi}{k}e^{-k} + \frac{2\pi}{k}I_0(k)\right) \\
&= -\frac{1}{k}\frac{\lambda}{I_0(k)}\left(e^{-k} - I_0(k)\right) \\
&= \frac{\lambda}{k}\left(1 - \frac{e^{-k}}{I_0(k)}\right)
\end{aligned}
$$

We define the$p^{th}$ circular moment $\varphi_p$ of random variable $\theta$ with pdf$f(\theta)$, with$-\pi \le \theta \le \pi$ as$\varphi_p = \int_{-\pi}^{\pi} e^{ip\theta} f(\theta)d\theta$.

It is known (see Jammalamadaka and SenGupta (2001)) that if $\varphi_p^*(k)$ is the $p^{th}$ circular moment of von Mises distribution with pdf as given in (1.2) and $\varphi_p(k, \lambda)$is the$p^{th}$circular moment of the sine-skewed von Mises distribution with pdf as given in (1.3), then $\varphi_p(k, \lambda)$ and $\varphi_p^*(k)$ are related. Specifically, we have

$$\varphi_p(k, \lambda) = \varphi_p^*(k) + \frac{i\lambda}{2}\{\varphi_{p-1}^*(k) - \varphi_{p+1}^*(k)\}.$$

Here

$\varphi_p^*(k) = \int_{-\pi}^{\pi} e^{ip\theta} \frac{\theta e^{k\cos\theta}}{2\pi I_0(k)} d\theta$ and $\varphi_p(k,\lambda) = \int_{-\pi}^{\pi} e^{ip\theta} \frac{\theta e^{k\cos\theta}}{2\pi I_0(k)} (1+\lambda\sin\theta) d\theta.$

## 3. Characterizations

We shall now derive the two characterizations of the sine-skewed von Mises distribution. We will need the following two lemmas for the characterization of the sine-skewed von Mises distribution by truncated moment.

**Assumption $\boldsymbol{\mathcal{A}}$**
Let $\theta$ be an absolutely continuous random variable with cdf $F(\theta_1)$and pdf $f(\theta_1)$.Define $\alpha = \text{in f}\{\theta_1 | F(\theta_1) > 0\}$ and $\beta = \sup\{\theta_1 | F(\theta_1) < 1\}$. Assume that $E(\theta)$ exists and $f(\theta)$ is differentiable.

**Lemma 3.1:** If $E(\theta | \theta \le \theta_1) = g(\theta_1)\frac{f(\theta_1)}{F(\theta_1)}$, where $g(\theta_1)$is a continuous differentiablefunction in $\alpha \le \theta_1 \le \beta$, then

$$f(\theta) = c\exp\left(\int \frac{\theta - g'(\theta)d\theta}{g(\theta)}\right),$$

where $c$ is determined by the condition $\int_\alpha^\beta f(\theta_1)d\theta_1 = 1$.

**Lemma 3.2:** Under the Assumption $\boldsymbol{\mathcal{A}}$, if $E(\theta | \theta \ge \theta_1) = h(\theta_1)\frac{f(\theta_1)}{1-F(\theta_1)}$, where $h(\theta)$ is a continuous differentiable function in $\alpha \le \theta \le \beta$, then

$$f(\theta) = c\exp\left(-\int \frac{\theta + h'(\theta)}{h(\theta)} d\theta\right),$$

where $c$ is determined by the condition $\int_\alpha^\beta f(\theta)\, d\theta = 1$.

The proofs of these two lemmas are given in Shakil et al. (2018) and hence omitted.

The following two theorems give the characterizations of sine-skewed von Mises distribution by truncated first moment.

**Theorem 3.1:** Suppose that the random variable $\theta$ satisfies the conditions given in Assumption $\boldsymbol{\mathcal{A}}$, with pdf $f(\theta)$, cdf $F(\theta), \alpha = -\pi$ and $\beta = \pi$. Then $E(\theta | \theta \le \theta_1) = g(\theta_1)\tau(\theta_1)$, where

$$\tau(\theta_1) = \frac{f(\theta_1)}{F(\theta_1)},$$

$$g(\theta_1) = \frac{2\pi I_0(k)p(\theta_1)}{e^{k\cos\theta_1}(1+\lambda\sin\theta_1)}$$

and

$$p(\theta_1) = \frac{\theta_1^2 - \pi^2}{4\pi} + \frac{1}{\pi I_0(k)}\sum_{j=1}^{\infty} I_j(k)\left\{\frac{1}{j}\theta_1\sin\theta_1 + \frac{1}{j^2}\cos j\theta_1 - \frac{(-1)^j}{j^2}\right\}$$
$$- \frac{\lambda}{2k\pi I_0(k)}\left(\pi e^{-k} + \theta_1 e^{k\cos\theta_1}\right) + \frac{\lambda(\theta_1+\pi)}{2k\pi}$$
$$+ \frac{\lambda}{k\pi I_0(k)}\sum_{j=1}^{\infty} I_j(k)\frac{1}{j}\sin j\theta_1$$

if and only if $\theta$ has the sine-skewed von Mises distribution with pdf $f_\theta(\theta_1)$ as

$$f_\theta(\theta_1) = \frac{e^{k\cos\theta_1}}{2\pi I_0(k)}(1+\lambda\sin\theta_1),$$

$-\pi \le \theta_1 \le \pi, -1 \le \lambda \le 1$ and $k$is any real number.

**Proof:** Suppose that

$$f(\theta) = \frac{e^{k\cos\theta}}{2\pi I_0(k)}(1+\lambda\sin\theta),$$

then

$$g(\theta_1)f(\theta_1) \;=\; \int_{-\pi}^{\theta_1} \theta\frac{e^{k\cos\theta}}{2\pi I_0(k)}(1+\lambda\sin\theta)d\theta$$

$$= \int_{-\pi}^{\theta_1} \frac{\theta e^{k\cos\theta}}{2\pi I_0(k)}d\theta + + \lambda\int_{-\pi}^{\theta_1}\frac{\theta e^{k\cos\theta}}{2\pi I_0(k)}\sin\theta\, d\theta$$

$$= \int_{-\pi}^{\theta_1}\frac{\theta}{2\pi I_0(k)}\left\{I_0(k) + 2\sum_{j=1}^{\infty} I_j(k)\cos j\theta\, d\theta\right\} + \lambda\left(\frac{-\theta e^{k\cos\theta}}{2k\pi I_0(k)}\right)\Big|_{-\pi}^{\theta_1}$$
$$+ \lambda\int_{-\pi}^{\theta_1}\frac{e^{k\cos\theta}}{2k\pi I_0(k)}d\theta$$

$$= \frac{\theta_1^2-\pi^2}{4\pi}+\frac{1}{\pi I_0(k)}\sum_{j=1}^{\infty} I_j(k)\left\{\frac{1}{j}\theta_1\sin\theta_1+\frac{1}{j^2}\cos j\theta_1-\frac{(-1)^j}{j^2}\right\}+\lambda\frac{(\theta_1+\pi)}{2k\pi}$$
$$+\frac{\lambda}{k\pi I_0(k)}\sum_{j=1}^{\infty} I_j(k)\frac{1}{j}\sin j\theta_1$$
$$-\frac{\lambda}{2k\pi I_0(k)}\left(\pi e^{-k}+\theta_1 e^{k\cos\theta_1}\right)$$

$= p(\theta_1)$, say.

Thus,

$$g(\theta_1)=\frac{2\pi I_0(k)p(\theta_1)}{e^{k\cos\theta_1}(1+\lambda\sin\theta_1)}.$$

Next, suppose that

$$g(\theta_1)=\frac{2\pi I_0(k)p(\theta_1)}{e^{k\cos\theta_1}(1+\lambda\sin\theta_1)}.$$

Then

$$g'(\theta_1)=\theta_1-\frac{2\pi I_0(k)p(\theta_1)}{e^{k\cos\theta_1}(1+\lambda\sin\theta_1)}\left\{-k\sin\theta_1+\frac{\lambda\cos\theta_1}{1+\lambda\sin\theta_1}\right\}$$
$$=\theta_1-g(\theta_1)\left\{-k\sin\theta_1+\frac{\lambda\cos\theta_1}{1+\lambda\sin\theta_1}\right\}.$$

Hence,

$$\frac{\theta_1-g'(\theta_1)}{g(\theta_1)}=-k\sin\theta_1+\frac{\lambda\cos\theta_1}{1+\lambda\sin\theta_1}.$$

By Lemma 2.1

$$\frac{f'(\theta_1)}{f(\theta_1)}=-k\sin\theta_1+\frac{\lambda\cos\theta_1}{1+\lambda\sin\theta_1}$$

On integrating the above equation with respect to $\theta_1$, we obtain

$$f(\theta)=ce^{k\cos\theta}\{1+\lambda\sin\theta\}.$$

Using the condition $\int_{-\pi}^{\pi} f(\theta)d\theta=1$, we obtain

$$f(\theta)=\frac{e^{k\cos\theta}}{2\pi I_0(k)}\{1+\lambda\sin\theta\}.$$

**Theorem 3.2:** Suppose that the random variable $\theta$ satisfies the conditions of the Assumption $\boldsymbol{\mathcal{A}}$ with $\alpha=-\pi$ and $\beta=\pi$. Then

$$E(\theta|\theta\geq\theta_1)=\frac{2\pi I_0(k)h(\theta_1)}{e^{k\cos\theta_1}(1+\lambda\sin\theta_1)}r(\theta_1),$$

where

$$r(\theta_1) = \frac{f(\theta_1)}{1 - F(\theta_1)}$$

and

$$h(\theta_1) = \frac{2\pi I_0(k)\{E(\theta) - p(\theta_1)\}}{e^{k\cos\theta_1}(1 + \lambda\sin\theta_1)}$$

if and only if

$$f(\theta) = \frac{e^{k\cos\theta}}{2\pi I_0(k)}\{1 + \lambda\sin\theta\}.$$

**Proof:** If the pdf of the random variable $\theta$ is $f(\theta) = \frac{e^{k\cos\theta}}{2\pi I_0(k)}\{1 + \lambda\sin\theta\}$, then

$$f(\theta_1)h(\theta_1) = \int_{\theta_1}^{\pi} \theta f(\theta)d\theta = E(\theta) - p(\theta_1).$$

Thus

$$h(\theta_1) = \frac{2\pi I_0(k)\{E(\theta) - p(\theta_1)\}}{e^{k\cos\theta_1}(1 + \lambda\sin\theta_1)}.$$

Next, suppose that

$$h(\theta_1) = \frac{2\pi I_0(k)\{E(\theta) - p(\theta_1)\}}{e^{k\cos\theta_1}(1 + \lambda\sin\theta_1)},$$

then

$$\begin{aligned} h'(\theta_1) &= -\theta_1 - \frac{2\pi I_0(k)\{E(\theta) - p(\theta_1)\}}{e^{k\cos\theta_1}(1 + \lambda\sin\theta_1)}\left(-k\sin\theta_1 + \frac{\lambda\cos\theta_1}{1 + \lambda\sin\theta_1}\right) \\ &= -\theta_1 - h(\theta_1)\left(-k\sin\theta_1 + \frac{\lambda\cos\theta_1}{1 + \lambda\sin\theta_1}\right). \end{aligned}$$

Hence,

$$-\frac{\theta_1 + h'(\theta_1)}{h(\theta_1)} = -k\sin\theta_1 + \frac{\lambda\cos\theta_1}{1 + \lambda\sin\theta_1}.$$

Thus by Lemma 2.2, we have

$$\frac{f'(\theta_1)}{f(\theta_1)} = -k\sin\theta_1 + \frac{\lambda\cos\theta_1}{1 + \lambda\sin\theta_1}.$$

On integrating the above equation with respect to $\theta_1$, we obtain

$$f(\theta) = ce^{k\cos\theta}(1 + \lambda\sin\theta).$$

Using the condition $\int_{-\pi}^{\pi} f(\theta)d\theta = 1$, we get

$$f(\theta) = \frac{e^{k\cos\theta}}{2\pi I_0(k)}(1 + \lambda\sin\theta).$$

## 4. Concluding Remarks

The characterization of probability distribution plays an important role in probability, statistics and other related fields. Before a particular probability distribution model is applied to fit data in the real world, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by characterization. A probability distribution can be characterized by various methods. The characterization of probability distribution by truncated moments is one such method. It is hoped that the findings of the paper will useful for researchers in probability, statistics and other sciences.

## Acknowledgement

The authors thank the referees for helpful suggestions that improved the presentation of the paper.

## References

Abe, T. and Ley, C. (2017): A tractable, parsimonious and flexible model for cylindrical data, with applications. *Econometrics and Statistics*. **4**, 91-104.

Abe, T. and Pewsey, A. (2011): Sine-skewed circular distributions. *Statistical Papers*, **52**, 683-707.

Abramowitz, M. and Stegan, I. A. (1970): *Handbook of Mathematical Functions.* Dover Publications, Inc. New York, USA.

Arnold, B.C. and Bever, R. J. (2000): Hidden truncated models. *Sankhya, Series A*, **62**, 22-35.

Azzalini, A. (1985): A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics.* **12**,121-178

Azzalini, A. (2014): *The Skew Normal and Related Families*. Cambridge University Press, Cambridge, U.K.

Fernandez-Duran, J. (2004): Circular distributions based on nonnegative trigonometric sums. *Biometrics*. **60** (2); 499–503.

Gupta, A. K., Chang, F. C. and Huang, W.J. (2002): Some skew symmetric models. *Random Operators and Stochastic Equations.* **10**,133-140.

Hatami, M. and Alamatsaz, M. H. (2019): Skew-symmetric circular distributions and their structural properties. *Indian Journal of Pure and Applied Mathematics*. **50**; 953–969.

Jammalamadaka, S. R. and T. Kozubowski (2003): A new family of circular models: The wrapped Laplace distributions. *Advances and Applications in Statistics.* **3** (1); 77–103.

Jammalamadaka, S. Rao and SenGupta, A. (2002): *Topics in Circular Statistics*. World Scientific, Singapore.

Kim, S. and SenGupta, A. (2013): A three-parameter generalized von Mises distribution. *Statistical Papers.* **54**, 685-693.

Ley, C. and Verdebout, T. (2014): Simple optimal tests for circular reflective symmetry about a specified median direction. *Statistica Sinica,* **24** (3), 1319–1339.

Pewsey, A. (2000): The wrapped skew-normal distribution on the circle. *Communications in Statistics - Theory and Methods*. **29** (11); 2459-2472.

Phani, Y.; Girija, S; and Rao, A. D. (2012): Circular model induced by inverse stereographic projection on extreme-value distribution. *Engineering Science and Technology*. **2** (5); 881–888.

Shakil, M.; Ahsanullah, M. and Kibria, B.M. (2018): On the characterization of Chen's two-parameter exponential power life-testing distribution. *Journal of Statistical Theory and Applications.* **17** (3), 393-407.

Umbach, D. and Jammaladaka, S. R. (2009): Building symmetric in circular models. *Statistics and Probability Letters.* **79**; 659-663.

Yilmaz, A. (2018): Wrapped flexible skew Laplace distribution. *Istatistik.* **11**(3); 53-64.