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On Smoothed Nonparametric Estimation of Mixing Proportion under Fixed Design Regression Model

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ABSTRACT

In the present paper, the smoothed estimators are proposed for estimating mixing proportion in a mixed model based on independent and not identically distributed (non-iid) random samples of the existing estimators proposed by Boes(1966) - James(1978) called BJ estimator here and which was constructed for estimating mixing proportion based on independent and identically distribution random samples. The proposed smoothed estimators are based on known "kernel function" as described in the introduction. The following results of the smoothed estimators are studied under the non-iid setup such as (a) its small sample behavior is compared with the unsmoothed version (BJ estimator) based on their mean square errors (MSEs) by using Monte-Carlo simulation and established the percentage gain in precision of smoothed estimator over its unsmoothed version measured interms of their mse. (b) its large sample properties such as almost surely (a.s.) convergence and asymptotic normality of these estimators are established in the present work.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and not identically distributed (non-iid) random variables with continuous distribution functions $\{F_i(x), 1 \leq i \leq n\}$ and let H(x) be a continuous cdf of mixture of component cdfs $H_1(x), \ldots, H_m(x)$ ($m \geq 2$) such that

$$H(x) = \sum_{j=1}^{m} p_j H_j(x) \qquad (1.1)$$

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where $\{p_j;\ 1\leq j\leq m\}$ is a set of mixing proportions satisfying (i) $0<p_j<1$ (ii) $\sum_{j=1}^{m} p_j = 1$. Let $\bar{H}_n(x) = n^{-1}\sum_{i=1}^{n} F_i(x) \rightarrow H(x)$ as $n \rightarrow\infty$ and $\bar{H}_j(x) = n_j^{-1}\sum_{i=1}^{n_j} F_{ji}(x) \rightarrow H_j(x)$, $n_j\rightarrow\infty$, $j = 1, 2,\ldots,m$, $H(x)$, $H_j(x)$ are unknown distribution functions. The problem of nonparametric estimation of mixing proportions p_j in a mixture (1.1) of m=2 unknown distributions $H_j(x)$ are investigated based on independent random samples of sizes n, n_j generated from the fixed design regression models,

$$X_i = \beta t_i + \epsilon_i\ ,\ 1\leq i\leq n,\ \epsilon_i \sim \text{i.i.d.} F(x) \tag{1.2}$$

$$X_{ji} = \beta_j t_{ji} + \epsilon_{ji},\ 1\leq i\leq n_j,\ j = 1, 2,\ldots,m,\ \epsilon_{ij} \sim \text{i.i.d.} F_j(x) \tag{1.3}$$

where β's and t's are known reals satisfying the model conditions

$$\beta_j > 0,\ \sum_{i=1}^{n} t_i = 0 \text{ and } \frac{1}{n}\sum t_i^2 = o(n^{-1}) \tag{1.4}$$

Note that, t_i is the known real value of i^{th} observation and is taken as $t_i = \frac{i}{n^{\delta}}$, $i = \mp1, \mp2,\ldots, \mp n$, $\delta\geq\frac{3}{2}$ fulfill (1.4) and $\bar{H}_n(x) = n^{-1}\sum_{i=1}^{n} F_i(x) = F(x) + O(\frac{1}{n}\sum t_i^2) + \ldots = H(x) + o(n^{-1})$ and $\bar{H}_{n_j}(x) = n_j^{-1}\sum_{i=1}^{n_j} F_{ji}(x) \rightarrow H_j(x)$, j=1,2 as $n_j\rightarrow\infty$.

The applications where finite mixture distributions describe mixture populations for non-i.i.d. sequence of variables are given below:

Area	Characteristic X_{ji}	Distribution function $F_{ji}(x)$
Survival Analysis	life time of components produced by i^{th} machine operated by j^{th} foreman.	Life time distribution of various products
Nutritional Studies	weight for age/height, age/ weight for height of i^{th} infant of j^{th} origin or group.	Distribution of weight for age/height of i^{th} infant
Fisheries	Fish length or weight of i^{th} age of j^{th} fish.	Distribution of weight/length of i^{th} fish
Automobiles	Degree of satisfaction i^{th} customer due to j^{th} service type	Distribution of degree of customer satisfaction
Hospitals	Time taken for treatment of i^{th} patient by j^{th} type of treatment	Distribution of time of treatment of different patients

Frequently, the collected data are not from a randomly selected sample, but rather from patients, customers or other objects as they come in for service during a certain period of time. These samples are non-i.i.d. For instance, suppose a car dealer has collected data on the degree of customer's satisfaction and on the period of trouble-free time from all his customers for a given year. Obviously, the dealer has ignored several variables on the customer side, such as the driving habit. Other such non-i.i.d. examples are: Data collected from patients over in a period of time, patrons of a particular restaurant, and viewers of a certain show, etc.

In these examples the average population is not a well-defined fixed population but, for a large sample size n, it can be viewed as the representative population pertaining to an 'average' patient or customer. We present here another example of a non-i.i.d. situation in sample surveys where the average population is actually the population of interest. Consider a survey designed to obtain the national average such as mean/median of a variable X such as the real-estate price, annual income, auto insurance charge, etc. For logistic reasons, suppose the data are collected state or country-wise and are put together to estimate the national average. If the regional sample sizes are proportional to the corresponding regional population sizes, the average population corresponding to the combined sample is exactly the same as the national population average.

The mixing model with two component populations becomes,

$$H(x) = pH_1(x) + (1-p)H_2(x)$$

Here X_i is the characteristic with distribution function $F_i(x)$ assuming

$$\bar{F}_n(x) = n^{-1}\sum_{i=1}^{n} F_i(x) = \bar{H}_n(x) \rightarrow H(x)$$
$$\bar{F}_{jn}(x) = n^{-1}\sum_{i=1}^{n} F_{ji}(x) = \bar{H}_{jn}(x) \rightarrow H_j(x) \quad \text{as } n\rightarrow\infty$$

and according to Hosmer(1973) model I, n, n_1 and n_2 are independent random sample sizes from mixed and component populations selected in such a way that $n = n_1 + n_2$ with $n_1 = [\rho n]$, $n_2 = [(1-\rho)n]$, $0< \rho <1$.

For more details of such examples reader is referred to Choi and Bulgren(1968), Harris(1958), Blischke(1965), Fu(1968-Pattern Recognition), Vardi et al (1985-Pattern Recognition), Clark(1976-Geology), Macdonald and Pitcher(1979-Fisheries), Odell and Basu (1976-Remote sensing), Bruni et al(1983-Genetics), Merz(1980-Physics) and Christensen et al(1980-Nuclear Physics) etc.

i.i.d case:

The mixing proportion model for iid case is

$$F(x) = pG_1(x)+(1-p)G_2(x) \tag{1.5}$$

where F(x), G_j(x); j=1,2 are cdfs of mixed and component populations respectively. The following estimator is studied in the literature.

Boes-James (BJ) estimator: Let F_n(x)= n^{-1} $\sum_{i=1}^{n} I(X_i \leq x)$ be the empirical distribution function of a random sample X_i, 1≤i≤n from a mixture (1.5) of two known component d.f.s G_j; j=1,2 as Boes (1966) proposed estimator

$$p_{n,1}(x) = \frac{\tilde{F}_n(x) - G_2(x)}{G_1(x) - G_2(x)} \qquad (1.6)$$

and shown it is a minimax unbiased estimator and derived the Cramer-Rao lower bound. James(1978) considered the problem of estimating the mixing proportion in a mixture of two known normal distributions. He studied the simple estimators based on (a) the number of observations less than a fixed point r, (b) the numbers less than s and greater than t, and (c) the sample mean. Van Houwelingen(1987) used Boes estimator to estimate the mixing proportion by using frequency densities and obtained the Cramer-Rao lower bound. Jayalakshmi(2002) used Boes-James estimator of Hosmer(1973) model I sampling structure based on kernel based empirical distribution function $\hat{F}_n$(x) = n^{-1} $\sum_{i=1}^{n} K(\frac{x - X_i}{a_n})$ and established smoothing improves efficiency when the components are known.

As pointed out in Hall (1981), methods based on nonparametric density estimators involves some significant draw backs in their use in the field of estimation from mixed data such as

- specification of window width in kernel based estimators and their behavior is very sensitive to the choice of window width parameter and
- their mean square errors converge at a slower rate than order n^{-1} .

To avoid these draw backs, Hall (1981) proposed nonparametric estimators of mixing proportions in a finite mixture based on the usual empirical distribution functions.

In the present work, we propose new kernel based estimators, called smoothed nonparametric estimators of mixing proportion p in a mixture of two unknown component distribution (1.1) and established their superiority over those estimators, called the unsmoothed ones based on the usual empirical distribution function. Further, in order to overcome the above drawbacks

- identify a method of ‘optimal’ choice of band width, which is crucial, in the sense of minimum mean square error of such smoothed nonparametric estimators and

- establish the convergence of mean square errors of proposed smoothed estimators at the fastest rate of order n^{-1}.

we extend the idea of estimation of mixing proportion p in two directions:

- Component distributions are completely unknown and estimators thus proposed are nonparametric in true sense.
- The proposed nonparametric estimators are based on independent, but not identically distributed samples generated by the fixed design regression models described by (1.2)-(1.4).

The main object of the present paper is to confine attention to m=2 case in the model (1.1) and to construct nonparametric estimators based on the usual empirical and kernel based empirical distribution functions defined by

$$\widetilde{H}_n(x) = n^{-1}\sum_{i=1}^{n} I(X_i \le x),\ \widetilde{H}_j(x) = n_j^{-1}\sum_{i=1}^{n_j} I(X_{ji} \le x) \qquad (1.7)$$

$$\widehat{H}_n(x) = n^{-1}\sum_{i=1}^{n} K\left(\frac{x - X_i}{a_n}\right),\ \widehat{H}_j(x) = n_j^{-1}\sum_{i=1}^{n_j} K\left(\frac{x - X_{ji}}{a_n}\right),\ j = 1,2 \qquad (1.8)$$

$\{a_n\}$ being the smoothing sequence satisfying $0 < a_n \to 0$, $na_n \to \infty$ and proposed as follows:

A. Unsmoothed estimators of p: The proposed BJ type unsmoothed estimators of p based on the empirical distribution functions are defined, for fixed $x=x_0$, as

$$p_{n,1}(x_0) = \frac{\widetilde{H}_n(x_0) - H_2(x_0)}{H_1(x_0) - H_2(x_0)}, \qquad \tilde{p}_{n,1}(x_0) = \frac{\widetilde{H}_n(x_0) - \widetilde{H}_2(x_0)}{\widetilde{H}_1(x_0) - \widetilde{H}_2(x_0)} \qquad (1.9)$$

B. Smoothed estimators of p: The proposed BJ type smoothed estimators of p based on the kernel functions are defined, for fixed $x=x_0$, as

$$p_{n,2}(x_0) = \frac{\widehat{H}_n(x_0) - H_2(x_0)}{H_1(x_0) - H_2(x_0)}, \qquad \hat{p}_{n,2}(x_0) = \frac{\widehat{H}_n(x_0) - \widehat{H}_2(x_0)}{\widehat{H}_1(x_0) - \widehat{H}_2(x_0)} \qquad (1.10)$$

and study the small sample as well as large sample behavior of proposed nonparametric estimators. The results of the present investigations for the non-i.i.d. sequences are completely new in the literature, even in i.i.d.case as well.

In section 2, the asymptotics of certain empirical distribution functions defined therein are established which are utilized in later sections. Further, In section 3, a.s. representations of both nonparametric estimators of p and the main results concerning the asymptotics of present paper such as i) exact mean square errors(MSEs) ii) rates of a.s. convergence and iii) asymptotic normality of the nonparametric estimators are established. In section 4, the crucial choice of smoothing parameter 'a_n' in kernel based estimator $\hat{p}_{n,2}$(x) is discussed and its value is determined by employing minimum mean square criterion. Comparisons of both estimators based on their MSEs are made and established the superiority of smoothed one over the unsmoothed version in section 5. The simulation work is undertaken by Monte Carlo methods, established superiority of smoothed estimators empirically over unsmoothed ones in section 6 and comments are appended in section 7.

2. Asymptotics of Certain Empirical Functions

In the present section, we study the asymptotic behavior of certain empirical functions which are very much used in establishing the main results of the present paper. Firstly, define

$$\hat{\lambda}_n(x) = \hat{H}_n(x) - E\,\hat{H}_n(x) = n^{-1} \sum_{i=1}^{n} [K(\frac{x - X_i}{a_n}) - E\,K(\frac{x - X_i}{a_n})]$$

$$= n^{-1} \sum_{i=1}^{n} Z_{2i}(x) \qquad (2.1)$$

and study its asymptotics in the following.

Lemma 2.1: Assume the conditions on $\{F_i\}$, the kernel function k and the bandwidth sequence $\{a_n\}$ given below:

AI: i) $F_i(x)$ is uniformly continuous distribution function with finite q^{th} derivatives $F_i^{(q)}(x) < \infty$, $1 \le i \le n$ and $\bar{H}_n^{(q)}(x) = \frac{1}{n} \sum F_i^{(q)}(x)$, q = 2,4,6

ii) $\bar{H}_n(x) = n^{-1}\sum_{i=1}^{n} F_i(x) \to H(x)$ as $n \to \infty$

AII: i) The kernel function satisfies $\mu_{2j}(K) = \int_{-\infty}^{\infty} t^{2j}\, dK(t) \neq 0$ and $\mu_j(K) = \int_{-\infty}^{\infty} t^{j}\, dK(t) = 0$ for j=1,3,5…

ii) $\psi_j(K) = 2 \int_{-\infty}^{\infty} t^{j}\, K(t)dK(t) < \infty$, j = 0, 1, 2, 3

AIII: $\{a_n\}$ is a sequence of bandwidths such that

i) $0 < a_n \downarrow 0$; $na_n \to \infty$ as $n \to \infty$

ii) $na_n^4 \to 0$ as $n \to \infty$

then

a) $\Lambda_n(x) = Var\, \hat{\lambda}_n(x) = \lambda_n(x) - \frac{V_{nF}(x)}{n} - \frac{a_n}{n} \bar{H}_n^{(1)}(x)\, \psi_1(K) + O(\frac{a_n^2}{n})$

b) $b_n(x) = E\,\hat{H}_n(x) - \bar{H}_n(x) = \frac{a_n^2}{2!} \bar{H}_n^{(2)}(x)\mu_2(K) + \frac{a_n^4}{4!} \bar{H}_n^{(4)}(x)\mu_4(K) + o(a_n^4)$

c) $\hat{\lambda}_n(x) = O(\frac{\log n}{n})^{\frac{1}{2}}$ a.s.

where $\lambda_n(x) = \frac{\bar{H}_n(x)(1 - \bar{H}_n(x))}{n}$, $V_{nF}(x) = n^{-1} \sum_{i=1}^{n} (F_i(x) - \bar{H}_n(x))^2 > 0$

Proof: From (2.1),

$$n\Lambda_n(x) = n\mathrm{Var}\,\hat{\lambda}_n(x) = \mathrm{Var}(n^{-1/2}\sum_{i=1}^{n} Z_{2i}(x))$$

$$= n^{-1}\sum_{i=1}^{n} E[\{K(\frac{x-X_i}{a_n}) - E\,K(\frac{x-X_i}{a_n})\}]^2$$

$$= n^{-1}\sum_{i=1}^{n}\{E\,K^2(\frac{x-X_i}{a_n}) - E^2\,K(\frac{x-X_i}{a_n})\}$$

$$= n^{-1}\sum_{i=1}^{n}[I_{1i}(x) - I_{2i}^2(x)] \qquad (2.2)$$

$$I_{1i}(x) = E\,K^2(\frac{x-X_i}{a_n}) = \int K^2(\frac{x-u}{a_n})\,dF_i(u) = \int F_i(x-a_nt)dK^2(t)$$

$$= F_i(x)\int dK^2(t) - F_i^{(1)}(x)a_n\int t dK^2(t) + \frac{a_n^2}{2!}F_i^{(2)}(x)\int t^2 dK^2(t)$$

$$- \frac{a_n^3}{3!}F_i^{(3)}(x)\int t^3 dK^2(t) + \frac{a_n^4}{4!}F_i^{(4)}(x)\int t^4 dK^2(t) + o(a_n^4)$$

$$= F_i(x)\,\psi_0(k) - a_n F_i^{(1)}(x)\,\psi_1(K) + \frac{a_n^2}{2!}F_i^{(2)}(x)\,\psi_2(K) - \frac{a_n^3}{3!}F_i^{(3)}(x)$$

$$\psi_3(K) + \frac{a_n^4}{4!}F_i^{(4)}(x)\,\psi_4(K) + o(a_n^4) \qquad (2.3)$$

where $\psi_o(K) = 2\int_{-\infty}^{\infty} K(t)\,dK(t) = 2\int_0^1 y\,dy = 1$ while

$$I_{2i}(x) = E\,K(\frac{x-X_i}{a_n}) = \int F_i(x - a_nt)\,dK(t)$$

$$= \{F_i(x) + \frac{a_n^2}{2!}F_i^{(2)}(x)\int_{-\infty}^{\infty} t^2 dK(t) + \frac{a_n^4}{4!}F_i^{(4)}(x)\int_{-\infty}^{\infty} t^4\,dK(t) +$$

$$o(a_n^4)\} \quad = F_i(x) + \frac{a_n^2}{2}F_i^{(2)}(x)\,\mu_2(K) + \frac{a_n^4}{4!}F_i^{(4)}(x)\,\mu_4(K) + o(a_n^4) \qquad (2.4)$$

From (2.3) and (2.4),(2.2) becomes,

$$n\Lambda_n(x) = n^{-1}\sum_{i=1}^{n}\{[F_i(x) - a_n F_i^{(1)}(x)\psi_1(K) + \frac{a_n^2}{2}F_i^{(2)}(x)\psi_2(K) - \frac{a_n^3}{3!}$$

$$F_i^{(3)}(x)\psi_3(K) + \frac{a_n^4}{4!}F_i^{(4)}(x)\psi_4(K) + o(a_n^4)]$$

$$- (F_i(x) + \frac{a_n^2}{2}F_i^{(2)}(x)\,\mu_2(K) + \frac{a_n^4}{4!}F_i^{(4)}(x)\,\mu_4(K) + o(a_n^4))^2\}$$

$$= n^{-1}\sum_{i=1}^{n} F_i(x)(1 - F_i(x)) - a_n\bar{H}_n^{(1)}(x)\,\psi_1(K)$$

$$+ a_n^2\{\frac{1}{2}\bar{H}_n^{(2)}(x)\,\psi_2(K) - n^{-1}\sum_{i=1}^{n} F_i(x)F_i^{(2)}(x)\mu_2(K)\} +$$

$$\frac{a_n^3}{3!}\bar{H}_n^{(3)}(x)\psi_3(K)$$

$$- a_n^4\{\frac{1}{4}\bar{H}_i^{(2)^2}(x)\mu_2^2(K) - \frac{1}{4!}\bar{H}_n^{(4)}(x)\psi_4(K)\} + o(a_n^4/n)\}$$

where $n^{-1}\sum_{i=1}^{n} F_i(x)(1 - F_i(x)) = \bar{F}_n(x) - \bar{F}_n^2(x) - (n^{-1}\sum_{i=1}^{n} F_i^2(x) - \bar{F}_n^2(x)) =$ $\bar{H}_n(x)(1-\bar{H}_n(x)) - V_{nF}(x)$

so that

$$\Lambda_n(x) = \frac{\bar{H}_n(x)(1-\bar{H}_n(x)) - V_{nF}(x)}{n} - \frac{a_n}{n}\bar{H}_n^{(1)}(x)\psi_1(K) + \frac{a_n^2}{n}\eta_{n2} + o(a_n^2/n) \quad (2.5)$$

where $\eta_{n2} = \frac{1}{2}\bar{H}_n^{(2)}(x)\,\psi_2(K) - n^{-1}\sum_{i=1}^{n} F_i(x)F_i^{(2)}(x)\mu_2(K)$

$$\Lambda_n(x) = E\,\hat{\lambda}_n^2(x) = \lambda_n(x) - \frac{V_{nF}(x)}{n} - \frac{a_n}{n}\bar{H}_n^{(1)}(x)\psi_1(K) + O(\frac{a_n^2}{n})$$

which proves part a. Further to prove part b, by definition

$$b_n(x) = E\,\hat{H}_n(x) - \bar{H}_n(x) = \frac{1}{n}\sum E\,K\left(\frac{x-X_i}{a_n}\right) - \bar{H}_n(x)$$

$$= \frac{1}{n}\sum \int F_i(x - a_n t)\,dK(t) - \bar{H}_n(x)$$

$$= \frac{1}{n}\sum \{F_i(x) + \frac{a_n^2}{2!}F_i^{(2)}(x)\int_{-\infty}^{\infty} t^2 dK(t) + \frac{a_n^4}{4!}F_i^{(4)}(x)\int_{-\infty}^{\infty} t^4 dK(t) + o(a_n^4)\} - \bar{H}_n(x)$$

$$= \frac{a_n^2}{2!}\bar{H}_n^{(2)}(x)\mu_2(K) + \frac{a_n^4}{4!}\bar{H}_n^{(4)}(x)\mu_4(K) + o(a_n^4)$$

yielding

$$b_n(x) = \frac{a_n^2}{2!}\bar{H}_n^{(2)}(x)\mu_2(K) + \frac{a_n^4}{4!}\bar{H}_n^{(4)}(x)\mu_4(K) + o(a_n^4) \qquad (2.6)$$

which proves part b.

In order to prove part c relating to the rates of a.s. convergence of $\hat{\lambda}_n(x)$ as n $\to\infty$, from (2.1) we have,

$$\hat{\lambda}_n(x) = n^{-1}\sum_{i=1}^{n} Z_{2i}(x) = \bar{Z}_{n2}(x)$$

$$\sigma_i^2 = E\,(Z_{2i}^2(x)) = F_i(x)(1-F_i(x)) - a_n F_i^{(1)}(x)\psi_1(K) + O(a_n^2)$$

$$\sigma_{n2}^2 = \frac{1}{n}\sum E\left[Z_{2i}^2\right] = \frac{\sum F_i(x)(1-F_i(x))}{n} - \frac{a_n\psi_1(K)\sum F_i^{(1)}(x)}{n} + O(a_n^2)$$

$$= \bar{H}_n(x)(1-\bar{H}_n(x)) - V_{nF}(x) - a_n\bar{H}_n^{(1)}(x)\psi_1(K) + O(a_n^2) = C < \infty$$

By applying Bernstein (1946) inequality to $\{Z_{2i}(x)\}$ with M = 2,

$$P(n^{-1}\sum_{i=1}^{n} Z_{2i}(x) > t) \le \exp\left(-\frac{\frac{nt^2}{2}}{C + \frac{2}{3}t}\right) \qquad (2.7)$$

By setting $t = \left(\frac{4C\log n}{n}\right)^{\frac{1}{2}}$

$$\text{R.H.S. of (2.7)} = \exp\left[-\frac{\frac{n4C\log n}{2n}}{C + \frac{2}{3}\left(\frac{4C\log n}{n}\right)^{\frac{1}{2}}}\right]$$

$$= \exp\left[-\frac{\frac{n4C\log n}{2n}}{C\left(1 + \frac{2}{3C}\left(\frac{4\log n}{n}\right)^{\frac{1}{2}}\right)}\right]$$

$$= \exp\left[\frac{-2\log n}{1 + \frac{2}{3C}\left(\frac{4\log n}{n}\right)^{\frac{1}{2}}}\right]$$

$$= n^{-2} \quad \text{for sufficiently large n.}$$

$$\Rightarrow \sum_{n\ge1}^{\infty} P(\bar{Z}_{n2}(x) > t) \le \sum_{n\ge1}^{\infty} n^{-2} < \infty$$

By Borel – Cantelli lemma, we conclude that $\bar{Z}_{n2} \overline{\overline{a.s.}}\ O\left(\frac{\log n}{n}\right)^{1/2}$ as n$\to\infty$

$$\hat{\lambda}_n(\text{x}) = \bar{Z}_{n2}(\text{x}) \;\overline{\overline{a.s.}}\; \text{O}(\frac{logn}{n})^{1/2}$$

Similarly, define $\hat{\lambda}_j(\text{x}) = \widehat{\text{H}}_\text{j}(\text{x}) - \text{E}\,\widehat{\text{H}}_\text{j}(\text{x})$, $b_j(\text{x}) = \text{E}\,\widehat{\text{H}}_\text{j}(\text{x}) - \overline{\text{H}}_\text{j}(\text{x})$, j=1,2

Corollary 2.2: Under the conditions of Lemma 2.1,

i. $\Lambda_j(\text{x}) = \text{Var}\,\hat{\lambda}_j(\text{x}) = \lambda_j(\text{x}) - \frac{\text{V}_{\text{jF}}(x)}{n_j} - \frac{a_n}{n_j}\,\overline{H}_j^{(1)}(\text{x})\,\psi_1(\text{K}) + \text{O}(\frac{a_n^2}{n_j})$

ii. $b_j(\text{x}) = \frac{\text{a}_\text{n}^2}{2!}\,\overline{H}_j^{(2)}(\text{x})\mu_2(\text{K}) + \frac{\text{a}_\text{n}^4}{4!}\,\overline{H}_j^{(4)}(\text{x})\mu_4(\text{K}) + \text{o}(a_n^4)$

iii. $b_{n2}(\text{x}) = b_n(\text{x}) - b_2(\text{x}) = \frac{\text{a}_\text{n}^2}{2!}\,[\overline{H}_n^{(2)}(\text{x}) - \overline{H}_2^{(2)}(\text{x})]\mu_2(\text{K}) + \frac{\text{a}_\text{n}^4}{4!}\,[\overline{H}_n^{(4)}(\text{x}) - \overline{H}_2^{(4)}(\text{x})]\mu_4(\text{K}) + \text{o}(a_n^4)$

iv. $b_{12}(\text{x}) = b_1(\text{x}) - b_2(\text{x}) = \text{O}(a_n^2)$

v. $\hat{\lambda}_j(\text{x}) = \text{O}(\frac{\log n}{n_j})^{\frac{1}{2}}$ a.s.

where for j=1,2, $\lambda_j(\text{x}) = \frac{\overline{H}_j(\text{x})\left(1 - \overline{H}_j(\text{x})\right)}{n_j}$, $\text{V}_{\text{jF}}(\text{x}) = n_j^{-1}\sum_{i=1}^{n}(F_{ji}(\text{x}) - \overline{H}_j(\text{x}))^2 > 0$

Proof: Proof follows exactly on the similar line of argument as for Lemma 2.1.

Further, define $\tilde{\lambda}_n(\text{x}) = \widetilde{\text{H}}_\text{n}(\text{x}) - \text{E}\,\widetilde{\text{H}}_\text{n}(\text{x})$, $\tilde{\lambda}_j(\text{x}) = \widetilde{\text{H}}_\text{j}(\text{x}) - \text{E}\,\widetilde{\text{H}}_\text{j}(\text{x})$, j=1,2

Corollary 2.3: Assume the uniform continuity on $\{\text{F}_\text{i}(\text{x})\}$. Then

i. $\text{Var}\,\tilde{\lambda}_n(\text{x}) = \lambda_n(\text{x}) - \frac{\text{V}_{\text{nF}}(x)}{n} = \frac{\text{M}_\text{n}}{n}$

ii. $\text{Var}\,\tilde{\lambda}_j(\text{x}) = \lambda_j(\text{x}) - \frac{\text{V}_{\text{jF}}(x)}{n_j} = \frac{\text{M}_\text{j}}{n_j}$, j = 1,2

iii. $\tilde{\lambda}_n(\text{x}) = \text{O}(\frac{\log n}{n})^{\frac{1}{2}}$ a.s.; $\tilde{\lambda}_j(\text{x}) = \text{O}(\frac{\log n}{n_j})^{\frac{1}{2}}$ a.s.

Proof: The result follows by adopting the similar line of argument as in the proof of Lemma 2.1.

3. A.S. Representations to $\hat{p}_{n,2}(\text{x})$ and $\tilde{p}_{n,1}(\text{x})$

In order to establish the asymptotics of $\hat{p}_{n,2}(\text{x})$ and $\tilde{p}_{n,1}(\text{x})$, first we obtain their a.s. representations in the following results.

Theorem 3.1: Under the conditions of Lemma 2.1, with probability 1,

$$\hat{p}_{n,2}(x) - p = d_{12}^{-1}(x)b_{n2}(x) + \hat{T}_n(x) + \epsilon_n$$

where $\hat{T}_n(x) = d_{12}^{-1}(x)[\hat{\lambda}_n(x) - (1-p)\hat{\lambda}_2(x) - p\hat{\lambda}_1(x)]$, $\epsilon_n = - d_{12}^{-2}(x)[(1-p)\hat{\lambda}_2^2(x) - p\hat{\lambda}_1^2(x)] + O(\frac{\log n}{n})$

$d_{12}(x) = H_1(x) - H_2(x)$ and $b_{n2}(x)$ as defined in Corollary 2.2.

Proof: First consider $\hat{p}_{n,2}(x)$ given in (1.10) and let $D_n(x) = \widehat{H}_1(x) - \widehat{H}_2(x)$

Then, $D_n(x)\hat{p}_{n,2}(x) = \widehat{H}_n(x) - \widehat{H}_2(x)$

$$= \widehat{H}_n(x) - E\,\widehat{H}_n(x) + E\,\widehat{H}_n(x) - \overline{H}_n(x) + \overline{H}_n(x) - H(x)$$
$$+ H(x) - H_2(x) + H_2(x) - \overline{H}_2(x) + \overline{H}_2(x) - E\,\widehat{H}_2(x) + E\,\widehat{H}_2(x) - \widehat{H}_2(x)$$
$$= \hat{\lambda}_n(x) - \hat{\lambda}_2(x) + b_n(x) + \tau_n(x) + p d_{12}(x) - \tau_2(x) - b_2(x)$$
$$= \hat{\lambda}_n(x) - \hat{\lambda}_2(x) + b_{n2}(x) + \tau_{n2}(x) + p d_{12}(x)$$
$$=: \hat{\lambda}_n(x) - \hat{\lambda}_2(x) + C_n(x) \quad (3.1)$$

where $\tau_n(x) = \overline{H}_n(x) - H(x)$, $\tau_j(x) = \overline{H}_j(x) - H_j(x)$; j=1,2, $\tau_{n2}(x) = \tau_n(x) - \tau_2(x)$, $d_{12}(x) = H_1(x) - H_2(x)$, $C_n(x) = b_{n2}(x) + \tau_{n2}(x) + p d_{12}(x)$ and further, we have as in (3.1)

$$D_n(x) = \widehat{H}_1(x) - \widehat{H}_2(x)$$
$$= \widehat{H}_1(x) - E\,\widehat{H}_1(x) + E\,\widehat{H}_1(x) - \overline{H}_1(x) + \overline{H}_1(x) - H_1(x) + H_1(x) - H_2(x) + H_2(x) - \overline{H}_2(x)$$
$$+ \overline{H}_2(x) - E\,\widehat{H}_2(x) + E\,\widehat{H}_2(x) - \widehat{H}_2(x)$$
$$= \hat{\lambda}_1(x) - \hat{\lambda}_2(x) + b_1(x) + \tau_1(x) + d_{12}(x) - \tau_2(x) - b_2(x)$$
$$= \hat{\lambda}_1(x) - \hat{\lambda}_2(x) + b_{12}(x) + \tau_{12}(x) + d_{12}(x)$$
$$=: \hat{\lambda}_1(x) - \hat{\lambda}_2(x) + D_{12}(x) \quad (3.2)$$

where $b_{12}(x) = b_1(x) - b_2(x)$, $\tau_{12}(x) = \tau_1(x) - \tau_2(x)$, $D_{12}(x) = b_{12}(x) + \tau_{12}(x) + d_{12}(x) = d_{12}(x) + O(a_n^2)$

From (3.1) and (3.2),

$$\hat{p}_{n,2}(x) = D_n^{-1}(x)(\hat{\lambda}_n(x) - \hat{\lambda}_2(x) + C_n(x))$$
$$= (\hat{\lambda}_n(x) - \hat{\lambda}_2(x) + C_n(x))[\hat{\lambda}_1(x) - \hat{\lambda}_2(x) + D_{12}(x)]^{-1}$$
$$= D_{12}^{-1}(x)[\hat{\lambda}_n(x) - \hat{\lambda}_2(x) + C_n(x)][1 - \frac{\hat{\lambda}_2(x) - \hat{\lambda}_1(x)}{D_{12}(x)}]^{-1} \quad (3.3)$$
$$= D_{12}^{-1}(x)[\hat{\lambda}_n(x) - \hat{\lambda}_2(x) + C_n(x)]\,[1 + \frac{\hat{\lambda}_2(x) - \hat{\lambda}_1(x)}{D_{12}(x)} + \frac{(\hat{\lambda}_2(x) - \hat{\lambda}_1(x))^2}{D_{12}^2(x)} + \ldots\ldots]$$
$$= T_{n1}(x) + T_{n2}(x) + T_{n3}(x) \quad (3.4)$$

From (3.1) and the regression condition (1.4)

$$T_{n1}(\text{x}) = \text{d}_{12}{}^{-1}(\text{x})[\hat{\lambda}_n(\text{x}) - \hat{\lambda}_2(\text{x}) + b_{n2}(\text{x}) + \text{p}d_{12}(\text{x}) + \text{o}(n^{-1})]$$
$$= \text{p} + \frac{b_{n2}(\text{x})}{d_{12}(\text{x})} + \frac{\hat{\lambda}_n(\text{x}) - \hat{\lambda}_2(\text{x})}{d_{12}(\text{x})} + \text{o}(n^{-1})$$
$$T_{n2}(\text{x}) = \text{d}_{12}{}^{-2}(\text{x})[\hat{\lambda}_n(\text{x}) - \hat{\lambda}_2(\text{x}) + b_{n2}(\text{x}) + \text{p}d_{12}(\text{x}) + \text{o}(n^{-1})](\hat{\lambda}_2(\text{x}) - \hat{\lambda}_1(\text{x}))$$
$$= \frac{p(\hat{\lambda}_2(\text{x}) - \hat{\lambda}_1(\text{x}))}{d_{12}(\text{x})} + \text{d}_{12}{}^{-2}(\text{x})[b_{n2}(\text{x})(\hat{\lambda}_2(\text{x}) - \hat{\lambda}_1(\text{x}))]$$
$$+ \text{d}_{12}{}^{-2}(\text{x})[\hat{\lambda}_n(\text{x})\hat{\lambda}_2(\text{x}) - \hat{\lambda}_n(\text{x})\hat{\lambda}_1(\text{x}) - \hat{\lambda}_2^2(\text{x}) + \hat{\lambda}_2(\text{x})\,\hat{\lambda}_1(\text{x})]$$
$$= \frac{p(\hat{\lambda}_2(\text{x}) - \hat{\lambda}_1(\text{x}))}{d_{12}(\text{x})} - \text{d}_{12}{}^{-2}(\text{x})\hat{\lambda}_2^2(\text{x}) + \epsilon_{n1}$$

where in view of Corollary 2.2,

$$\epsilon_{n1} = \text{d}_{12}{}^{-2}(\text{x})[b_{n2}(\text{x})(\hat{\lambda}_2(\text{x}) - \hat{\lambda}_1(\text{x})) + \hat{\lambda}_n(\text{x})\hat{\lambda}_2(\text{x}) - \hat{\lambda}_n(\text{x})\hat{\lambda}_1(\text{x}) + \hat{\lambda}_2(\text{x})\hat{\lambda}_1(\text{x})]$$
$$= \text{O}(a_n^2(\frac{\log n}{n})^{1/2} \vee \frac{\log n}{n}) \text{ a.s.}$$
$$T_{n3}(\text{x}) = \text{d}_{12}{}^{-3}(\text{x})[\hat{\lambda}_n(\text{x}) - \hat{\lambda}_2(\text{x}) + b_{n2}(\text{x}) + \text{p}d_{12}(\text{x}) + \text{o}(n^{-1})][\hat{\lambda}_2^2(\text{x}) + \hat{\lambda}_1^2(\text{x}) - 2\hat{\lambda}_2(\text{x})\hat{\lambda}_1(\text{x})]$$
$$= \frac{\text{p}(\hat{\lambda}_2^2(\text{x}) + \hat{\lambda}_1^2(\text{x}))}{\text{d}_{12}^2(\text{x})} + \epsilon_{n2}$$

where $\epsilon_{n2} = \text{d}_{12}{}^{-3}(\text{x})[\hat{\lambda}_n(\text{x}) - \hat{\lambda}_2(\text{x}) + b_{n2}(\text{x}) + \text{O}(n^{-1})][\hat{\lambda}_2^2(\text{x}) + \hat{\lambda}_1^2(\text{x}) - 2\hat{\lambda}_2(\text{x})\hat{\lambda}_1(\text{x}) – 2\text{p}\hat{\lambda}_2(\text{x})\hat{\lambda}_1(\text{x})]$

$$\overline{\overline{a.s.}}\ \text{O}(a_n^2\, (\frac{\log n}{n}) \vee (\frac{\log n}{n})^{3/2})$$

in view of Lemma 2.1, Corollary 2.2 and thus (3.4) becomes

$$\hat{p}_{n,2}(\text{x}) = \text{p} + \frac{b_{n2}(\text{x})}{d_{12}(\text{x})} + \text{d}_{12}{}^{-1}(\text{x})[\hat{\lambda}_n(\text{x}) – (1 - \text{p})\hat{\lambda}_2(\text{x}) - \text{p}\hat{\lambda}_1(\text{x})]$$
$$-\text{d}_{12}{}^{-2}(\text{x})[(1\text{-p})\hat{\lambda}_2^2(\text{x}) - \text{p}\hat{\lambda}_1^2(\text{x})] + \epsilon_{n1} + \epsilon_{n2}$$
$$=: \text{d}_{12}{}^{-1}(\text{x})b_{n2}(\text{x}) + \hat{T}_n(\text{x}) + \epsilon_n \qquad (3.5)$$

where $\epsilon_n = -\ \text{d}_{12}{}^{-2}(\text{x})[(1\text{-p})\hat{\lambda}_2^2(\text{x}) - \text{p}\hat{\lambda}_1^2(\text{x})] + \epsilon_{n1} + \epsilon_{n2}$ and $\hat{T}_n(\text{x})$, $b_{n2}(\text{x})$ as defined in the statement.

Theorem 3.2: Under the conditions of Corollary 2.3,

$$\tilde{p}_{n,1}(\text{x}) - \text{p} = \tilde{T}_n(\text{x}) + \text{O}(\frac{\log n}{n})$$

where $\tilde{T}_n(\text{x}) = \text{d}_{12}{}^{-1}(\text{x})[\tilde{\lambda}_n(\text{x}) – (1 - \text{p})\tilde{\lambda}_2(\text{x}) - \text{p}\tilde{\lambda}_1(\text{x})]$

Proof: The result follows by adopting the similar line of argument as in the proof of Theorem 3.1 and using Corollary 2.3.

Asymptotics of $\hat{p}_{n,2}$(x) and $\tilde{p}_{n,1}$(x): Now we consider the small and large sample behavior of the nonparametric estimators $\hat{p}_{n,2}(\text{x})$ and $\tilde{p}_{n,1}(\text{x})$ of mixing proportion p in the mixing model (1.1) for m=2 by utilizing the results established in section 2.

3A. Mean square errors of $\widehat{p}_{n,2}$(x) and $\widetilde{p}_{n,1}$(x): Now we establish the variance and bias of $\hat{p}_{n,2}$(x) to compute its Mean square errors in the following theorem.

Theorem 3.3: Under the conditions of Lemma 2.1,

i. Var $\hat{p}_{n,2}(x) = d_{12}^{-2}(x)[\Lambda_n(x) + (1 - p)^2\Lambda_2(x) + p^2\Lambda_1(x)] + O(\frac{a_n^2}{n})$

ii. Bias $\hat{p}_{n,2}(x) = \frac{b_{n2}(x)}{d_{12}(x)} + O(\frac{a_n^2}{n})$

where $\Lambda_n(x)$, $\Lambda_j(x)$, $b_{n2}(x)$, $d_{12}(x)$ are as defined in Lemma 2.1 and Corollary 2.2.

Proof: From a.s. representation of $\hat{p}_{n,2}$(x) in (3.5),

$$\hat{p}_{n,2}(x) = p + d_{12}^{-1}(x)b_{n2}(x) + \hat{T}_n(x) + \epsilon_n$$

In order to compute variance, first we consider,

$$E\ \hat{p}_{n,2}(x) = p + d_{12}^{-1}(x)b_{n2}(x) + E\ \hat{T}_n(x) + E\ \epsilon_n + O(n^{-1})$$

Since $E\ \hat{\lambda}_n(x) = E(\widehat{H}_n(x) - E\widehat{H}_n(x)) = 0$; $E\ \hat{\lambda}_j(x) = E(\widehat{H}_j(x) - E\widehat{H}_j(x)) = 0$

$$E(\hat{\lambda}_n(x)\hat{\lambda}_2(x)) = E\hat{\lambda}_n(x)E\hat{\lambda}_2(x) = 0, E\hat{\lambda}_j^3(x) = E\hat{\lambda}_n^3(x) = E\hat{\lambda}_j^4(x) = E\hat{\lambda}_n^4(x) = O(n^{-2}) \quad (3.6)$$

since samples of sizes n, n_1, n_2 are independent and from Lemma 2.1 and Corollary 2.3,

$$E\ \hat{p}_{n,2}(x) = p + \frac{b_{n2}(x)}{d_{12}(x)} - d_{12}^{-2}(x)[(1- p)\Lambda_2(x) + p\Lambda_1(x)] + o(n^{-1}) \quad (3.7)$$

$$\text{Bias } \hat{p}_{n,2}(x) = E\ \hat{p}_{n,2}(x) - p$$

$$= \frac{b_{n2}(x)}{d_{12}(x)} + O(n^{-1}) \quad (3.8)$$

By Theorem 3.1 and (3.6),

$$d_{12}^{2}(x)\text{Var } \hat{p}_{n,2}(x) = E\ [\hat{p}_{n,2}(x) - E\ \hat{p}_{n,2}(x)]^2$$

$$= E\ [\hat{T}_n(x) + (\epsilon_n - E\ \epsilon_n)]^2$$

$$= E\ \hat{T}_n^2(x) + 2\ E\ \hat{T}_n(x)(\epsilon_n - E\ \epsilon_n) + E(\epsilon_n - E\ \epsilon_n)^2$$

where $E\ \hat{T}_n^2(x) = [E\ \hat{\lambda}_n^2(x) + (1 - p)^2E\hat{\lambda}_2^2(x) + p^2E\hat{\lambda}_1^2(x)]/d_{12}^{2}(x) = [\Lambda_n(x) + (1-p)^2\Lambda_2(x) + p^2\Lambda_1(x)]/d_{12}^{2}(x)$

$2\,E\,\hat{T}_n(x)(\epsilon_n - E\,\epsilon_n) = -2d_{12}^{-3}(x)E[-(1-p)^2\hat{\lambda}_2^3(x) - p^2\hat{\lambda}_1^3(x)] = O(n^{-2})$

$E(\epsilon_n - E\,\epsilon_n)^2 = O(n^{-2})$ in view of (3.6), so that

$$\mathrm{Var}\,\hat{p}_{n,2}(x) = d_{12}^{-2}(x)[\Lambda_n(x) + (1-p)^2\Lambda_2(x) + p^2\Lambda_1(x)] + O(n^{-2}) \tag{3.9}$$

Now by definition of bias and from (3.7)

$$\mathrm{Bias}^2\,\hat{p}_{n,2}(x) = (E\,\hat{p}_{n,2}(x) - p)^2$$

$$= \frac{b_{n2}^2(x)}{d_{12}^2(x)} + O(n^{-2}) \tag{3.10}$$

Theorem 3.4: Under the conditions of Lemma 2.1,

$$\mathrm{MSE}\ \hat{p}_{n,2}(x) = \frac{b_{n2}^2(x)}{d_{12}^2(x)} + d_{12}^{-2}(x)[\Lambda_n(x) + (1-p)^2\Lambda_2(x) + p^2\Lambda_1(x)] + O(n^{-2})$$

where Λ_n, Λ_j are as defined in Lemma 2.1 and Corollary 2.2.

Proof: By definition of mean square error,

$$\mathrm{MSE}\ \hat{p}_{n,2}(x) = \mathrm{Var}\ \hat{p}_{n,2}(x) + \mathrm{Bias}^2\ \hat{p}_{n,2}(x)$$

By Theorem 3.3,

$$\mathrm{MSE}\ \hat{p}_{n,2}(x) = d_{12}^{-2}(x)[\Lambda_n(x) + (1-p)^2\Lambda_2(x) + p^2\Lambda_1(x))] + \frac{b_{n2}^2(x)}{d_{12}^2(x)} + O(n^{-2})$$

$$= \frac{b_{n2}^2(x)}{d_{12}^2(x)} + d_{12}^{-2}(x)[\Lambda_n(x) + (1-p)^2\Lambda_2(x) + p^2\Lambda_1(x)] + O(n^{-2}) \tag{3.11}$$

In the similar way the MSEs of $\tilde{p}_{n,1}(x)$, $p_{n,1}(x)$ and $p_{n,2}(x)$ are derived in the following Corollaries.

Corollary 3.5: Under the condition of Corollary 2.3,

$$\mathrm{MSE}\ \tilde{p}_{n,1}(x) = d_{12}^{-2}(x)[\lambda_n(x) + (1-p)^2\lambda_2(x) + p^2\lambda_1(x))]$$

where $\lambda_n(x) = E\tilde{\lambda}_n^2(x) = \frac{M_n(x)}{n}$, $\lambda_j(x) = E\tilde{\lambda}_j^2(x) = \lambda_j(x) = \frac{M_j(x)}{n_j}$, j=1,2, $M_n(x)$, $M_j(x)$ are as given in Corollary 2.3.

The following result gives exact expressions for the mean square errors of BJ type estimators $p_{n,1}(x)$, $p_{n,2}(x)$ defined in (1.9)-(1.10) when component distributions are known under non-iid situations accepted to publish an article of Ramakrishnaiah et al(2019).

Corollary 3.6:

$$\mathrm{MSE}(p_{n,1}(x)) = d_{12}^{-2}(x)\left[\frac{\bar{H}_n(x)[1-\bar{H}_n(x)]}{n} - \frac{V_{nF}(x)}{n}\right]$$

MSE($p_{n,2}$(x)) = MSE($p_{n,1}$(x)) - d_{12}^{-2}(x)[$\frac{a_n}{n}\overline{H}_n^{(1)}(x)\psi_1$(K) - $\frac{a_n^4}{4}\overline{H}_n^{(2)^2}(x)\mu_2^2$(K)] + O($\xi_{n,0}$(x) a_n^2) + o(a_n^4)

where V_{nF}(x)= $n^{-1}\sum(F_i(x) - \overline{H}_n(x))^2>0$, ψ_1(K)= 2$\int t$K(t)dK(t)>0, $\xi_{n,0}$(x)=$\overline{H}_n$(x) – H(x)→0 as n→∞

3B. Asymptotic normality of $\hat{p}_{n,2}$(x) and $\tilde{p}_{n,1}$(x): We now consider the limiting distribution of nonparametric smoothed estimator $\hat{p}_{n,2}$(x) of p. By applying Lyapunov CLT to the sequence $\{\hat{Z}_{2i}\}$ defined in (2.1) of independent random variables, we establish the asymptotic normality of $\hat{p}_{n,2}$(x) in the following theorems.

Lemma 3.7: Under the conditions of Lemma 2.1,

i. $\sqrt{n}\,\hat{\lambda}_n(x) \xrightarrow{L}$ N (0, σ^2) as n → ∞

where $\sigma^2 = \lim_{n\to\infty} n^{-1}\sum[\,F_i\,(x)(1\text{-}F_i(x)) - a_n F_i^{(1)}(x)\psi_1(K) + O(a_n^2)]$

ii. $\sqrt{n_j}\,\hat{\lambda}_j(x) \xrightarrow{L}$ N (0,σ_j^2), j = 1, 2 as $n_j \to \infty$

where $\sigma_j^2 = \lim_{n_j\to\infty} {n_j}^{-1}\sum[\,F_{ji}(x)(1\text{-}F_{ji}(x))\text{-}\; a_n F_{ji}^{(1)}(x)\psi_1(K)\text{+ } O(a_n^2)]$

Proof: Note from Lemma 2.1 and Corollary 2.2,

$$\hat{\lambda}_n(x) = n^{-1}\sum_{i=1}^{n} Z_{2i},\ |Z_i| \le 2\,\|k\| = M < \infty,\ \sigma_i^2 = \text{Var}\, Z_{2i}$$

$$\frac{s_{n2}^2}{n} = \sum\frac{\sigma_i^2}{n} = n^{-1}\sum[F_i(x)(1\text{-}F_i(x)) - a_n F_i^{(1)}(x)\psi_1(K) + O(a_n^2)]$$

$$= \overline{H}_n(x)(1\text{-}\overline{H}_n(x)) - V_{nF}(x) - a_n\overline{H}_n^{(1)}(x)\psi_1(K) + O(a_n^2) <\infty$$

In order to apply Lyapunov CLT to the sequence $\{Z_{2i}\}$, consider the Lyapunov condition

$$\frac{1}{s_{n2}^3}\sum_1^n E\,|Z_{2i}|^3 \le \frac{n}{s_{n2}^3}\,[\,\sum_1^n E\,|Z_{2i}|^3/n] = O(\,\frac{n}{n^{3/2}}\,) \to 0 \text{ as } n\to\infty$$

Now Lyapunov condition is satisfied, Lyapunov CLT to the sequence $\{Z_{2i}\}$ holds.

$$\frac{\sum_1^n Z_{2i}}{s_{n2}} \to N(0,1)$$

i.e. $\frac{\sqrt{n}}{s_{n2}}\frac{\sum_1^n Z_{2i}}{\sqrt{n}} \xrightarrow{L}$ N(0, 1) as n → ∞

i.e. $\frac{\sum_1^n Z_{2i}}{\sqrt{n}} \xrightarrow{L}$ N(0, σ^2) as n → ∞

as $\frac{s_{n2}}{\sqrt{n}} = (\frac{\sum_1^n \sigma^2}{n})^{1/2} \to \sigma$ as n → ∞

i.e. $\sqrt{n}\,\hat{\lambda}_n(x) \xrightarrow{L}$ N (0, σ^2) as n → ∞

where σ^2 = H(x)(1 – H(x)) – V(x), V(x) = $\lim_{n\to\infty} V_{nF}(x) = \lim_{n\to\infty} n^{-1}\sum(F_i(x) - \bar{F}_n(x))^2 \to 0$

similarly, it is easy to establish

$$\sqrt{n_j}\,\hat{\lambda}_j(\text{x}) \xrightarrow{L} \text{N}(0,\sigma_j^2),\ \text{j} = 1, 2 \text{ as } n_j \to \infty$$

where $\sigma_j^2 = \lim_{n_j\to\infty} n_j^{-1} \sum[F_{ji}(\text{x})(1 - F_{ji}(\text{x})) - a_n F_{ji}^{(1)}(\text{x})\psi_1(\text{K}) + O(a_n^2)] = H_j(\text{x})(1 - H_j(\text{x}) - V_j(\text{x}))$; j=1,2

$V_j(\text{x}) = \lim_{n_j\to\infty} n_j^{-1} \sum(F_{ji}(\text{x}) - \bar{F}_j(\text{x}))^2 = 0$

Corollary 3.8: Under the conditions of Corollary 2.3,

i. $\sqrt{n}\,\tilde{\lambda}_n(\text{x}) \xrightarrow{L} \text{N}(0, t^2)$ as $n \to \infty$

where $t^2 = \lim_{n\to\infty} n^{-1} \sum[F_i(\text{x})(1 - F_i(\text{x})]$

ii. $\sqrt{n_j}\,\tilde{\lambda}_j(\text{x}) \xrightarrow{L} \text{N}(0,t_j^2)$, j = 1, 2 as $n_j \to \infty$

where $t_j^2 = \lim_{n_j\to\infty} n_j^{-1} \sum[F_{ji}(\text{x})(1 - F_{ji}(\text{x})]$

Theorem 3.9: under the conditions of Lemma 2.1,

$$\sqrt{n}\,(\hat{p}_{n,2}(\text{x}) - \text{p}) \to \text{N}(0, V_2(\text{x})) \text{ as } n \to \infty$$

where $V_2(\text{x}) = d_{12}^{-2}(\text{x}) \lim_{n\to\infty}[n\,\Lambda_n(\text{x}) + (1-p)^2 n_2\Lambda_2(\text{x}) + p^2 n_1\Lambda_1(\text{x})]$

Proof: Recall from a.s. representation of $\hat{p}_{n,2}(\text{x})$ in (3.5),

$$\hat{p}_{n,2}(\text{x}) - \text{p} = \frac{b_{n2}(\text{x})}{\text{d}_{12}(x)} + \text{d}_{12}{}^{-1}(\text{x})[\hat{\lambda}_n(\text{x}) - (1-\text{p})\hat{\lambda}_2(\text{x}) - \text{p}\hat{\lambda}_1(\text{x}))] + \text{O}\left(\frac{\log n}{n}\right)$$

Letting $\underline{c}' = d_{12}^{-1}(\text{x})[1 \quad -(1-p) \quad -p]$, $\underline{Z}' = [\sqrt{n}\hat{\lambda}_n(\text{x}) \quad \sqrt{n_2}\hat{\lambda}_2(\text{x}) \quad \sqrt{n_1}\hat{\lambda}_1(\text{x})] = (z_n \quad z_2 \quad z_1)^1$

$$\sqrt{n}\,(\hat{p}_{n,2}(\text{x}) - \text{p}) = \underline{c}'\underline{Z} + \text{O}\left(a_n^2\ \text{V}\left(\frac{\log n}{n}\right)\right) \tag{3.12}$$

From Lemma 3.7,

i) $\text{Z}_\text{n} = \sqrt{n}(\hat{\lambda}_n(\text{x}))$

$= \sqrt{n}(\widehat{H}_\text{n}(\text{x}) - \text{E}\widehat{H}_\text{n}(\text{x})) \xrightarrow{L} \text{N}(0, \sigma^2)$ as $n \to \infty$

$\sigma^2 = lim_{n\to\infty}\ \text{n var}(\hat{\lambda}_n(\text{x})) = lim_{n\to\infty}\ \text{n}\Lambda_n$

ii) $\text{Z}_\text{j} = \sqrt{n_j}\,(\widehat{H}_\text{j}(\text{x}) - \text{E}\widehat{H}_\text{j}(\text{x}))$

$= \sqrt{n_j}\,(\hat{\lambda}_j(\text{x})) \xrightarrow{L} \text{N}(0, \sigma_j^2)$ as $n_j \to \infty$

$\sigma_j^2 = lim_{n\to\infty}\ \text{n}_j \text{ var}(\hat{\lambda}_j(\text{x}))$

$= lim_{n\to\infty}\ \text{n}_j\Lambda_j$

Since $\text{n}_1 \sim \rho\text{n}$ and $\text{n}_2 \sim (1-\rho)\text{n}$, the components of $\underline{Z}$ are independent and asymptotically normal, then (3.12) becomes,

$$\sqrt{n}(\hat{p}_{\text{n},2} - \text{p}) = \underline{c}'\underline{Z} + \text{O}\left(a_n^2\ \text{V}\left(\frac{\log n}{n}\right)\right) \text{ a.s.} \to \text{N}(0, V_2(\text{x}))$$

$$V_2(\text{x}) = lim_{n\to\infty}\ \underline{c}'\ \Sigma_n\ \underline{c}$$

where $\underline{c}'\Sigma_n\underline{c} = d_{12}^{-2}\ [1 \quad -(1-p) \quad -p]\begin{bmatrix} n\Lambda_n & & \\ & n_2\Lambda_2 & \\ & & n_1\Lambda_1 \end{bmatrix}\begin{bmatrix} 1 \\ -(1-p) \\ -p \end{bmatrix}$

$$= d_{12}^{-2}\,[n\Lambda_n + (1-p)^2 n_2\Lambda_2 + p^2 n_1\Lambda_1]$$

Applying Lyapunov CLT to the sequence of independent random variables, we establish,

$$\sqrt{n}\,(\hat{p}_{n,2}(x) - p) \to N\,(0, V_2(x)) \text{ as } n \to \infty$$

where $V_2(x) = d_{12}^{-2}\, lim_{n\to\infty}\,[n\Lambda_n + (1-p)^2 n_2\Lambda_2 + p^2 n_1\Lambda_1]$

Theorem 3.10: Assume the uniform continuous distribution functions $\{F_i(x): 1\leq i\leq n\}$. Then,

$$\sqrt{n}\,(\tilde{p}_{n,1}(x) - p) \xrightarrow{\ d\ } N\,(0, V_1(x)) \text{ as } n \to \infty$$

where $V_1(x) = d_{12}^{-2}\, lim_{n\to\infty}\,[n\lambda_n(x) + (1-p)^2 n_2\lambda_2(x) + p^2 n_1\lambda_1(x)]$, $n\lambda_n(x) = \bar{H}_n(x)(1 - \bar{H}_n(x)) - V_{nF}(x)$ and $n_j\lambda_j(x) = \bar{H}_j(x)(1 - \bar{H}_j(x)) - V_{n_j}(x)$; j=1,2, $V_n(x)$, $V_{n_j}(x)$ are defined in Lemma 2.1 and Corollary 2.2.

Proof: The result follows by adopting the similar line of argument as in the proof of Lemma 3.7 and using Corollary 3.8.

Corollary 3.11: Under the conditions AI – AIII on $\{F_i\}$, the kernel function k and the sequence $\{a_n\}$,

$$\sqrt{n}\,(p_{n,2}(x) - p) \xrightarrow{\ L\ } N(0, \frac{\tau^2}{d_{12}^2(x)}) \text{ as } n \to \infty$$

where $\tau^2 = \lim_{n\to\infty}[\bar{F}_n(x)(1 - \bar{F}_n(x)) - V_{nF}(x)]$, $V_{nF}(x) = n^{-1}\sum(F_i(x) - \bar{F}_n(x))^2 > 0$

Proof: On the similar line of argument as in the proof of Theorem 3.9 and using Corollary 3.6, the result follows.

Corollary 3.12: Under the conditions of Lemma 2.1 on $\{F_i(x)\}$,

$$\sqrt{n}\,(p_{n,1}(x) - p) \xrightarrow{\ L\ } N(0, \frac{\tau^2}{d_{12}^2(x)}) \text{ as } n \to \infty$$

Proof: The result follows by the similar arguments of the proof of Corollary 3.7 and using Theorem 3.10.

3C. Rates of strong convergence of $\hat{p}_{n,2}(x)$ and $\tilde{p}_{n,1}(x)$: Now we establish a.s. convergence of the nonparametric BJ type smoothed estimators $\hat{p}_{n,2}(x)$ and $\tilde{p}_{n,1}(x)$ defined by (1.9)-(1.10) under non-i.i.d. setup in the following result:

Theorem 3.13: Under the conditions of Lemma 2.1 on $\{F_i(x)\}$ and the kernel function k,

$$\hat{p}_{n,2}(x) - p \overline{\overline{a.s.}}\ O(\frac{\log n}{n})^{\frac{1}{2}} \text{ as } n\to\infty$$

Proof: Recall from (3.5),

$\hat{p}_{n,2}$(x) - p = $d_{12}{}^{-1}$(x)b_{n2}(x)+$d_{12}{}^{-1}$(x)[$\hat{\lambda}_n$(x) –(1 - p)$\hat{\lambda}_2$(x)- p$\hat{\lambda}_1$(x))] + O($\frac{\log n}{n}$ V a_n^2 ($\frac{\log n}{n}$))

(3.13)

Let $\hat{T}_n$ = $\hat{\lambda}_n$(x) – (1 - p)$\hat{\lambda}_2$(x) - p$\hat{\lambda}_1$(x) then

$$d_{12}(x)(\hat{p}_{n,2}(x) - p) \overline{\overline{a.s.}}\, b_{n2}(x) + \hat{T}_n + O(\frac{\log n}{n})$$

For any $\omega_n > 0$,

$$[|\hat{T}_n| > \omega_n] \Rightarrow |\hat{\lambda}_n(x)| + (1 - p)|\hat{\lambda}_2(x)| + p\,|\hat{\lambda}_1(x)| > \omega_n$$

$$P(|\hat{T}_n| > \omega_n) \leq P(|\hat{\lambda}_n(x)| > \frac{\omega_n}{3}) + P(|\hat{\lambda}_2(x)| > \frac{\omega_n}{3(1-p)}) + P(|\hat{\lambda}_1(x)| > \frac{\omega_n}{3p})$$

(3.14)

By Lemma 2.1(c) and Corollary 2.2(v), we have, for ω_n = O($\frac{\log n}{n}$)$^{1/2}$,

$$P(|\hat{T}_n| > \omega_n) < n^{-2}$$

By Borel – Cantelli lemma,

$$\hat{T}_n = O(\frac{\log n}{n})^{1/2} \text{ a.s. as } n \to \infty$$

i.e. $\hat{p}_{n,2}(x) - p\ \overline{\overline{a.s.}}\ O(\frac{\log n}{n})^{\frac{1}{2}}$ as n→∞

Corollary 3.14: Under the conditions of Corollary 2.3,

$$\tilde{p}_{n,1}(x) - p = O(\frac{\log n}{n})^{1/2} \text{ as } n \to \infty$$

Proof: The proof follows exactly on the similar line of argument as for the proof of Theorem 3.13.

Theorem 3.15: Under the conditions of Lemma 2.1,

i. $p_{n,2}(x) - p = O(\frac{\log n}{n})^{\frac{1}{2}}$ a.s.

and by conditions of Corollary 2.3

ii. $p_{n,1}(x) - p = O(\frac{\log n}{n})^{\frac{1}{2}}$ a.s.

Proof: The proof follows exactly on the similar line of argument as for the proof of Corollary 2.3.

4. Optimal Bandwidth $a_{n,opt}$

As mentioned in section 1, the selection of band width 'a_n' in kernel based smoothed nonparametric estimators of mixing proportion is very crucial and we now a method obtaining the 'optimal' value for smoothing parameter 'a_n' in the construction of kernel based nonparametric estimator $\hat{p}_{n,2}$(x) in (1.10). We select the optimal $a_{n,opt}$ as that a_n for which MSE ($\hat{p}_{n,2}$(x)) is the minimum. Solving the equation $\frac{\partial \text{MSE}\,\hat{p}_{n,2}(\text{x})}{\partial a_n}$ = 0 for a_n, we have from Theorem 3.4 and Corollary 3.5,

$$\text{i.e. M} = \text{d}_{12}{}^2(\text{x})\text{MSE}\,\hat{p}_{n,2}(\text{x})$$

$$= \frac{M_n(x)+(1-p)^2M_2(x)+p^2M_1(x)}{n} - \frac{a_n}{n}\,\xi_{n,1}(\text{x}) + a_n^4\xi_{n,2}^2(\text{x})$$

$$\frac{\partial \text{M}}{\partial a_n} = 0 = -\frac{1}{n}\,\xi_1(\text{x}) + 4a_n^3\,\xi_{n,2}^2(\text{x})$$

where from Lemma 2.1(a) and Corollary 2.2(i),

$$\xi_{n,1}(\text{x}) = [\overline{H}_n^{(1)}(\text{x}) + \frac{n}{n_2}(1\text{-p})^2\,\overline{H}_2^{(1)}(\text{x}) + \frac{n}{n_1}\text{p}^2\overline{H}_1^{(1)}(\text{x})]\psi_1(\text{K}) > 0$$

$$\xi_{n,2}(\text{x}) = \frac{1}{2}[\overline{H}_n^{(2)}(\text{x}) - \overline{H}_2^{(2)}(\text{x})]\mu_2(\text{K})$$

so that

$$a_{n,opt} = [\frac{\xi_{n,1}(x)}{4\xi_{n,2}^2(x)}]^{1/3}\cdot n^{-1/3} \qquad (4.1)$$

5. Comparisons Between The Estimators

We first compare the performance of proposed smoothed estimator $\hat{p}_{n,2}$(x) with Boes-James type estimator $\tilde{p}_{n,1}$(x), when H_1(x), H_2(x) are unknown based on the minimum mean square error (MSE) criterion under non-i.i.d. setup and establish clear superiority of smoothed estimator in the sense of smaller MSE.

MSE $\hat{p}_{n,2}$(x): From Theorem 3.4 and Lemma 2.1, we have as n → ∞,

$$d_{12}^2(\text{x})\text{MSE}\,\hat{p}_{n,2}(\text{x}) = b_{n2}^2(\text{x}) + [\Lambda_n(\text{x}) + (1\text{-p})^2\Lambda_2(\text{x}) + p^2\Lambda_1(\text{x})] + \text{O}(n^{-1})$$

$$= \frac{M_n(x)+(1-p)^2M_2(x)+p^2M_1(x)}{n} - \frac{a_n}{n}\,\xi_{n,1}(\text{x}) + a_n^4\xi_{n,2}^2(\text{x}) + \text{O}(n^{-1}) \qquad (5.1)$$

where $\xi_{n,1}$(x) and $\xi_{n,2}$(x) are greater than 0 and defined in (4.1).

$$\text{MSE}\,\tilde{p}_{n,1}(\text{x}) = \text{d}_{12}{}^{-2}(\text{x})[\lambda_n(\text{x}) + (1\text{-p})^2\lambda_2(\text{x}) + \text{p}^2\lambda_1(\text{x}))]$$

$$= \frac{M_n(x)+(1-p)^2M_2(x)+p^2M_1(x)}{nd_{12}^2(\text{x})} \qquad (5.2)$$

From (5.1) and (5.2),

$$\text{MSE}\,\hat{p}_{n,2}(\text{x}) < \text{MSE}\,\tilde{p}_{n,1}(\text{x})$$

If

$$\frac{a_n}{n}\,\xi_{n,1}(\text{x}) > a_n^4\xi_{n,2}^2(\text{x}) \qquad (5.3)$$

for finite values of n. Since both the terms in the above inequality are always positive and na_n^4 →0 for moderate values of n,

$$a_n\xi_{n,1}(\text{x}) > \text{n}a_n^4\xi_{n,2}^2(\text{x})$$

always holds.

The percentage gain in precision of $\hat{p}_{n,2}$(x) over $\tilde{p}_{n,1}$(x) is

$$\frac{MSE\ \tilde{p}_{n,1}(x) - MSE\ \hat{p}_{n,2}(x)}{MSE\ \tilde{p}_{n,1}(x)} \text{ X } 100$$

$$\text{i.e. } \frac{\frac{a_n}{n}\,\xi_{n,1}(x) - a_n^4\,\xi_{n,2}^2(x)}{MSE\ \tilde{p}_{n,1}(x)} \text{ X } 100$$

is always positive in view of (5.3).

6. Monte Carlo Simulation

A simulation study is carried out in the estimation of p by $\hat{p}_{n,2}(x_0)$ and $\tilde{p}_{n,1}(x_0)$ when two component distributions are unknown and are estimated by using empirical distribution function and kernel distribution function for Normal and Exponential populations. The procedure is given in appendix A.

Table 6.1: Simulation results of $\tilde{p}_{n,1}(x_0)$ and $\hat{p}_{n,2}(x_0)$ for different sets N of sample size n=48 with p=0.5 and X_0 = -2.5, -1.5, 0, 1.5, 2.5 and 2 of Normal distributions and X_0 = 1.5, 2.5, 3, 3.5, 4, 4.5 and 2 of Exponential distributions.

p=0.5	**N**	**H_1(x)=N(βt_{1i},0.5^2), H_2(x)=N(βt_{2i},3^2), H(x)=N(βt_i,$(2.151)^2$)**					**H_1(x)=Exp(2), H_2(x)=Exp(3), H(x)=Weibull(1.25,k=0.5)**					
		$\tilde{p}_{n,1}(x_0)$	**$\hat{p}_{n,2}(x_0)$**	**$\widehat{MSE}$**			**p=0.5**	**$\tilde{p}_{n,1}(x_0)$**	**$\hat{p}_{n,2}(x_0)$**	**$\widehat{MSE}$**		
				$\tilde{p}_{n,1}(x_0)$	**$\hat{p}_{n,2}(x_0)$**	**efficiency**				**$\tilde{p}_{n,1}(x_0)$**	**$\hat{p}_{n,2}(x_0)$**	**Efficiency**
X_0=2	**10**	0.40	0.54	0.015	0.007	55.69	X_0=2	0.841	0.404	0.0039	0.0022	43.90
	50	0.33	0.48	0.021	0.017	19.20		0.854	0.418	0.0063	0.0037	41.03
	100	0.34	0.48	0.020	0.018	10.04		0.855	0.416	0.0063	0.0039	37.24
	200	0.34	0.47	0.020	0.017	15.55		0.855	0.417	0.0063	0.0038	39.14
	300	0.34	0.48	0.020	0.016	18.61		0.854	0.417	0.0063	0.0038	39.78
	400	0.34	0.48	0.020	0.015	22.79		0.854	0.417	0.0063	0.0038	40.09
	500	0.34	0.48	0.020	0.016	21.44		0.854	0.417	0.0063	0.0038	40.28
X_0=-2.5	**10**	0.426	0.565	0.026	0.005	81.88	X_0=1.5	0.834	0.313	0.0049	0.0028	42.949
	50	0.438	0.502	0.031	0.012	61.38		0.843	0.325	0.0072	0.0048	33.356
	100	0.419	0.49	0.03	0.015	49.51		0.842	0.323	0.0072	0.0045	37.448
	200	0.429	0.482	0.029	0.017	40.67		0.843	0.325	0.0072	0.0047	35.462
	300	0.425	0.485	0.029	0.016	42.55		0.843	0.325	0.0072	0.0047	34.758
	400	0.424	0.488	0.029	0.016	44.31		0.843	0.325	0.0072	0.0047	34.407
	500	0.426	0.486	0.029	0.016	43.26		0.843	0.325	0.0072	0.0048	34.176
X_0=-1.5	**10**	0.2108	0.6042	0.013	0.002	82.33	X_0=2.5	0.744	0.268	0.0172	0.0131	23.801
	50	0.0841	0.2752	0.019	0.01	48.24		0.772	0.289	0.0202	0.0104	48.313
	100	0.0824	0.2592	0.018	0.013	27.97		0.774	0.29	0.0201	0.0104	48.412
	200	0.0848	0.2504	0.018	0.013	30.94		0.773	0.29	0.0243	0.0104	57.251
	300	0.0878	0.248	0.017	0.014	17.77		0.773	0.29	0.0202	0.0104	48.347

	400	0.0871	0.2442	0.017	0.015	10.48		0.772	0.289	0.0202	0.0104	48.444
	500	0.0888	0.2447	0.017	0.015	10.52		0.772	0.289	0.0202	0.0104	48.417
$X_0=0$	**10**	0.328	0.881	0.031	0.014	54.018	$X_0=3.0$	0.7645	0.3013	0.0062	0.00256	58.68
	50	0.331	0.803	0.024	0.021	13.019		0.7517	0.31	0.0051	0.00392	23.18
	100	0.341	0.821	0.025	0.022	9.119		0.7508	0.3079	0.0051	0.00366	27.76
	200	0.332	0.819	0.027	0.023	13.553		0.7504	0.3069	0.0051	0.00352	30.27
	300	0.332	0.825	0.028	0.024	14.153		0.7502	0.3069	0.005	0.00356	29.24
	400	0.334	0.824	0.027	0.024	13.058		0.7503	0.3067	0.005	0.00356	29.51
	500	0.332	0.823	0.028	0.024	13.891		0.7503	0.3067	0.0051	0.00363	28.2
$X_0=1.5$	**10**	0.286	0.518	0.008	0.01023	-28.74	$X_0=3.5$	0.7074	0.274	0.0064	0.0045	29.61
	50	0.271	0.477	0.021	0.01549	26.35		0.7133	0.2777	0.0057	0.0045	20.5
	100	0.279	0.458	0.02	0.01674	16.52		0.7115	0.2773	0.0057	0.0044	23.47
	200	0.285	0.473	0.018	0.01559	11.33		0.7098	0.2776	0.0056	0.0045	20.53
	300	0.272	0.478	0.017	0.01527	12.29		0.7102	0.2779	0.0057	0.0045	21.42
	400	0.28	0.477	0.017	0.01523	9.46		0.7104	0.2785	0.0057	0.0045	21.27
	500	0.28	0.478	0.017	0.01552	9.74		0.7105	0.279	0.0058	0.0045	21.24
$X_0=2.5$	**10**	0.395	0.533	0.0155	0.0074	52.15	$X_0=4.0$	0.6745	0.1907	0.0082	0.0054	33.96
	50	0.31	0.509	0.0235	0.015	36.28		0.6706	0.1927	0.0078	0.006	23.05
	100	0.311	0.505	0.0237	0.0158	33.16		0.6729	0.1924	0.0083	0.006	27.34
	200	0.31	0.506	0.0235	0.0155	34.04		0.6741	0.1922	0.0086	0.0061	29.26
	300	0.31	0.507	0.0235	0.0155	34.08		0.6745	0.192	0.0086	0.0061	29.68
	400	0.31	0.508	0.0235	0.0153	34.84		0.6747	0.1919	0.0087	0.0061	29.95
	500	0.311	0.508	0.0241	0.0156	35.58		0.6748	0.1919	0.0087	0.0061	30.17

7. Comments

It shows when the component Normal and Exponential populations with parameters respectively are $N(\beta t_{1i},0.25)$, $N(\beta t_{2i},9)$ and $Exp(2)$, $Exp(3)$ and weight function has taken as empirical distribution function, the mean value of estimate $\tilde{p}_{n,1}(x_0)$ and $\hat{p}_{n,2}(x_0)$ is close to its actual value p**.** The simulation results show $\widehat{\boldsymbol{MSE}}$ for nonparametric smoothed estimator is less than unsmoothed estimator for different values of x, uniformly for all samples. So the smoothed estimator is best estimator in terms of minimum MSE. The average gain in efficiency due to smoothing is lying between 3% to 82% for different sets N of size n.

Appendix A

A simulation study is carried out to estimate mixing proportion p when two component distributions are unknown and are estimated by using empirical distribution function and kernel distribution function. A random samples of sizes n_1=24 and n_2=24 are generated from the two component mixture of the normal populations with parameters $(\mu_1, \mu_2) = (\beta t_{1i}, \beta t_{2i})$ and $(\sigma_1^2, \sigma_2^2) = (0.5^2, 3^2)$ and with parameters $(\theta_1,\theta_2) = (2,3)$ for Exponential populations. The mixed sample of size $n = n_1+n_2 = 48$ is generated from the normal population with parameters $(\mu = p\mu_1 + q\mu_2) = (\beta t_i = p\beta t_{1i}+q\beta t_{2i})$ and $\sigma^2 = (p\sigma_1^2+q\sigma_2^2)$ and in case of Exponential population, the mixed sample is drawn from Weibull population with shape parameter k lessthan 1. Since Weibull distribution with shape parameter k<1 arises as a mixture of Exponential distributions (Jewel 1982), the samples of sizes n, n_1, n_2 are independent. Taking p=q= 0.5, β=0.1 and $t_i = \mp \frac{i}{n^\delta}$; j = 1, 2, δ =1.5 are selected in such a way that $\sum t_i = 0$ and $\sum t_i^2 \to 0$. The present simulation study is to estimate nonparametric estimators such as $\hat{p}_{n,2}(x_0)$ and $\tilde{p}_{n,1}(x_0)$ for x=x_0 at 2 and -2.5, -1.5, 0, 1.5, 2.5 are defined as follows.

$$\tilde{p}_{n,1}(x_0) = \frac{\tilde{H}_n(x_0) - \tilde{H}_2(x_0)}{\tilde{H}_1(x_0) - \tilde{H}_2(x_0)} \quad \text{and} \quad \hat{p}_{n,2}(x_0) = \frac{\hat{H}_n(x_0) - \hat{H}_2(x_0)}{\hat{H}_1(x_0) - \hat{H}_2(x_0)}$$

where $\tilde{H}_n(x_0)$, $\tilde{H}_j(x_0)$ are estimated by the usual empirical distribution function such as,

$$\tilde{H}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x), \qquad \tilde{H}_j(x) = \frac{1}{n_j}\sum_{i=1}^{n_j} I(X_{i,j} \le x),\ j = 1, 2;$$

If $I(X_i \le x)$**,** assign 1 other wise 0.

$$\tilde{p}_{n,1}(x_0) = \frac{\frac{1}{n}\sum_{i=1}^{n} I(X_i \le x_0) - \frac{1}{n_2}\sum_{i=1}^{n_2} I(X_{2i} \le x_0)}{\frac{1}{n_1}\sum_{i=1}^{n_1} I(X_{1i} \le x_0) - \frac{1}{n_2}\sum_{i=1}^{n_2} I(X_{2i} \le x_0)} \tag{7.1}$$

$\hat{H}_n(x_0)$, $\hat{H}_j(x_0)$ are estimated by kernel functions defined as,

$$\hat{H}_n(x_0) = n^{-1}\sum_{i=1}^{n} K\left(\frac{x_0 - X_i}{a_n}\right) \text{ and } \hat{H}_j(x_0) = n_j^{-1}\sum_{i=1}^{n_j} K\left(\frac{x_0 - X_{i,j}}{a_n}\right),\ j = 1, 2$$

Here we used Epanechnikov kernel function as $K(u) = \frac{3}{4}(1 - u^2);\ |u| \le 1$.

The Distribution function of Epanechnikov kernel function is

$$K\left(\frac{x_0 - X_i}{a_n}\right) = \int_{-1}^{\frac{x_0 - X_i}{a_n}} k(t)dt = \frac{3}{4}\int_{-1}^{\frac{x_0 - X_i}{a_n}} (1 - u^2)\,du$$

$$= \frac{3}{4}\left(u - \frac{u^3}{3}\right)_{-1}^{\frac{x_0 - X_i}{a_n}}$$

$$= \frac{3}{4}\left[\frac{x_0 - X_i}{a_n} - \frac{1}{3}\left(\frac{x_0 - X_i}{a_n}\right)^3 - \left(-1 + \frac{1}{3}\right)\right]$$

$$= \frac{3}{4}\left[\frac{x_0 - X_i}{a_n} - \frac{1}{3}\left(\frac{x_0 - X_i}{a_n}\right)^3 + \frac{2}{3}\right]$$

Numerator = $\widehat{H}_n(x_0) - \widehat{H}_2(x_0)$

$$= n^{-1}\sum_{i=1}^{n} K\left(\frac{x_0 - X_i}{a_n}\right) - n_2^{-1}\sum_{i=1}^{n_2} K\left(\frac{x_0 - X_{i1}}{a_n}\right)$$

$$= \frac{3}{4}\, n^{-1} \sum_{i=1}^{n}\left[\frac{x_0 - X_i}{a_n} - \frac{1}{3}\left(\frac{x_0 - X_i}{a_n}\right)^3 + \frac{2}{3}\right] - \frac{3}{4}\, n_2^{-1} \sum_{i=1}^{n_2}\left[\frac{x_0 - X_{2i}}{a_n} - \frac{1}{3}\left(\frac{x_0 - X_{2i}}{a_n}\right)^3 + \frac{2}{3}\right]$$

$$= \frac{3}{4a_n} x_0 - \frac{3}{4a_n}\bar{x} - \frac{1}{4a_n^3} n^{-1}\sum_{i=1}^{n}(x_0^3 - 3\,x_0^2\,x_i + 3\,x_i^2\,x_0 - x_i^3)$$
$$- \frac{3}{4a_n} x_0 + \frac{3}{4a_n}\bar{x}_2 + \frac{1}{4a_n^3} n_2^{-1}\sum_{i=1}^{n_2}(x_0^3 - 3x_0^2x_{2i} + 3x_{2i}^2x_0 - x_{2i}^3)$$

$$= \frac{3}{4a_n}\left[(\bar{x}_2 - \bar{x}) + \frac{1}{3a_n^2}\left(n^{-1}\sum_{i=1}^{n} x_i^3 - n_2^{-1}\sum_{i=1}^{n_2} x_{2i}^3\right) + \frac{x_0}{a_n^2}\left(n_2^{-1}\sum_{i=1}^{n_2} x_{2i}^2 - n^{-1}\sum_{i=1}^{n} x_i^2\right) + \frac{x_0^2}{a_n^2}(\bar{x} - \bar{x}_2)\right]$$

$$= \frac{3}{4a_n}[a_1 + a_2x_0 + a_3x_0^2]$$

Denominator = $\widehat{H}_1(x_0) - \widehat{H}_2(x_0)$

$$= n_1^{-1}\sum_{i=1}^{n_1} K\left(\frac{x_0 - X_{i1}}{a_n}\right) - n_2^{-1}\sum_{i=1}^{n_2} K\left(\frac{x_0 - X_{i2}}{a_n}\right)$$

$$= \frac{3}{4}\, n_1^{-1} \sum_{i=1}^{n_1}\left[\frac{x_0 - X_{1i}}{a_n} - \frac{1}{3}\left(\frac{x_0 - X_{1i}}{a_n}\right)^3 + \frac{2}{3}\right] - \frac{3}{4}\, n_2^{-1} \sum_{i=1}^{n_2}\left[\frac{x_0 - X_{2i}}{a_n} - \frac{1}{3}\left(\frac{x_0 - X_{2i}}{a_n}\right)^3 + \frac{2}{3}\right]$$

$$= \frac{3}{4a_n} x_0 - \frac{3}{4a_n}\bar{x}_1 - \frac{1}{4a_n^3} n_1^{-1}\sum_{i=1}^{n}(x_0^3 - 3\,x_0^2\,x_{1i} + 3\,x_{1i}^2\,x_0 - x_{1i}^3)$$
$$- \frac{3}{4a_n} x_0 + \frac{3}{4a_n}\bar{x}_2 + \frac{1}{4a_n^3} n_2^{-1}\sum_{i=1}^{n_2}(x_0^3 - 3x_0^2x_{2i} + 3x_{2i}^2x_0 - x_{2i}^3)$$

$$= \frac{3}{4a_n}\left[(\bar{x}_2 - \bar{x}_1) + \frac{1}{3a_n^2}\left(n_1^{-1}\sum_{i=1}^{n} x_{1i}^3 - n_2^{-1}\sum_{i=1}^{n_2} x_{2i}^3\right)\right.$$
$$\left. + \frac{x_0}{a_n^2}\left(n_2^{-1}\sum_{i=1}^{n_2} x_{2i}^2 - n_1^{-1}\sum_{i=1}^{n} x_{1i}^2\right) + \frac{x_0^2}{a_n^2}(\bar{x}_1 - \bar{x}_2)\right]$$

$$= \frac{3}{4a_n}[b_1 + b_2x_0 + b_3x_0^2]$$

$$\hat{p}_{n,2}(x_0) = \frac{a_1 + a_2x_0 + a_3x_0^2}{b_1 + b_2x_0 + b_3x_0^2} \qquad (7.2)$$

where $a_1 = (\bar{x}_2 - \bar{x}) + \frac{1}{3a_n^2}\left(n^{-1}\sum_{i=1}^{n} x_i^3 - n_2^{-1}\sum_{i=1}^{n_2} x_{2i}^3\right)$, $a_2 = \frac{1}{a_n^2}\left(n_2^{-1}\sum_{i=1}^{n_2} x_{2i}^2 - n^{-1}\sum_{i=1}^{n} x_i^2\right)$, $a_3 = \frac{1}{a_n^2}(\bar{x} - \bar{x}_2)$, $b_1 = (\bar{x}_2 - \bar{x}_1) + \frac{1}{3a_n^2}\left(n_1^{-1}\sum_{i=1}^{n} x_{1i}^3 - n_2^{-1}\sum_{i=1}^{n_2} x_{2i}^3\right)$, $b_2 = \frac{1}{a_n^2}\left(n_2^{-1}\sum_{i=1}^{n_2} x_{2i}^2 - n_1^{-1}\sum_{i=1}^{n} x_{1i}^2\right)$, $b_3 = \frac{1}{a_n^2}(\bar{x}_1 - \bar{x}_2)$

The computational procedure is as follows.

1. Generate n uniform random numbers between (0,1).
2. Generate the cumulative distribution function of Normal and Exponential distribution by taking different means and variances.
3. Generate the mixed normal and Weibull observations by taking different means and variance.

4. Calculate (7.1) and (7.2) by taking different values of X_0.

Generated the N=500 sets of samples of sizes n=48, n_1= 24 and n_2=24, so that $n=n_1+n_2$. The unsmoothed and smoothed estimators $\tilde{p}_{n,1}(x_0)$ and $\hat{p}_{n,2}(x_0)$ are computed and compared their mean square errors. The no. of sets is ignored when the value of p does not belong to 0 and 1. The results are presented in table 6.1.

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