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Tests for the Shape Parameter of Family of Lifetime Distributions under Progressive Censoring

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ABSTRACT

This article presents method of constructing tests for the hypothesis on shape parameter β of the family of lifetime distributions under progressive censoring. We derive three test statistics $Q_{(sd)}$, F and Q_1. The performance of these statistics, in terms of power, is studied through simulation for different values of shape parameter of Gompertz distribution, Burr-XII distribution, bathtub distribution, and Weibull distribution. For these distributions, it is concluded that either a test $Q_{(sd)}$, or test Q_1 performs better than test F in case of large samples whereas test F performs better than other tests $Q_{(sd)}$ and Q_1 in case of small samples when the shape parameter is not equal to the value under null hypothesis.

1. Introduction

A two parameter family of lifetime distributions plays an important role in data analysis in the field of biostatistics, when the data are censored. Gompertz distribution, bathtub distribution, Weibull distribution and, Burr-XII distribution, are some of the distributions used to analyze lifetime data. The Weibull distribution is quite popular as a life testing model due to various shapes of the probability density function and has the advantage of having a closed form of cumulative distribution function. This model is quite flexible and has been used very effectively for analyzing lifetime data. The Gompertz distribution was introduced by Gompertz (1825). This distribution plays an important role in modeling human mortality and fitting actuarial tables. In the area of lifetime analysis, the two-parameter Burr-XII distribution with uni-modal failure rate function will be more appropriate when the failure factor of the product is fatigue or aging. The bathtub distribution was introduced by Chen (2000). This distribution is widely used to fit well to the real life data which have bathtub shaped.

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Censoring is very common in studies or experiments involving lifetime data since the experimenter may have to terminate the test before all items have failed due to time limit or economic reason. The two most common forms of censoring schemes are type-I and type-II. In type-I censoring scheme, the experimental time is fixed, but the number of observed failures is a random variable. In type-II censoring scheme, the number of observed failures is fixed, but the experimental time is a random variable. The advantage of such a scheme over type-I censoring is that one knows exactly how many failures will be observed ahead of time; however, the time to test possibly unbounded, and could be on average much larger than in type-I censoring. Under type-II right censoring scheme n units are placed on test at time zero, experiment is terminated after first m (≤ n) failure items are observed. Principal advantage of such censoring is that it may take much less time for the first m failures of n item to occur as compared to observing failure of all n items.

Among the different censoring schemes, the progressive censoring scheme has received considerable attention in the last few years. Under this scheme n units are placed on test at time zero, the test is continued till m failures are observed. When the first failure is observed, r_1 of the surviving units are randomly selected and removed. At the second observed failure, r_2 of the surviving units are randomly selected and removed. This experiment terminates at the time when the m^{th} failure is observed and the remaining $r_m = \left(n - m - \sum_{i=1}^{m-1} r_i\right)$ surviving units are all removed. Balakrishnan (2007) provided details on progressive censoring scheme and on its different applications. For some of earlier research under progressive censored data, we refer to Balakrishnan and Aggarwala (2000), Balakrishnan et al. (2001), and Balakrishnan and Cramer (2008). Shanubhogue and Muralidharan (2004) have obtained conditional test for shape parameter of Weibull distribution based on quadratic form under type-II censored scheme. The paper does not consider hypotheses testing problem in the case of progressively type-II censoring scheme because of the complexity of finding distribution of test statistic obtained. In this article we propose a computationally simple method of constructing tests for testing shape parameter of family of lifetime distributions as $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$ using progressive type-II censored sample.

In this work, we derived conditional joint distribution of generalized spacings $Z_1, Z_2, \ldots, Z_m$ given their sum T = t, using joint distribution of progressive type-II censored sample from the family of distributions. We suggest a test based on a quadratic form constructed using conditional means and covariances. To arrive at an unconditional setup we modify the statistic Q by replacing t by T. Further, we constructed two more test statistics F and Q_1, and the performances of these are compared based on simulated powers.

The rest of this paper is organized as follows. In Section 2, we introduce a new model of family of distributions and shows that few distributions already proposed in the literature are members of the family. Section 3 presents derivations of the test statistics and test procedure. Simulation studies for testing shape parameter of the family of distributions are done, simulated cut-off points and the powers of the tests are discussed in Section 4. The last section deals with conclusions regarding the usefulness of the tests.

2. Model

In this section, we introduce new model for family of lifetime distributions and shown that few distributions already proposed in the literature are members of the family. Let Y be a random variable having distribution belonging to a family of distributions. The probability distribution function (pdf) and cumulative distribution function (cdf) of family of distributions is as

$$f(x;\theta,\beta) = \frac{a(x,\beta)[h(\theta)]^{\,d(x,\beta)}}{g(\theta)}; \;\; x > 0, \theta, \beta > 0$$

$$F(x;\theta,\beta) = \left\{1 - [h(\theta)]^{\,d(x,\beta)}\right\} \qquad (2.1)$$

where, $g(\theta) = \int_0^\infty a(x,\beta)[h(\theta)]^{d(x,\beta)}\,dx$, and $a(x,\beta)$ is the derivative of $d(x,\beta)$. Some of the members of the above family are listed as below.

Case I: Gompertz Distribution

Consider $a(x,\beta) = e^{\beta x}$, $d(x,\beta) = \frac{e^{\beta x}-1}{\beta}$, $h(\theta) = e^{-\theta}$ and $g(\theta) = \frac{1}{\theta}$, then the density function can be written as,

$$f(x;\theta,\beta) = \theta\, e^{\beta x}\, e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}; \;\; x > 0, \;\; \theta, \beta > 0$$

The cdf is

$$F(x;\theta,\beta) = [1 - e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}] \qquad (2.2)$$

These are the pdf and cdf of two parameter Gompertz distribution. Here, β and θ are shape and scale parameters respectively. When $\beta = 0$, Gompertz distribution reduces to exponential distribution. Hence, test for β is of interest.

Case II: Burr-XII Distribution

For $a(x,\beta) = \frac{\beta x^{\beta-1}}{[1+x^\beta]}$, $d(x,\beta) = \log\left[1 + x^\beta\right]$, $h(\theta) = e^{-\theta}$ and $g(\theta) = \frac{1}{\theta}$ then the density function can be written as,

$$f(x;\theta,\beta) = \frac{\theta\beta x^{\beta-1}}{[1+x^\beta]}\, e^{-\theta\{\log[1+x^\beta]\}}; \;\; x > 0, \;\; \theta, \beta > 0$$

The above density function can be simplified as,

$$f(x;\theta,\beta) = \theta\beta x^{\beta-1}\left[1+x^{\beta}\right]^{-(\theta+1)};\ \ x>0,\ \ \theta,\beta>0$$

The cdf is

$$F(x;\theta,\beta) = [1-(\,1+x^{\beta})^{-\theta}] \tag{2.3}$$

These are the pdf and cdf of two parameter Burr-XII distribution. Here, β and θ are shape and scale parameters respectively. When $\beta=1$, Burr-XII distribution reduces to Pareto distribution. Hence, test for β is of interest.

Case III: Weibull Distribution

For $a(x,\beta)=\beta x^{\beta-1},\ d(x,\beta)=x^{\beta},\ \ h(\theta)=\ e^{-\,\theta}$ and $g(\theta)=\frac{1}{\theta},$ then the density function can be written as,

$$f(x;\theta,\beta) = \theta\beta x^{\beta-1}\,e^{-\,\theta x^{\beta}};\ \ x>0,\ \ \theta,\beta>0;$$

The cdf is

$$F(x;\theta,\beta) = (1-e^{-\,\theta x^{\beta}}) \tag{2.4}$$

These are the pdf and cdf of two parameter Weibull distribution. Here, $\beta$and θ are shape and scale parameters respectively. For $\beta=1$ Weibull distribution reduces to exponential distribution. Hence, test for β is of interest.

Case IV: Bathtub Distribution

If $a(x,\beta)=\beta\,e^{x^{\beta}}x^{\beta-1}\,,\ d(x,\beta)=\ e^{x^{\beta}}-1,\ h(\theta)=\ e^{-\theta}$ and $g(\theta)=\frac{1}{\theta},$ then the density function can be written as,

$$f(x;\theta,\beta) = \theta\beta\,e^{x^{\beta}}x^{\beta-1}\,e^{-\theta(\,e^{x^{\beta}}-1)};\ \ x>0,\ \ \theta,\beta>0$$

and the cdf is

$$F(x;\theta,\beta) = (1-\ e^{-\theta(\,e^{x^{\beta}}-1)}) \tag{2.5}$$

These are the pdf and cdf of two parameter bathtub distribution. Here, $\beta$and θ are shape and scale parameters respectively.

3. Derivation of the test

In this section we derive the test statistics using progressive type-II censored sample. Let $(X_1,r_1),\left(X_{2,}r_2\right),\dots,\left(X_{m,}r_m\right)$, where $X_1<X_2<\cdots<X_m$ and $m\le n$, denote a progressive type-II censored sample based on *n* experimental units

whose life times follow the generalised inverted family of distributions as given in (2.1), With pre-determined number of removals, say, $r_{1,}\ r_{2\,,}\ \ldots, r_m$ the joint probability density function for $X_1,\ X_2, \ldots, X_m,\ m \le n$ is given by(see Cohen, A.C. (1963))

$$f_{\underline{X}}(\underline{x};\theta) = C \prod_{i=1}^{m} f(x_i;\theta)\,[1 - F(x_i,\theta)]^{r_i}$$

$$f_{\underline{\mathrm{X}}}(\underline{\mathrm{x}};\theta,\beta) = C\ \frac{\prod_{i=1}^{m} a(x_i,\beta)\ [h(\theta)]^{[\sum_{i=1}^{m}(1+r_i)d(x_i,\beta)]}}{[g(\theta)]^m},\ m \le n,\ x > 0, \theta, \beta > 0 \qquad (3.1)$$

where, $C = n\,(n - r_1 - 1)(n - r_1 - r_2 - 2)\ \ldots\ (n\ - r_1 - r_2 \ldots - r_{m-1} - m + 1), 0 \le r_i \le (n - m - r_1 - r_2 \ldots - r_{i-1}),\ i = 1, 2, \ldots, m$ and $a(x,\beta)$ is the derivative of $d(x,\beta)$.

Let$Y_i = d(x_i,\beta)$, where $\beta > 0$ is known and $Y_1 <\ Y_2 < \cdots < Y_m$. Making the transformation,

$$Z_i = (n - i + 1 - r_1 - r_2 \ldots . - r_{i-1})[\,Y_i - Y_{i-1}], \quad i = 1, 2, 3, \ldots \ldots, m \qquad (3.2)$$

The Jacobean of transformation is

$$|J| = \frac{1}{n\,(n - r_1 - 1)(n - r_1 - r_2 - 2)\ \ldots\ (n\ - r_1 - r_2 \ldots - r_{m-1} - m + 1)\prod_{i=1}^{m} a(x_i,\ \beta)}$$

We have= $\sum_{i=1}^{m}(\,1 + r_i)d(x_i,\beta)$. Hence from (3.1) the joint distribution of Z_i's is

$$f_{\underline{Z}}(\underline{z};\theta) = \frac{[h(\theta)]^{\sum_{i=1}^{m} z_i}}{[g(\theta)]^m} \qquad (3.3)$$

$$= \prod_{i=1}^{m} \frac{[h(\theta)]^{z_i}}{g(\theta)}$$

This proves that generalized spacings $Z_1,\ Z_2, \ldots, Z_m$are i.i.d. random variables.

Let $T = \sum_{i=1}^{m} Z_i = \sum_{i=1}^{m}(\,1 + r_i)d(x_i,\beta)$. Then the joint pdf of $(Z_1, Z_2, \ldots, Z_{m-1},\ T\,)$ is obtained as

$$f(z_1, z_2, \ldots, z_{m-1}, t;\theta) = \frac{[h(\theta)]^t}{[g(\theta)]^m}\ ; \sum_{i=1}^{m-1} z_i < t < \infty, z_i > 0, i = 1, 2, \ldots, m-1 \qquad (3.4)$$

Integrating out other variables, we obtain the joint distribution of (Z_i, T) as

$$f_{Z_i,T}(z_i,t;\theta) = \frac{[h(\theta)]^t}{[g(\theta)]^m (m-2)!}(t - z_i)^{m-2};\ 0 < z_i < t < \infty, \qquad (3.5)$$

Similarly, the joint distribution of (Z_i, Z_j, T) is

$$f_{Z_i,Z_j,T}\left(z_i, z_j, t; \theta\right) = \frac{[h(\theta)]^t}{[g(\theta)]^m (m-3)!} (t - z_i - z_j)^{m-3};\ 0 < z_i + z_j < t < \infty. \tag{3.6}$$

and the marginal pdf of T is

$$f_T(t;\theta) = \frac{[h(\theta)]^t}{[g(\theta)]^m (m-1)!} t^{m-1}\ ; \quad t > 0, \tag{3.7}$$

Now, from (3.4) and (3.7), the conditional pdf of $Z_1, Z_2, \dots, Z_{m-1}$ given $\mathrm{T} = \mathrm{t}$ is obtained as

$$f_{\underline{Z}|T}(\underline{z}|t) = \frac{(m-1)!}{t^{m-1}};\ 0 < z_1 \leq z_2 \leq \cdots \leq z_{m-1} < t \tag{3.8}$$

It is seen that this conditional density does not depend on the nuisance parameter θ. Thus $T = \sum_{i=1}^{m}(1 + r_i) d(x_i, \beta)$ is the sufficient statistic for θ. To develop test for the parameter of interest β in the presence of nuisance parameter θ, we construct a quadratic form using the conditional distribution (3.8) as

$$Q = (\underline{Z} - \underline{\mu})' \textstyle\sum^{-1} (\underline{Z} - \underline{\mu}) \tag{3.9}$$

where, $\underline{Z}' = (Z_1, Z_2, \dots, Z_{m-1}), \underline{\mu}' = (\mu_1, \mu_2, \dots, \mu_{m-1});\ \mu_i = E(Z_i|T = t)$ and

$\sum = ((\sigma_{ij}))$ is the variance-covariance matrix of $(\underline{Z}\big|\mathrm{T} = \mathrm{t})$. We have the

conditional distributions as

$$f_{Z_i|T}(z_i|t) = \frac{(m-1)}{t}\left(1 - \frac{z_i}{t}\right)^{m-2};\ 0 < z_i < t \tag{3.10}$$

and $f_{Z_i,Z_j|T}\left(z_i, z_j\middle|t\right) = \frac{(m-1)(m-2)}{t^2}\left(1 - \frac{z_i}{t} - \frac{z_j}{t}\right)^{m-3};\ 0 < z_i + z_j < t.$ (3.11)

Using above distributions, we have, $\mathrm{E}(Z_i|\mathrm{T} = \mathrm{t}) = \frac{t}{m}$, $\mathrm{E}(Z_i, Z_j|\mathrm{T} = \mathrm{t}) = \frac{t^2}{m(m+1)}$,

$\mathrm{E}(Z_i^2|\mathrm{T} = \mathrm{t}) = \frac{2t^2}{m(m+1)}$, $\mathrm{E}(Z_i^4|\mathrm{T} = \mathrm{t}) = \frac{24t^4}{m(m+1)(m+2)(m+3)}$ and

$$\mathrm{E}(Z_i^2, Z_j^2|\mathrm{T} = \mathrm{t}) = \frac{4t^4}{m(m+1)(m+2)(m+3)} \tag{3.12}$$

Therefore, we have $\underline{\mu} = E(\underline{Z}|\mathrm{T}=\mathrm{t}) = \frac{t}{m}E_{m-1}$ and the conditional covariances are as $\sigma_{ii} = \frac{t^2(m-1)}{m^2(m+1)}$, $\sigma_{\mathrm{ij}} = -\frac{t^2}{m^2(m+1)}$ and the variance-covariance matrix of $(\underline{Z}|\mathrm{T}=\mathrm{t})$ is

$$\sum = \frac{t^2}{m(m+1)}\begin{bmatrix} \left(1-\frac{1}{m}\right) & -\frac{1}{m} & -\frac{1}{m} & & -\frac{1}{m} \\ -\frac{1}{m} & \left(1-\frac{1}{m}\right) & -\frac{1}{m} & \cdots & -\frac{1}{m} \\ -\frac{1}{m} & -\frac{1}{m} & \left(1-\frac{1}{m}\right) & & -\frac{1}{m} \\ & \vdots & & \ddots & \vdots \\ -\frac{1}{m} & -\frac{1}{m} & -\frac{1}{m} & \cdots & \left(1-\frac{1}{m}\right) \end{bmatrix}$$

$$= \frac{t^2}{m(m+1)}\left[I_{m-1} - \frac{E_{(m-1),(m-1)}}{m}\right]$$

and $\sum^{-1} = \frac{m(m+1)}{t^2}\left[I_{m-1} + E_{(m-1),(m-1)}\right]$

Substituting the values of $\underline{\mu}$ and $\sum^{-1}$ in the expression for Q we get,

$$Q = (\underline{Z} - \underline{\mu})' \sum^{-1} (\underline{Z} - \underline{\mu})$$

$$= \frac{m(m+1)}{t^2}\left(\underline{Z} - \frac{t}{m}E_{m-1}\right)' \left(I_{m-1} + E_{(m-1),(m-1)}\right)\left(\underline{Z} - \frac{t}{m}E_{m-1}\right)$$

$$= \frac{m(m+1)}{t^2}\left[\sum_{i=1}^{m-1}\left(Z_i - \frac{t}{m}\right)^2 + \left(Z_m - \frac{t}{m}\right)^2\right]$$

$$= m(m+1)\left[\frac{\sum_{i=1}^{m} Z_i^2}{t^2} - \frac{1}{m}\right] \tag{3.13}$$

To arrive at an unconditional setup we modify the statistic Q by replacing t by T. Thus the statistic Q in terms of type-II progressive censored data is

$$Q = m(m+1)\left[\frac{\sum_{i=1}^{m} Z_i^2}{T^2} - \frac{1}{m}\right]$$

$$= m(m+1)\left[\frac{\sum_{i=1}^{m}(n-i+1-r_1-r_2 \ldots . -r_{i-1})^2[\,Y_i - Y_{i-1}]^2}{[\,\sum_{i=1}^{m}(\,1+r_i)\,Y_i\,]^2} - \frac{1}{m}\right]$$

$$= m(m+1)\left[\frac{\sum_{i=1}^{m}(n-i+1-r_1 \ldots - r_{i-1})^2[d(X_i,\beta)-d(X_{i-1},\ \beta)]^2}{[\sum_{i=1}^{m}(1+r_i)d(X,\ \beta)\ ]^2}-\frac{1}{m}\right]$$

For the hypothesis of testing $H_0{:}\,\beta = \beta_0$ against $H_1{:}\,\beta \neq \beta_0$ under H_0, on simplification, the statistic Q can written as

$$Q = m(m+1)\left[\frac{\sum_{i=1}^{m}(n-i+1-r_1-r_2 \ldots - r_{i-1})^2[d(X_i,\beta_0)-d(X_{i-1},\beta_0)]^2}{[\sum_{i=1}^{m}(1+r_i)d(X_i,\beta_0)\ ]^2}-\frac{1}{m}\right] \qquad (3.14)$$

The exact mean and variance of Q under H_0 are

$$E_{H_0}(Q) = m(m+1)E_{H_0}\left(\frac{\sum_{i=1}^{m} Z_i^2}{T^2}-\frac{1}{m}\right)$$

$$= m(m+1)\left[EE\left(\frac{\sum_{i=1}^{m} Z_i^2}{t^2}|T=t\right)-\frac{1}{m}\right] \qquad (3.15)$$

Using values from (3.12) and substituting it in the above expression, on simplification, the mean of Q under H_0 is given by

$$E_{H_0}(Q)\ = m-1 \qquad (3.16)$$

Let $V = \sum_{i=1}^{m} Z_i^2$, then variance of Q is

$$Var_{H_0}(Q) = m^2(m+1)^2\ Var_{H_0}\left(\frac{\sum_{i=1}^{m} Z_i^2}{T^2}-\frac{1}{m}\right)$$

$$= m^2(m+1)^2\left[E_{H_0}\left(\frac{V^2}{T^4}\right)-\{E_{H_0}\left(\frac{V}{T^2}\right)\}^2\right]$$

$$= m^2(m+1)^2\left[EE\left(\left(\frac{\sum_{i=1}^{m} Z_i^4+\sum\sum_{i\neq j}^{m} Z_i^2 Z_j^2}{T^4}\right| T=t\right)\right)-\{EE\left(\frac{\sum_{i=1}^{m} Z_i^2}{T^2}\right| T=t)\}^2\right]$$

Using values from (3.12) and putting in the above expression, on simplification, the variance of Q under H_0 is given by

$$Var_{H_0}(Q) = \frac{4m^2(m-1)}{(m+2)(m+3)} \qquad (3.17)$$

Then the standardized statistic is given by $Q_{(sd)} = \frac{Q_{H_0}-E_{H_0}(Q)}{\sqrt{Var_{H_0}(Q)}}$

Substituting values from (3.16) and (3.17) in the expression for $Q_{(sd)}$ and on simplification, we obtain the test statistics as,

$$Q_{(sd)} = \sqrt{\frac{(m+2)(m+3)}{m-1}}\left[\frac{(m+1)\{\sum_{i=1}^{m}(n-i+1-r_1-r_2\ldots.-r_{i-1})^2[d(X_i,\beta_0)-d(X_{i-1},\beta_0)]^2\}}{2[\sum_{i=1}^{m}(1+r_i)d(X_i,\beta_0)]^2}-1\right] \tag{3.18}$$

Asymptotic null distribution of *Q*:

Define $\underline{W_i} = (Z_{i,}\ Z_i^2)'$, where $Z_i = (n-i+1-r_1-r_2 \ldots - r_{i-1})[d(x_i,\beta_0) - d(x_{i-1},\beta_0)]$, $i = 2,3,\ldots,m$. It is noted that $\{\underline{W_i},\ i = 1,2,3,\ldots,m\}$ are iid random vectors in view of the iid nature of $Z_i,\ i = 1,2,\ldots,m$. We have $\underline{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ where, $V_1 = \frac{\sum_{i=1}^{m} Z_i}{m}$ and $V_2 = \sum_{i=1}^{m} Z_i^2/m$. Now, applying Multivariate CLT, the asymptotic distribution of $\sqrt{m}\left(\underline{V} - E(\underline{V})\right)$ is bivariate normal with zero mean and dispersion matrix $D(\underline{V})$. Also, we have $E(\underline{V}) = (\theta, 2\theta^2)$ and $D(\underline{V}) = \begin{bmatrix} \theta^2 & 4\theta^3 \\ 4\theta^3 & 20\theta^4 \end{bmatrix}$. Then the asymptotic distribution of $g(V_1,V_2) = V_2/V_1^2$ is normal with mean 2 and variance 4/m. Hence

$$Q = (m+1)[V_2/V_1^2 - 1] \sim N(m+1, 4(m+1)^2/m) \tag{3.19}$$

Here, we propose the two more test statistics by replacing β by the null hypothesis value β_0. For $g(\theta) = \frac{1}{\theta}$ and $h(\theta) = e^{-\theta}$, the equation (3.1) can be written as

$$f_{\underline{\mathrm{x}}}(\underline{\mathrm{x}};\theta,\beta) = C\,\theta^m\left[\prod_{\mathrm{i}=1}^{\mathrm{m}} a(x_i,\beta)\right]e^{[-\theta\sum_{\mathrm{i}=1}^{\mathrm{m}}(1+r_i)d(x_i,\beta)]}, m \le n, x > 0, \theta, \beta > 0$$

where, $C = n\,(n-r_1-1)(n-r_1-r_2-2)\ \ldots\ (n-r_1-r_2 \ldots - r_{m-1} - m + 1)$

(3.20)

Here, we discuss method of finding MLE of θ. The log-likelihood function is given by

$$L = logC + mlog\theta + \sum_{\mathrm{i}=1}^{\mathrm{m}} loga(x_i,\beta) - \theta\sum_{\mathrm{i}=1}^{\mathrm{m}}(1+r_i)d(x_i,\beta)\,.$$

For known β, the MLE of θ is the solution of $\frac{dL}{d\theta} = \frac{m}{\theta} - \sum_{\mathrm{i}=1}^{\mathrm{m}}(1+r_i)d(x_i,\beta) = 0$, which is given by

$$\hat{\theta} = \frac{m}{\sum_{i=1}^{m}(1+r_i)d(x_i,\beta)} \tag{3.21}$$

The second statistic is based on the ratio of two independently chi-square distributed variables. For known β,$\theta d(X_1,\beta) < \theta d(X_2,\beta) < \cdots < \theta d(X_m,\beta)$ is a sample from the i.i.d standard exponential distribution. Thomas and Wilson (1972) showed that if $\theta d(X_1,\beta) < \theta d(X_2,\beta) < \cdots < \theta d(X_m,\beta)$ are ordered statistics from the standard exponential distribution then the generalized spacings $Z_1, Z_2, \ldots, Z_m$ are i.i.d. standard exponential random variables where, $Z_i = (n-i+1-r_1-r_2 \ldots - r_{i-1})[\theta d(X_i,\beta) - \theta d(X_{i-1},\beta)]$ and $\sum_{i=1}^{m} Z_i = \theta\sum_{i=1}^{m}(1+r_i)d(X_i,\beta)$. Using these generalized spacing's, we have $U = 2[\sum_{i=2}^{m} Z_i] = 2[\sum_{i=1}^{m}(1+r_i)\theta d(X_i,\beta) - n\theta d(X_1,\beta)]$ and $V = 2Z_1 = 2n\theta d(X_1,\beta)$ are chi-square distributed with 2(m-1) and 2 degrees of freedom respectively. Since U and V depends on two disjoint set of independent random variables, therefore the distributions of U and V are also independent. The corresponding statistics is

$$F = F(\beta_0) = \frac{\frac{U}{2(m-1)}}{\frac{V}{2}} = \frac{\sum_{i=1}^{m}(1+r_i)d(X_i,\beta_0) - nd(X_1,\beta_0)}{n(m-1)d(X_1,\beta_0)} \tag{3.22}$$

It can be seen that F(β_0) has F distribution with 2(m-1) and 2 degrees of freedom.

The third test statistic is proposed using a quadratic form constructed based on the dispersion matrix Σ of $\underline{Z}$. Which is

$$Q_1 = (\underline{Z} - \underline{\mu})' \Sigma^{-1}(\underline{Z} - \underline{\mu}) \tag{3.23}$$

Here, $\mu_i = E(Z_i) = 1, \sigma_{ii} = var(Z_i) = 1, \sigma_{ij} = cov(Z_i, Z_j) = 0$, $\underline{\mu} = \mathrm{E}(\underline{Z}) = E_m$

and $\Sigma = I_m$ Hence

$$Q_1 = \sum_{i=1}^{m}(Z_i - 1)^2$$

$$= \sum_{i=1}^{m} Z_i^{\,2} - 2\sum_{i=1}^{m} Z_i + m$$

$$= \theta^2 \sum_{i=1}^{m}\{(n-i+1-r_1-r_2 \ldots - r_{i-1})[d(x_i,\beta) - d(x_{i-1},\beta)]\}^2$$

$$- 2\theta\left\{\sum_{i=1}^{m}(1+r_i)d(x_i,\beta)\right\} + m$$

Replacing θ by reciprocal of it's MLE using (3.21), the test statistic is as $Q_1 = \left[\frac{\sum_{i=1}^{m}(1+r_i)d(X_i,\beta_0)}{m}\right]^2 [\sum_{i=1}^{m}\{(n-i+1-r_1-r_2 \ldots - r_{i-1})[d(X_i,\beta_0) -$

$$d(X_{i-1},\beta_0)]\}^2] - 2\frac{\left\{\sum_{i=1}^{m}(1+r_i)d(X_i,\beta_0)\right\}^2}{m} + m \quad (3.24)$$

Suppose that we would like to test$H_0: \beta = \beta_0$ against $H_1: \beta \neq \beta_0$ with the level of significance α. Since we could not find the expression for mean of any of these statistics under alternative hypothesis we propose test procedure based on certain assumptions. Suppose for all values of $\beta \neq \beta_0$, the mean of statistics, say L, is increasing with β for $\beta \neq \beta_0$, then the critical region is in the upper tail and hence null hypothesis is rejected if L > L_α (upper α-thquantile). Suppose the value of statistics decreases (or increases) for $\beta < \beta_0$ and increases otherwise (or decreases) for $\beta > \beta_0$. Then test procedure is to reject $H_0: \beta = \beta_0$ in favour of $H_1: \beta \neq \beta_0$ if L <$L_{\alpha/2}$or L >$L_{1-\alpha/2}$. Here L is a typical statistics and L_γ is the γ–thquantile of the distribution of L.

4. Power Study for some well-known distributions

4.1 Tests for Shape Parameter of Gompertz Distribution

Consider a ordered sample $X_1, X_2, \ldots, X_n$, from the Gompertz distribution defined by the pdf

$$f(x;\theta,\beta) = \theta\, e^{\beta x}\, e^{-\frac{\theta}{\beta}(e^{\beta x}-1)};\ \ x > 0,\ \ \theta,\beta > 0$$

where, β and θ are shape and scale parameters respectively. Here, we are interested to test $H_0: \beta = \beta_0$ against $H_1: \beta \neq \beta_0$ by treating θ as nuisance parameter. Using $d(X_i,\beta)$=$\frac{(e^{\beta x_i-1})}{\beta}$ given in (2.2),$T = \sum_{i=1}^{m}(1+r_i)\frac{(e^{\beta x_i-1})}{\beta}$is a complete sufficient statistic for θ. Hence the test statistic based on standardized version of the statistic Q under H_0 is given by

$$Q_{(sd)} = \sqrt{\frac{(m+2)(m+3)}{m-1}}\left[\frac{(m+1)\{\sum_{i=1}^{m}(n-i+1-r_1-r_2 \ldots - r_{i-1})^2[\frac{(e^{\beta_0 X_{i-1}})}{\beta_0} - \frac{(e^{\beta_0 X_{i-1-1}})}{\beta_0}]^2\}}{2[\sum_{i=1}^{m}(1+r_i)\frac{(e^{\beta_0 X_{i-1}})}{\beta_0}]^2} - 1\right]$$

Similarly, the other two test statistics under $H_0$are as

$$F = F(\beta_0) = \frac{\sum_{i=1}^{m}(1+r_i)\frac{(e^{\beta_0 X_{i-1}})}{\beta_0} - n\frac{(e^{\beta_0 X_{1-1}})}{\beta_0}}{n(m-1)\frac{(e^{\beta_0 X_{1-1}})}{\beta_0}}$$

and

$$Q_1 = \left[\frac{\sum_{i=1}^{m}(1+r_i)\frac{(e^{\beta_0 X_{i-1})}}{\beta_0}}{m}\right]^2 \left[\sum_{i=1}^{m}\left\{(n-i+1-r_1-r_2 \ldots - r_{i-1})\left[\frac{(e^{\beta_0 X_{i}-1)}}{\beta_0} - \frac{(e^{\beta_0 X_{i-1}-1)}}{\beta_0}\right]\right\}^2\right] - 2\frac{\left\{\sum_{i=1}^{m}(1+r_i)\frac{(e^{\beta_0 X_{i-1})}}{\beta_0}\right\}^2}{m} + m$$

Suppose that we would like to test $H_0 : \beta = 1$ against $H_1 : \beta \neq 1$ with the level of significance α = 0.05. Using the algorithm described in Balakrishnan and Sandhu (1995), we construct progressively type-II censored samples from Gompertz distribution with censoring proportion $p = 0.1$. The Tables 1 and 2 gives the direction of the test procedures and simulated cut-off points. The powers of test at 5% level of significance for different values of n and β is given in tables 3 and 4 respectively.

Table 1: Simulated means of the statistics for different values of β for p = 0.1.

n	Test	β=0.3	β=0.4	β=0.6	β=0.8	β=1	β=1.2	β=1.4	β=1.6	β=1.8	β=2
20	$Q_{(sd)}$	6.73	3.87	1.34	0.34	0.01	0.03	0.28	0.67	1.16	1.71
	Q_1	17.92	17.40	14.51	14.76	27.20	57.23	106.33	173.48	256.41	352.32
30	$Q_{(sd)}$	9.34	5.20	1.71	0.40	0.00	0.08	0.45	1.01	1.71	2.52
	Q_1	26.89	26.00	20.54	18.93	35.47	79.17	153.67	259.13	393.51	553.51
40	$Q_{(sd)}$	11.80	6.44	2.04	0.46	0.01	0.14	0.63	1.36	2.28	3.34
	Q_1	35.85	34.65	27.04	23.92	44.69	102.49	203.78	350.49	541.43	773.26
50	$Q_{(sd)}$	14.02	7.54	2.33	0.50	0.00	0.17	0.74	1.61	2.70	3.98
	Q_1	44.81	43.30	33.50	28.67	53.38	124.66	251.60	438.10	684.30	987.51
60	$Q_{(sd)}$	16.24	8.63	2.59	0.53	-0.01	0.20	0.87	1.88	3.17	4.68
	Q_1	53.77	51.96	40.08	33.75	62.49	147.46	301.04	529.87	836.12	1218.20
80	$Q_{(sd)}$	20.85	10.96	3.21	0.65	0.00	0.26	1.08	2.32	3.92	5.83
	Q_1	71.70	69.27	53.13	43.37	79.20	189.77	393.74	703.21	1124.90	1660.32

Table 2: The percentage points of the distribution of Test statistics for p = 0.1.

n	Test	1%	5%	95%	99%
20	$Q_{(sd)}$	-1.4738	-1.2002	1.8427	3.3046
	Q_1	5.3258	6.7093	90.0191	191.3381
30	$Q_{(sd)}$	-1.5838	-1.2486	1.8274	3.3463
	Q_1	9.0164	10.6718	102.9052	179.4333
40	$Q_{(sd)}$	-1.5984	-1.2773	1.8074	3.1959
	Q_1	13.2257	15.1679	118.2104	195.5479
50	$Q_{(sd)}$	-1.6560	-1.3006	1.8269	3.0983
	Q_1	17.3127	20.1142	127.5421	204.7737
60	$Q_{(sd)}$	-1.6923	-1.3084	1.8276	3.0453
	Q_1	21.7082	24.8927	145.2131	219.1663
80	$Q_{(sd)}$	-1.7486	-1.3101	1.7540	3.0129
	Q_1	31.1924	35.6043	164.7771	231.1849

The critical values $F_{.025}(2(\mathrm{m}-1),2)$ and $F_{.975}(2(\mathrm{m}-1),2)$ are used to compute power of test F.

Table 3: Power of Tests for $\beta < 1$ alternatives at α = 0.05.

n	Censoring proportion	Progressive sampling plan	Tests	β=0.3	β=0.4	β=0.6	β=0.8
20	0.1	m=18 and r_9=1, r_{18}=1	$Q_{(sd)}$	0.88	0.66	0.28	0.10
			F	0.96	0.78	0.31	0.10
			Q_1	0.00	0.00	0.00	0.01
	0.2	m=16 and r_9=1, r_{16}=3	$Q_{(sd)}$	0.86	0.64	0.29	0.09
			F	0.93	0.76	0.30	0.09
			Q_1	0.00	0.00	0.00	0.01
30	0.1	m=27 and r_9=1, r_{18}=1, r_{27}=1	$Q_{(sd)}$	0.96	0.80	0.36	0.11
			F	0.99	0.86	0.35	0.10
			Q_1	0.00	0.00	0.00	0.00
	0.2	m=24 and r_9=1, r_{18}=1, r_{24}=4	$Q_{(sd)}$	0.92	0.77	0.34	0.10
			F	0.95	0.83	0.33	0.09
			Q_1	0.00	0.00	0.00	0.00

40	0.1	m=36 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1	$Q_{(sd)}$	0.99	0.88	0.42	0.13
			F	0.99	0.90	0.38	0.10
			Q_1	0.00	0.00	0.00	0.00
	0.2	m=32 and r_9=1, r_{18}=1, r_{27}=1, r_{32}=5	$Q_{(sd)}$	0.96	0.86	0.40	0.12
			F	0.96	0.88	0.35	0.09
			Q_1	0.00	0.00	0.00	0.00
50	0.1	m=45 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1	$Q_{(sd)}$	1.00	0.93	0.49	0.13
			F	0.99	0.93	0.40	0.10
			Q_1	0.00	0.00	0.00	0.00
	0.2	m=40 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{40}=6	$Q_{(sd)}$	1.00	0.90	0.47	0.12
			F	0.98	0.89	0.38	0.10
			Q_1	0.00	0.00	0.00	0.00
60	0.1	m=54 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1	$Q_{(sd)}$	1.00	0.96	0.52	0.13
			F	0.99	0.94	0.43	0.11
			Q_1	0.00	0.00	0.00	0.00
	0.2	m=48 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{48}=7	$Q_{(sd)}$	1.00	0.93	0.50	0.12
			F	0.99	0.91	0.41	0.11
			Q_1	0.00	0.00	0.00	0.00
80	0.1	m=72 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{72}=1	$Q_{(sd)}$	1.00	0.99	0.63	0.17
			F	1.00	0.97	0.47	0.11
			Q_1	0.00	0.00	0.00	0.00
	0.2	m=64 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{64}=9	$Q_{(sd)}$	1.00	0.97	0.61	0.16
			F	1.00	0.95	0.45	0.11
			Q_1	0.00	0.00	0.00	0.00

Table 4: Power of Tests for $\beta > 1$ alternatives at α = 0.05.

n	Censoring Proportion	Progressive sampling plan	Tests	β=1.2	β=1.4	β=1.6	β=1.8	β=2
20	0.1	m=18 and r_9=1, r_{18}=1	$Q_{(sd)}$	0.06	0.09	0.16	0.26	0.37
			F	0.08	0.15	0.27	0.38	0.50
			Q_1	0.18	0.41	0.66	0.84	0.93
	0.2	m=16 and r_9=1, r_{16}=3	$Q_{(sd)}$	0.06	0.09	0.14	0.24	0.34
			F	0.07	0.14	0.25	0.36	0.47

			Q_1	0.17	0.40	0.64	0.81	0.90
30	0.1	m=27 and r_9=1, r_{18}=1, r_{27}=1	$Q_{(sd)}$	0.06	0.12	0.22	0.37	0.53
			F	0.09	0.19	0.34	0.47	0.60
			Q_1	0.24	0.57	0.84	0.96	0.99
	0.2	m=24 and r_9=1, r_{18}=1, r_{24}=4	$Q_{(sd)}$	0.06	0.11	0.21	0.35	0.51
			F	0.08	0.18	0.32	0.45	0.58
			Q_1	0.22	0.55	0.81	0.93	0.97
40	0.1	m=36 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1	$Q_{(sd)}$	0.07	0.16	0.31	0.50	0.68
			F	0.11	0.24	0.39	0.54	0.68
			Q_1	0.29	0.69	0.92	0.99	1.00
	0.2	m=32 and r_9=1, r_{18}=1, r_{27}=1, r_{32}=5	$Q_{(sd)}$	0.07	0.15	0.30	0.48	0.66
			F	0.10	0.23	0.37	0.52	0.66
			Q_1	0.28	0.67	0.90	0.97	1.00
50	0.1	m=45 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1	$Q_{(sd)}$	0.07	0.18	0.35	0.58	0.78
			F	0.11	0.26	0.43	0.60	0.72
			Q_1	0.37	0.81	0.97	1.00	1.00
	0.2	m=40 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{40}=6	$Q_{(sd)}$	0.07	0.17	0.34	0.57	0.76
			F	0.10	0.25	0.42	0.59	0.70
			Q_1	0.36	0.80	0.95	0.99	1.00
60	0.1	m=54 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1	$Q_{(sd)}$	0.08	0.20	0.42	0.66	0.84
			F	0.13	0.29	0.48	0.64	0.76
			Q_1	0.40	0.86	0.99	1.00	1.00
	0.2	m=48 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{48}=7	$Q_{(sd)}$	0.08	0.19	0.40	0.65	0.83
			F	0.12	0.28	0.46	0.62	0.75
			Q_1	0.39	0.85	0.97	0.99	1.00

80	0.1	m=72 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{72}=1	$Q_{(sd)}$	0.09	0.25	0.55	0.80	0.94
			F	0.14	0.32	0.53	0.69	0.81
			Q_1	0.51	0.95	1.00	1.00	1.00
	0.2	m=64 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{64}=9	$Q_{(sd)}$	0.09	0.24	0.55	0.79	0.93
			F	0.13	0.31	0.52	0.68	0.80
			Q_1	0.50	0.94	1.00	1.00	1.00

4.2 Tests for Shape Parameter of Weibull Distribution

Consider a ordered sample $X_1, X_2, \dots, X_n$from the Webuill distribution is defined by the pdf

$$f(x;\theta,\beta) = \theta\beta x^{\beta-1}\, e^{-\theta x^{\beta}};\;\; x > 0,\;\; \theta,\beta > 0$$

where, β and θ are the shape and scale parameters respectively. Here, we are interested to test $H_0{:}\,\beta = \beta_0$ againstH_1: $H_1{:}\,\beta \neq \beta_0$ by treating θ as nuisance parameter. Using $d(X_i,\beta)$=x_i^{β}given in (2.4), $T = \sum_{i=1}^{m}(1 + r_i)x_i^{\beta}$ is a complete sufficient statistic for θ. Hence the test statistic based on standardized version of the statistic Q is given by

$$Q_{(sd)} = \sqrt{\frac{(m+2)(m+3)}{m-1}}\left[\frac{(m+1)\{\sum_{i=1}^{m}(n-i+1-r_1-r_2 \dots -r_{i-1})^2[X_i^{\beta_0} - X_{i-1}^{\beta_0}]^2\}}{2[\sum_{i=1}^{m}(1+r_i)X_i^{\beta_0}]^2} - 1\right]$$

Similarly, the other two test statistics under H_0 are as

$$F = F(\beta_0) = \frac{\sum_{i=1}^{m}(1+r_i)X_i^{\beta_0} - nX_1^{\beta_0}}{n(m-1)X_1^{\beta_0}}$$

and

$$Q_1 = \left[\frac{\sum_{i=1}^{m}(1+r_i)X_i^{\beta_0}}{m}\right]^2 \left[\sum_{i=1}^{m}\left\{(n-i+1-r_1-r_2 \dots - r_{i-1})\left[X_i^{\beta_0} - X_{i-1}^{\beta_0}\right]\right\}^2\right] - 2\frac{\left\{\sum_{i=1}^{m}(1+r_i)X_i^{\beta_0}\right\}^2}{m} + m$$

Suppose that we would like to test $H_0{:}\,\beta = 1$ against $H_1{:}\,\beta \neq 1$ with the level of significance α = 0.05. Using the algorithm described in Balakrishnan and Sandhu (1995), we construct progressively type-II censored samples from Weibull distribution with censoring proportion $p = 0.1$. The Tables 5 and 6 gives the direction of the test procedures and simulated cut-off points. The powers of tests at 5% level of significance for different values of n and β are given in Tables 7 and 8 respectively.

Table 5: Simulated means of the statistics for different values of β for p=0.1.

n	Test	β=0.4	β=0.6	β=0.8	β=1	β=1.2	β=1.4	β=1.6	β=1.8	β=2
20	$Q_{(sd)}$	5.20	1.69	0.40	0.01	0.05	0.33	0.78	1.33	1.95
	Q_1	36512.69	358.38	59.07	27.20	20.61	19.88	21.40	24.12	27.61
30	$Q_{(sd)}$	7.14	2.14	0.47	0.00	0.10	0.53	1.18	2.00	2.93
	Q_1	27185.49	383.05	71.34	35.47	28.65	29.16	32.82	38.38	45.33
40	$Q_{(sd)}$	8.92	2.53	0.53	0.01	0.18	0.75	1.61	2.69	3.93
	Q_1	31343.50	459.96	87.46	44.69	37.29	39.14	45.26	54.19	65.31
50	$Q_{(sd)}$	10.48	2.85	0.57	0.00	0.20	0.88	1.91	3.22	4.75
	Q_1	37387.20	537.68	103.00	53.38	45.32	48.46	57.04	69.43	84.94
60	$Q_{(sd)}$	11.99	3.14	0.60	-0.01	0.24	1.04	2.26	3.82	5.65
	Q_1	43507.24	623.71	119.65	62.49	53.75	58.35	69.75	86.12	106.72
80	$Q_{(sd)}$	15.08	3.81	0.74	0.00	0.31	1.29	2.81	4.78	7.15
	Q_1	49792.57	756.73	148.38	79.20	69.61	77.12	94.02	118.27	149.10

Table 6: The percentage points of the distribution of Test statistics for p = 0.1.

n	Test	1%	5%	95%	99%
20	$Q_{(sd)}$	-1.4738	-1.2002	1.8427	3.3046
	Q_1	5.3258	6.7093	90.0191	191.3381
30	$Q_{(sd)}$	-1.5838	-1.2486	1.8274	3.3463
	Q_1	9.0164	10.6718	102.9052	179.4333
40	$Q_{(sd)}$	-1.5984	-1.2773	1.8074	3.1959
	Q_1	13.2257	15.1679	118.2104	195.5479
50	$Q_{(sd)}$	-1.6560	-1.3006	1.8269	3.0983
	Q_1	17.3127	20.1142	127.5421	204.7737
60	$Q_{(sd)}$	-1.6923	-1.3084	1.8276	3.0453

	Q_1	21.7082	24.8927	145.2131	219.1663
80	$Q_{(sd)}$	-1.7486	-1.3101	1.7540	3.0129
	Q_1	31.1924	35.6043	164.7771	231.1849

The critical values $F_{.025}(2(\mathrm{m}-1),2)$ and $F_{.975}(2(\mathrm{m}-1),2)$ are used to compute power of test F.

Table 7: Power of Tests for $\beta < 1$ alternatives at α = 0.05.

n	Censoring proportion	Progressive sampling plan	Tests	β=0.2	β=0.4	β=0.6	β=0.8
20	0.1	m=18 and r_9=1, r_{18}=1	$Q_{(sd)}$	0.99	0.78	0.35	0.11
			F	1.00	0.85	0.35	0.11
			Q_1	0.96	0.78	0.43	0.16
	0.2	m=16 and r_9=1, r_{16}=3	$Q_{(sd)}$	0.97	0.76	0.34	0.10
			F	0.98	0.84	0.34	0.10
			Q_1	0.94	0.77	0.42	0.15
30	0.1	m=27 and r_9=1, r_{18}=1, r_{27}=1	$Q_{(sd)}$	1.00	0.90	0.44	0.12
			F	1.00	0.92	0.41	0.11
			Q_1	0.99	0.89	0.55	0.20
	0.2	m=24 and r_9=1, r_{18}=1, r_{24}=4	$Q_{(sd)}$	1.00	0.87	0.43	0.11
			F	1.00	0.90	0.40	0.11
			Q_1	0.97	0.87	0.54	0.19
40	0.1	m=36 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1	$Q_{(sd)}$	1.00	0.95	0.52	0.14
			F	1.00	0.95	0.44	0.11
			Q_1	1.00	0.94	0.63	0.22
	0.2	m=32 and r_9=1, r_{18}=1, r_{27}=1, r_{32}=5	$Q_{(sd)}$	1.00	0.93	0.50	0.13
			F	1.00	0.93	0.42	0.11
			Q_1	1.00	0.92	0.61	0.21
50	0.1	m=45 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1	$Q_{(sd)}$	1.00	0.98	0.58	0.14
			F	1.00	0.97	0.47	0.11
			Q_1	1.00	0.97	0.71	0.25
	0.2	m=40 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{40}=6	$Q_{(sd)}$	1.00	0.96	0.56	0.12
			F	1.00	0.95	0.45	0.11
			Q_1	1.00	0.95	0.69	0.24

60	0.1	m=54 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1	$Q_{(sd)}$	1.00	0.99	0.64	0.15
			F	1.00	0.98	0.48	0.12
			Q_1	1.00	0.99	0.76	0.25
	0.2	m=48 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{48}=7	$Q_{(sd)}$	1.00	0.97	0.63	0.14
			F	1.00	0.96	0.46	0.12
			Q_1	1.00	0.97	0.74	0.24
80	0.1	m=72 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{72}=1	$Q_{(sd)}$	1.00	1.00	0.74	0.19
			F	1.00	0.99	0.53	0.13
			Q_1	1.00	1.00	0.84	0.31
	0.2	m=64 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{64}=9	$Q_{(sd)}$	1.00	1.00	0.71	0.18
			F	1.00	0.97	0.52	0.13
			Q_1	1.00	1.00	0.82	0.30

Table 8: Power of Tests for $\beta > 1$ alternatives at α = 0.05.

n	Censoring proportion	Progressive sampling plan	Tests	β=1.2	β=1.4	β=1.6	β=1.8	β=2
20	0.1	m=18 and r_9=1, r_{18}=1	$Q_{(sd)}$	0.06	0.10	0.18	0.29	0.43
			F	0.09	0.19	0.33	0.46	0.58
			Q_1	0.02	0.01	0.01	0.02	0.02
	0.2	m=16 and r_9=1, r_{16}=3	$Q_{(sd)}$	0.06	0.09	0.17	0.28	0.41
			F	0.08	0.18	0.32	0.45	0.56
			Q_1	0.02	0.01	0.01	0.02	0.02
30	0.1	m=27 and r_9=1, r_{18}=1, r_{27}=1	$Q_{(sd)}$	0.07	0.13	0.27	0.44	0.61
			F	0.10	0.24	0.41	0.56	0.69
			Q_1	0.02	0.02	0.02	0.03	0.05
	0.2	m=24 and r_9=1, r_{18}=1, r_{24}=4	$Q_{(sd)}$	0.07	0.12	0.26	0.43	0.59
			F	0.09	0.23	0.40	0.55	0.67
			Q_1	0.02	0.02	0.02	0.03	0.04
40	0.1	m=36 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1	$Q_{(sd)}$	0.08	0.18	0.36	0.57	0.76
			F	0.13	0.28	0.46	0.63	0.76
			Q_1	0.02	0.02	0.03	0.05	0.09
	0.2	m=32 and r_9=1, r_{18}=1, r_{27}=1, r_{32}=5	$Q_{(sd)}$	0.07	0.17	0.35	0.56	0.74
			F	0.12	0.27	0.45	0.62	0.74

			Q_1	0.02	0.02	0.03	0.05	0.09
50	0.1	m=45 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1	$Q_{(sd)}$	0.07	0.21	0.42	0.66	0.85
			F	0.13	0.31	0.52	0.68	0.81
			Q_1	0.02	0.02	0.04	0.08	0.14
	0.2	m=40 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{40}=6	$Q_{(sd)}$	0.07	0.20	0.40	0.64	0.83
			F	0.12	0.30	0.49	0.65	0.80
			Q_1	0.02	0.02	0.04	0.08	0.13
60	0.1	m=54 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1	$Q_{(sd)}$	0.09	0.24	0.49	0.74	0.90
			F	0.15	0.36	0.56	0.73	0.83
			Q_1	0.02	0.02	0.04	0.09	0.18
	0.2	m=48 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{48}=7	$Q_{(sd)}$	0.09	0.23	0.48	0.72	0.88
			F	0.14	0.35	0.55	0.71	0.81
			Q_1	0.02	0.02	0.04	0.09	0.18
80	0.1	m=72 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{72}=1	$Q_{(sd)}$	0.10	0.31	0.63	0.86	0.97
			F	0.16	0.38	0.60	0.77	0.87
			Q_1	0.02	0.02	0.07	0.16	0.32
	0.2	m=64 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{64}=9	$Q_{(sd)}$	0.09	0.30	0.62	0.85	0.96
			F	0.15	0.36	0.59	0.75	0.86
			Q_1	0.02	0.02	0.07	0.15	0.30

4.3 Tests for Shape Parameter of Burr-XII distribution

Consider a ordered sample $X_1, X_2, \dots, X_n$ from the Burr-XII distribution is defined by the pdf.

$$f(x;\theta,\beta) = \theta\beta x^{\beta-1}\left[1 + x^{\beta}\right]^{-(\theta+1)};\ \ x > 0,\ \ \theta,\beta > 0$$

where, β and θ are shape and scale parameters respectively. Here, we are interested for testing of $H_0\!: \beta = \beta_0$ against $H_1\!: \beta \neq \beta_0$ by treating θ as nuisance parameter. Using $d(X_i,\beta)$=log$\left[1 + x_i^{\beta}\right]$ given in (2.3), $T = \sum_{i=1}^{m}(1 + r_i)\log\left[1 + x_i^{\beta}\right]$is a complete sufficient statistic for θ. Hence the test statistics based on standardized version of the statistic Q under H_0 is given by

$$Q_{(sd)} = \sqrt{\frac{(m+2)(m+3)}{m-1}}\left[\frac{(m+1)\{\sum_{i=1}^{m}(n-i+1-r_1-r_2 \ldots -r_{i-1})^2[\log(1+X_i^{\beta_0}) - \log(1+X_{i-1}^{\beta_0})]^2\}}{2[\sum_{i=1}^{m}(1+r_i)\log(1+X_i^{\beta_0})]^2} - 1\right]$$

Similarly, the other two test statistics under $H_0$are as

$$F = F(\beta_0) = \frac{\sum_{i=1}^{m}(1+r_i)\log(1+X_i^{\beta_0}) - n\log(1+X_1^{\beta_0})}{n(m-1)\log(1+X_1^{\beta_0})}$$

and

$$Q_1 = \left[\frac{\sum_{i=1}^{m}(1+r_i)\log(1+X_i^{\beta_0})}{m}\right]^2\left[\sum_{i=1}^{m}\left\{(n-i+1-r_1-r_2 \ldots - r_{i-1})\left[\log(1+X_i^{\beta_0}) - \log(1+X_{i-1}^{\beta_0})\right]\right\}^2\right] - 2\frac{\left\{\sum_{i=1}^{m}(1+r_i)\log(1+X_i^{\beta_0})\right\}^2}{m} + m$$

Suppose that we would like to test $H_0: \beta = 1$ against $H_1: \beta \neq 1$ with the level of significance $\alpha = 0.05$. We construct progressively type-II censored samples from Burr-XII distribution with censoring proportion $p = 0.1$. The Tables 9 and 10 gives the direction of the test procedures and cut-off points. The powers of test for different values of n and β is given in Tables 11 and 12 respectively.

Table 9: Simulated means of the statistics for different values of β for p = 0.1.

n	Test	β=0.4	β=0.6	β=0.8	β=1	β=1.2	β=1.4	β=1.6	β=1.8	β=2
20	$Q_{(sd)}$	2.32	1.03	0.31	0.01	0.02	0.28	0.71	1.27	1.93
	Q_1	801.00	156.94	53.75	27.20	18.76	16.10	15.71	16.36	17.54
30	$Q_{(sd)}$	2.70	1.19	0.34	0.00	0.07	0.45	1.10	2.00	2.96
	Q_1	1000.22	195.84	67.84	35.47	25.77	23.42	24.05	38.38	28.94
40	$Q_{(sd)}$	3.00	1.32	0.37	0.01	0.13	0.66	1.52	2.66	4.02
	Q_1	1223.17	239.85	83.79	44.69	33.47	31.43	33.24	37.00	41.89
50	$Q_{(sd)}$	3.27	1.43	0.39	0.00	0.15	0.78	1.82	3.21	4.90
	Q_1	1450.30	283.96	99.30	53.38	40.61	38.91	41.99	47.60	54.78
60	$Q_{(sd)}$	11.99	3.14	0.60	-0.01	0.24	1.04	2.26	3.82	5.65
	Q_1	1707.24	323.71	119.65	62.49	53.75	58.35	69.75	86.12	106.72
80	$Q_{(sd)}$	4.00	1.76	0.47	0.00	0.25	1.18	2.73	4.87	7.54
	Q_1	2082.62	408.90	144.27	79.20	62.35	62.17	69.76	81.96	97.37

Table 10: The percentage points of the distribution of Test statistics for p = 0.1.

n	Test	1%	5%	95%	99%
20	$Q_{(sd)}$	-1.4738	-1.2002	1.8427	3.3046
	Q_1	5.3258	6.7093	90.0191	191.3381
30	$Q_{(sd)}$	-1.5838	-1.2486	1.8274	3.3463
	Q_1	9.0164	10.6718	102.9052	179.4333
40	$Q_{(sd)}$	-1.5984	-1.2773	1.8074	3.1959
	Q_1	13.2257	15.1679	118.2104	195.5479
50	$Q_{(sd)}$	-1.6560	-1.3006	1.8269	3.0983
	Q_1	17.3127	20.1142	127.5421	204.7737
60	$Q_{(sd)}$	-1.6923	-1.3084	1.8276	3.0453
	Q_1	21.7082	24.8927	145.2131	219.1663
80	$Q_{(sd)}$	-1.7486	-1.3101	1.7540	3.0129
	Q_1	31.1924	35.6043	164.7771	231.1849

The critical values $F_{.025}(2(\mathrm{m}-1),2)$ and $F_{.975}(2(\mathrm{m}-1),2)$ are used to compute power of test F.

Table 11: Power of Tests for $\beta < 1$ alternatives at α = 0.05.

n	Censoring proportion	Progressive sampling plan	Tests	β=0.2	β=0.4	β=0.6	β=0.8
20	0.1	m=18 and r_9=1, r_{18}=1	$Q_{(sd)}$	0.79	0.49	0.23	0.10
			F	0.97	0.83	0.35	0.11
			Q_1	0.97	0.79	0.41	0.15
	0.2	m=16 and r_9=1, r_{16}=3	$Q_{(sd)}$	0.77	0.48	0.22	0.09
			F	0.95	0.81	0.34	0.10
			Q_1	0.96	0.78	0.40	0.14
30	0.1	m=27 and r_9=1, r_{18}=1, r_{27}=1	$Q_{(sd)}$	0.88	0.58	0.27	0.10
			F	0.98	0.89	0.40	0.11
			Q_1	1.00	0.89	0.52	0.19
	0.2	m=24 and r_9=1, r_{18}=1, r_{24}=4	$Q_{(sd)}$	0.86	0.57	0.26	0.10
			F	0.96	0.87	0.39	0.10
			Q_1	0.99	0.87	0.51	0.18
40	0.1	m=36 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1	$Q_{(sd)}$	0.92	0.66	0.30	0.10
			F	0.98	0.92	0.43	0.11

			Q_1	1.00	0.94	0.60	0.21
	0.2	m=32 and r_9=1, r_{18}=1, r_{27}=1, r_{32}=5	$Q_{(sd)}$	0.91	0.65	0.30	0.10
			F	0.97	0.91	0.41	0.10
			Q_1	1.00	0.93	0.57	0.20
50	0.1	m=45 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1	$Q_{(sd)}$	0.96	0.72	0.32	0.11
			F	0.98	0.95	0.46	0.11
			Q_1	1.00	0.97	0.69	0.24
	0.2	m=40 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{40}=6	$Q_{(sd)}$	0.95	0.70	0.31	0.10
			F	0.97	0.91	0.45	0.11
			Q_1	1.00	0.96	0.67	0.23
60	0.1	m=54 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1	$Q_{(sd)}$	0.98	0.82	0.38	0.12
			F	1.00	0.98	0.48	0.12
			Q_1	1.00	0.99	0.76	0.25
	0.2	m=48 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{48}=7, r_{48}=7	$Q_{(sd)}$	0.97	0.81	0.37	0.11
			F	1.00	0.97	0.46	0.11
			Q_1	1.00	0.97	0.75	0.23
80	0.1	m=72 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{72}=1	$Q_{(sd)}$	1.00	0.87	0.43	0.13
			F	1.00	0.98	0.53	0.13
			Q_1	1.00	1.00	0.83	0.30
	0.2	m=64 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{64}=9	$Q_{(sd)}$	1.00	0.86	0.41	0.13
			F	1.00	0.96	0.52	0.13
			Q_1	1.00	1.00	0.81	0.29

Table 12: Power of Tests for $\beta > 1$ alternatives at α = 0.05.

n	Censoring proportion	Progressive sampling plan	Tests	β=1.2	β=1.4	β=1.6	β=1.8	β=2
20	0.1	m=18 and r_9=1, r_{18}=1	$Q_{(sd)}$	0.05	0.09	0.17	0.28	0.42
			F	0.09	0.19	0.35	0.49	0.63
			Q_1	0.02	0.01	0.00	0.00	0.00
	0.2	m=16 and r_9=1, r_{16}=3	$Q_{(sd)}$	0.05	0.09	0.17	0.27	0.41
			F	0.08	0.18	0.34	0.47	0.61
			Q_1	0.01	0.01	0.00	0.00	0.00

30	0.1	m=27 and r_9=1, r_{18}=1, r_{27}=1	$Q_{(sd)}$	0.06	0.12	0.25	0.44	0.61
			F	0.11	0.25	0.43	0.56	0.74
			Q_1	0.01	0.00	0.00	0.03	0.01
	0.2	m=24 and r_9=1, r_{18}=1, r_{24}=4	$Q_{(sd)}$	0.06	0.11	0.24	0.43	0.60
			F	0.09	0.24	0.42	0.55	0.72
			Q_1	0.01	0.00	0.00	0.03	0.01
40	0.1	m=36 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1	$Q_{(sd)}$	0.07	0.16	0.34	0.55	0.75
			F	0.13	0.30	0.50	0.68	0.80
			Q_1	0.01	0.01	0.00	0.01	0.01
	0.2	m=32 and r_9=1, r_{18}=1, r_{27}=1, r_{32}=5	$Q_{(sd)}$	0.07	0.15	0.33	0.54	0.74
			F	0.12	0.28	0.48	0.67	0.79
			Q_1	0.00	0.00	0.00	0.01	0.01
50	0.1	m=45 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1	$Q_{(sd)}$	0.07	0.19	0.40	0.64	0.85
			F	0.14	0.33	0.55	0.72	0.84
			Q_1	0.01	0.01	0.01	0.01	0.02
	0.2	m=40 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{40}=6	$Q_{(sd)}$	0.07	0.18	0.39	0.62	0.83
			F	0.13	0.32	0.54	0.70	0.82
			Q_1	0.00	0.01	0.01	0.01	0.02
60	0.1	m=54 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1	$Q_{(sd)}$	0.09	0.24	0.49	0.74	0.90
			F	0.15	0.36	0.56	0.73	0.83
			Q_1	0.02	0.02	0.04	0.09	0.12
	0.2	m=48 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{48}=7	$Q_{(sd)}$	0.09	0.24	0.48	0.73	0.89
			F	0.14	0.35	0.55	0.71	0.82
			Q_1	0.02	0.02	0.04	0.09	0.10
80	0.1	m=72 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{72}=1	$Q_{(sd)}$	0.09	0.28	0.60	0.85	0.97
			F	0.17	0.40	0.64	0.80	0.90
			Q_1	0.01	0.02	0.05	0.08	0.09
	0.2	m=64 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{64}=9	$Q_{(sd)}$	0.09	0.28	0.59	0.84	0.96
			F	0.16	0.39	0.62	0.79	0.89
			Q_1	0.01	0.01	0.02	0.04	0.07

4.4 Tests for Shape Parameter of bathtub Distribution

Consider a ordered sample $X_1, X_2, \dots, X_n$ from the bathtub distribution defined by the pdf.

$$f(x;\theta,\beta) = \theta\beta x^{\beta-1}\left[1+x^{\beta}\right]^{-(\theta+1)};\ \ x>0,\ \ \theta,\beta>0;$$

where, β and θ are shape and scale parameters respectively. Here, we are interested to test $H_0\!:\beta=\beta_0$ against $H_1\!:\beta\neq\beta_0$ by treating θ as a nuisance parameter. Using $d(X_i,\beta)$=($e^{x_i^{\beta}}-1$) given in (2.5), $T=\sum_{i=1}^{m}(1+r_i)(e^{x_i^{\beta}}-1)$is a complete sufficient statistic for θ. Hence the test statistics based on standardized version of the statistic Q under H_0 is given by

$$Q_{(sd)} = \sqrt{\frac{(m+2)(m+3)}{m-1}}\left[\frac{(m+1)\{\sum_{i=1}^{m}(n-i+1-r_1-r_2\dots-r_{i-1})^2[e^{X_i^{\beta_0}}-e^{X_{i-1}^{\beta_0}}]^2\}}{2[\sum_{i=1}^{m}(1+r_i)(e^{X_i^{\beta_0}}-1)]^2}-1\right]$$

Similarly, the other two test statistics under $H_0$are as

$$F=F(\beta_0)=\frac{\sum_{i=1}^{m}(1+r_i)(e^{X_i^{\beta_0}}-1)-n(e^{X_1^{\beta_0}}-1)}{n(m-1)(e^{X_1^{\beta_0}}-1)}$$

$$Q_1=\left[\frac{\sum_{i=1}^{m}(1+r_i)(e^{X_i^{\beta_0}}-1)}{m}\right]^2\left[\sum_{i=1}^{m}\left\{(n-i+1-r_1-r_2\dots-r_{i-1})\left[e^{X_i^{\beta_0}}-e^{X_{i-1}^{\beta_0}}\right]\right\}^2\right]-2\frac{\left\{\sum_{i=1}^{m}(1+r_i)(e^{X_i^{\beta_0}}-1)\right\}^2}{m}+m$$

Suppose that we would like to test $H_0\!:\beta=1$ against $H_1\!:\beta\neq1$ when θ = 1 with the level of significance α = 0.05. We construct progressively type-II censored samples from bathtub distribution with censoring proportion $p=0.1$. The Tables 13 and 14 gives the direction of the test procedures and cut-off points. The powers of test for different values of n and β is given in tables 15 and 16 respectively.

Table 13: Simulated means of the statistics for different values of β for p = 0.1.

n	Test	β=0.4	β=0.6	β=0.8	β=1	β=1.2	β=1.4	β=1.6	β=1.8	β=2
20	$Q_{(sd)}$	5.64	1.61	0.36	0.01	0.03	0.24	0.58	0.98	1.45
	Q_1	4304373.08	153.72	36.47	27.20	28.70	34.67	43.85	55.78	70.24
30	$Q_{(sd)}$	8.84	2.14	0.43	0.00	0.07	0.40	0.89	1.49	2.16

	Q_1	804260.27	139.42	43.46	35.47	39.61	50.01	65.59	86.03	111.16
40	$Q_{(sd)}$	12.35	2.64	0.49	0.01	0.13	0.57	1.22	2.00	2.90
	Q_1	1412239.81	166.57	53.35	44.69	51.26	66.39	89.04	119.08	156.47
50	$Q_{(sd)}$	15.97	3.08	0.54	0.00	0.15	0.67	1.44	2.39	3.48
	Q_1	1829599.55	198.22	62.82	53.38	62.20	81.78	111.26	150.77	200.50
60	$Q_{(sd)}$	19.93	3.51	0.57	-0.01	0.18	0.80	1.70	2.82	4.13
	Q_1	2064955.51	232.50	73.09	62.49	73.58	97.93	134.89	184.92	248.54
80	$Q_{(sd)}$	28.69	4.45	0.71	0.00	0.24	0.99	2.11	3.52	5.18
	Q_1	2047504.59	279.86	90.89	79.20	94.80	128.26	179.51	249.87	340.68

Table 14: The percentage points of the distribution of Test statistics for p = 0.1.

n	Test	1%	5%	95%	99%
20	$Q_{(sd)}$	-1.4738	-1.2002	1.8427	3.3046
	Q_1	5.3258	6.7093	90.0191	191.3381
30	$Q_{(sd)}$	-1.5838	-1.2486	1.8274	3.3463
	Q_1	9.0164	10.6718	102.9052	179.4333
40	$Q_{(sd)}$	-1.5984	-1.2773	1.8074	3.1959
	Q_1	13.2257	15.1679	118.2104	195.5479
50	$Q_{(sd)}$	-1.6560	-1.3006	1.8269	3.0983
	Q_1	17.3127	20.1142	127.5421	204.7737
60	$Q_{(sd)}$	-1.6923	-1.3084	1.8276	3.0453
	Q_1	21.7082	24.8927	145.2131	219.1663
80	$Q_{(sd)}$	-1.7486	-1.3101	1.7540	3.0129
	Q_1	31.1924	35.6043	164.7771	231.1849

The critical values $F_{.025}(2(\mathrm{m}-1),2)$ and $F_{.975}(2(\mathrm{m}-1),2)$ are used to compute power of test F.

Table 15: Power of Tests for $\beta < 1$ alternatives at α = 0.05.

n	Progressive Censoring	Progressive sampling plan	Tests	β=0.3	β=0.4	β=0.6	β=0.8
20	0.1	m=18 and r_9=1, r_{18}=1	$Q_{(sd)}$	0.90	0.74	0.32	0.11
			F	0.95	0.78	0.31	0.10
			Q_1	0.51	0.38	0.19	0.09
	0.2	m=16 and r_9=1, r_{16}=3	$Q_{(sd)}$	0.85	0.69	0.30	0.10
			F	0.90	0.74	0.29	0.08
			Q_1	0.45	0.34	0.17	0.08
30	0.1	m=27 and r_9=1, r_{18}=1, r_{27}=1	$Q_{(sd)}$	0.97	0.87	0.42	0.12
			F	0.99	0.86	0.35	0.10
			Q_1	0.68	0.51	0.23	0.09
	0.2	m=24 and r_9=1, r_{18}=1, r_{24}=4	$Q_{(sd)}$	0.93	0.84	0.38	0.10
			F	0.95	0.83	0.31	0.09
			Q_1	0.64	0.49	0.20	0.08
40	0.1	m=36 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1	$Q_{(sd)}$	0.99	0.94	0.50	0.13
			F	0.99	0.90	0.38	0.10
			Q_1	0.77	0.61	0.27	0.09
	0.2	m=32 and r_9=1, r_{18}=1, r_{27}=1, r_{32}=5	$Q_{(sd)}$	0.96	0.93	0.47	0.12
			F	0.95	0.86	0.35	0.09
			Q_1	0.74	0.58	0.24	0.08
50	0.1	m=45 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1	$Q_{(sd)}$	1.00	0.97	0.57	0.14
			F	1.00	0.93	0.41	0.10
			Q_1	0.85	0.69	0.31	0.10
	0.2	m=40 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{40}=6	$Q_{(sd)}$	1.00	0.94	0.55	0.12
			F	0.98	0.90	0.39	0.10
			Q_1	0.82	0.67	0.30	0.09
60	0.1	m=54 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1	$Q_{(sd)}$	1.00	0.99	0.62	0.14
			F	1.00	0.95	0.43	0.11
			Q_1	0.90	0.75	0.33	0.10
	0.2	m=48 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{48}=7	$Q_{(sd)}$	1.00	0.97	0.60	0.12
			F	0.99	0.92	0.41	0.10
			Q_1	0.86	0.71	0.31	0.09

80	0.1	m=72 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{72}=1	$Q_{(sd)}$	1.00	1.00	0.73	0.18
			F	1.00	0.98	0.47	0.11
			Q_1	0.95	0.83	0.40	0.10
	0.2	m=64 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{64}=9	$Q_{(sd)}$	1.00	0.98	0.69	0.15
			F	1.00	0.94	0.44	0.09
			Q_1	0.91	0.80	0.37	0.09

Table 16: Power of Tests for $\beta > 1$ alternatives at α = 0.05.

n	Censoring Proportion	Progressive sampling plan	Tests	β=1.2	β=1.4	β=1.6	β=1.8	β=2
20	0.1	m=18 and r_9=1, r_{18}=1	$Q_{(sd)}$	0.05	0.09	0.14	0.22	0.32
			F	0.08	0.15	0.26	0.37	0.48
			Q_1	0.05	0.07	0.11	0.17	0.24
	0.2	m=16 and r_9=1, r_{16}=3	$Q_{(sd)}$	0.05	0.08	0.12	0.20	0.29
			F	0.07	0.14	0.24	0.35	0.45
			Q_1	0.05	0.06	0.10	0.16	0.22
30	0.1	m=27 and r_9=1, r_{18}=1, r_{27}=1	$Q_{(sd)}$	0.06	0.11	0.20	0.33	0.47
			F	0.09	0.19	0.33	0.46	0.58
			Q_1	0.06	0.09	0.17	0.27	0.42
	0.2	m=24 and r_9=1, r_{18}=1, r_{24}=4	$Q_{(sd)}$	0.05	0.10	0.18	0.30	0.43
			F	0.08	0.17	0.30	0.43	0.53
			Q_1	0.06	0.08	0.15	0.25	0.38
40	0.1	m=36 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1	$Q_{(sd)}$	0.07	0.15	0.28	0.44	0.61
			F	0.11	0.23	0.38	0.53	0.67
			Q_1	0.06	0.12	0.22	0.39	0.57
	0.2	m=32 and r_9=1, r_{18}=1, r_{27}=1, r_{32}=5	$Q_{(sd)}$	0.06	0.13	0.25	0.41	0.55
			F	0.10	0.20	0.34	0.48	0.62
			Q_1	0.06	0.10	0.19	0.35	0.52
50	0.1	m=45 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1	$Q_{(sd)}$	0.07	0.16	0.32	0.52	0.71
			F	0.11	0.26	0.42	0.59	0.71
			Q_1	0.07	0.15	0.30	0.51	0.72
	0.2	m=40 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1,	$Q_{(sd)}$	0.06	0.14	0.29	0.48	0.66
			F	0.10	0.24	0.39	0.54	0.67

		r_{40}=6	Q_1	0.06	0.13	0.28	0.47	0.66
60	0.1	m=54 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1	$Q_{(sd)}$	0.08	0.18	0.38	0.59	0.79
			F	0.13	0.29	0.48	0.63	0.76
			Q_1	0.07	0.16	0.34	0.58	0.79
	0.2	m=48 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{48}=7	$Q_{(sd)}$	0.07	0.16	0.35	0.55	0.74
			F	0.11	0.27	0.45	0.60	0.70
			Q_1	0.06	0.15	0.31	0.54	0.74
80	0.1	m=72 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{72}=1	$Q_{(sd)}$	0.09	0.23	0.50	0.74	0.90
			F	0.14	0.32	0.52	0.68	0.80
			Q_1	0.08	0.22	0.48	0.75	0.92
	0.2	m=64 and r_9=1, r_{18}=1, r_{27}=1, r_{36}=1, r_{45}=1, r_{54}=1, r_{63}=1, r_{64}=9	$Q_{(sd)}$	0.08	0.21	0.46	0.70	0.87
			F	0.12	0.29	0.47	0.64	0.76
			Q_1	0.07	0.20	0.45	0.71	0.86

5. Conclusions

From the simulated power study, it is observed that among the proposed tests $Q_{(sd)}$, Q_1 and F for testing $H_0: \beta = \beta_0$ against $H_1: \beta \neq \beta_0$,

a) For testing shape parameter of Gompertz distribution, the tests $Q_{(sd)}$, and F performs well in identifying $\beta < \beta_0$ alternatives whereas test Q_1performs better than other tests F and $Q_{(sd)}$ in identifying $\beta > \beta_0$ alternatives. Therefore the test Q_1 is recommended for testing $\mathrm{H}_0: \beta = \beta_0$ against $\mathrm{H}_1: \beta > \beta_0$ and the tests $\mathrm{Q}_{(\mathrm{sd})}$ and F are recommended for testing $\mathrm{H}_0: \beta = \beta_0$ against $\mathrm{H}_1: \beta < \beta_0$ under progressively type-II censored sample.
b) For testing shape parameter of Weibull distribution, the tests $Q_{(sd)}$, and F performs well in identifying $\beta < \beta_0$ alternatives whereas test F performs better than other tests $Q_{(sd)}$ and Q_1 in identifying $\beta > \beta_0$ alternative. Therefore the test Q_1 is recommended for testing $\mathrm{H}_0: \beta = \beta_0$ against $\mathrm{H}_1: \beta > \beta_0$ and the tests $\mathrm{Q}_{(\mathrm{sd})}$ and F to be used for testing $\mathrm{H}_0: \beta = \beta_0$ against $\mathrm{H}_1: \beta < \beta_0$ under progressively type-II censored sample.
c) For testing shape parameter of Burr XII distribution, the tests Q_1, and F performs well in identifying $\beta < \beta_0$ alternatives whereas test F performs better than other tests $Q_{(sd)}$ and Q_1 in identifying $\beta > \beta_0$ alternatives in case of small sample and $Q_{(sd)}$ performs well in case of large sample. Therefore the test F is recommended for testing $\mathrm{H}_0: \beta = \beta_0$ against $\mathrm{H}_1: \beta > \beta_0$ and the tests Q_1 and F be used for testing $\mathrm{H}_0: \beta = \beta_0$ against $\mathrm{H}_1: \beta < \beta_0$ under progressively type-II censored sample.

d) For testing shape parameter of bathtub distribution, the tests $Q_{(sd)}$, and F performs well in identifying $\beta < \beta_0$ alternatives whereas test F performs better than other tests $Q_{(sd)}$ and Q_1 in identifying $\beta > \beta_0$ alternatives in case of small sample and tests $Q_{(sd)}$, Q_1 performs well in case of large sample. Therefore the test F is recommended for testing $\mathrm{H}_0\colon \beta = \beta_0$ against $\mathrm{H}_1\colon \beta > \beta_0$ and the tests $Q_{(sd)}$ and F be used for testing $\mathrm{H}_0\colon \beta = \beta_0$ against $\mathrm{H}_1\colon \beta < \beta_0$ under progressively type-II censored sample.

Thus, we recommend the test based on quadratic form i.e. test $Q_{(sd)}$ and test Q_1 over the test F in identifying either $\beta > \beta_0$ or $\beta < \beta_0$ alternatives for some member of the family of distributions.

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