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Missing Longitudinal Data Analysis with Covariance Structure

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ABSTRACT

Linear mixed model with covariance structures have become increasingly popular in longitudinal data analysis because of its wide applications. There is a problem of dealing with missing data in longitudinal analysis which need careful attention. In missing data analysis there are two different mechanisms referred to as missing completely at random and missing at random. In this paper estimation of parameters with distinct covariance structures using Maximum Likelihood and Restricted Maximum Likelihood methods are considered for the evaluation of models using six different information criteria with data missing at random. The study involves both the nested and non-nested covariance structure for comparison based on model selection information criteria. Also, evaluation using bootstrap method to identify the most plausible covariance structure for longitudinal models are studied.

1. Introduction

Most of the researchers involved in longitudinal studies face the problem of trying to get study subjects return for every follow-up visit as in the case of clinical studies. Thus, there is always an amount of missing data when looking at these type of studies. There are several reasons attributed to missing data including equipment malfunction, subjects are sick or factors like weather may prevent a visit or the data entered was incorrect.

Missing data have three important implications for longitudinal analysis. First, the data set is necessarily unbalanced over time since not all individuals have the same number of repeated measurement at a common set of occasions. Second, there will be a loss of information and a reduction in the precision. Finally, the

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validity should be established for any method of analysis which will require assumptions about the reasons for missingness, often referred as missing data mechanism. The main goal in this paper is the use of a linear mixed model setting and identify the ability on overall performance of covariance structure using information criteria.

In this paper, we have studied about fourteen heterogeneous covariance structures. These covariance structures are divided into two types namely, banded and non-banded. The non-banded covariance structure has been studied by various authors namely, Hussein and Marshadi (2007), Eyduran and Akbas (2010), Kincaid (2012) and Mohanraj and Srinivasan (2015a,b). The banded covariance structure was studied by Littell et al. (2000), and Fitzmaurice et al. (2004). The study involves using the bootstrap data under linear mixed model setup to estimate the parameters using Maximum Likelihood (ML) and Restricted Maximum Likelihood Methods (REML) approach and identify the best covariance structure model with the help of six information criteria. The idea of using the bootstrap is to improve the performance of a model selection rule introduced by Efron (1986), and bootstrap sample size is taken to be same as the size of the observed and unobserved samples.

Generally missing data analysis handles two different mechanisms namely, Missing Completely At Random (MCAR) and Missing At Random (MAR). The study is now restricted only to MAR approach. The MAR data has been studied by Allison (2002) and Enders (2010).

In missing data analysis three types of data sets namely, missing data with replacement, missing data with case deletion and imputation are involved. Again, the case deletion technique has two different deletion methods namely, listwise and pairwise deletion. The present study is restricted to the pairwise case deletion data. In pairwise deletion, only specific missing values from the analysis are removed and not the entire case. In other words, all available data is included. The pairwise deletion will result in different sample sizes for each correlation being the same were studied by various authors namely, Allison (2002), Little and Rubin (2002) and Howell (2008). The present study is focused on missing at random data in longitudinal study with pairwise deletion under linear mixed models.

2. Mixed Model Approach

In linear mixed effects model both fixed and random effects contribute linearly to the response function and the general form of such a model is,

$$Y = X\beta + Z\gamma + \in \tag{2.1}$$

Y is an $n\times 1$ vector of observations, β is an $p\times 1$ vector of fixed effects, X is an $n\times p$ design matrix for fixed effects, Z is a given $n\times q$ matrix, and γ is an unobservable random vector of dimensions $q\times 1$, $\in$ is an $n\times 1$ vector of residual and both γ and $\in$ are distributed as N(0,G) and N(0,R) respectively, i.e.,

$$E\begin{bmatrix}\gamma\\ \in\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad Var\begin{bmatrix}\gamma\\ \in\end{bmatrix}=\begin{bmatrix}G & 0\\ 0 & R\end{bmatrix}$$

Here $Y\sim \mathrm{N}(X\beta, V)$ and variance of Y is $V=ZGZ'+R$. The model V is setting up the random-effects design matrix Z and by specifying covariance structures for G and R. Simple random effects are a special case of the general mixed model specification with Z containing dummy variables, G containing variance components in a diagonal structure, and $R=\sigma^2 I_n$, where I_n denotes the $n\times n$ identity matrix. Further, general linear fixed effect model is a special case with $Z=0$ and $R=\sigma^2 I_n$. The ordinary least squares approach is no longer appropriate and Generalized Least Squares (GLS) is more appropriate by minimizing,

$$(Y-X\beta)'V^{-1}(Y-X\beta)$$

However, it requires knowledge of V and thereby knowledge of G and R. When lacking such information, a reasonable estimation for V is obtained and the same is substituted in the GLS for the minimization problem. The initial goal thus becomes finding a reasonable estimate for G and R. In many situations, the best approach is to use likelihood-based methods, exploiting the assumption that γ and $\in$ are normally distributed (Harville (1977) and Laird and Ware (1982)). Basically the variance components analysis applies to a mixed effect models, that is, one in which there are random effects and fixed effect is considered often as an adjunct to these procedures. Variance components analysis may be seen as a more computationally efficient procedure useful for models in special designs, such as split plot, univariate repeated measures, random block, and other mixed effect designs. Variance components procedure supports four methods of estimation: Analysis of Variance (ANOVA), ML, REML and Minimum Norm Quadratic Unbiased Estimator (MINQUE). But, the present study is restricted to the comparison of two likelihood estimation methods namely, ML and REML.

A favorable theoretical property of ML and REML are given in Little (1995). PROC MIXED constructs an objective function associated with ML or REML and maximizes it over all unknown parameters. Using calculus, it is possible to reduce this maximization problem over the only parameters in G and R. The corresponding log-likelihood functions are as follows:

$$l_{ML}(G,R): -\frac{1}{2}\log|V| - \frac{1}{2}r^T V^{-1} r - \frac{n}{2}\log(2\pi)$$

$$l_{REML}(G,R): -\frac{1}{2}\log|V| - \frac{1}{2}r^T V^{-1} r - \frac{1}{2}\log|X^T V^{-1} X| - \left(\frac{(n-p)}{2}\log(2\pi)\right)$$

where, $r = Y - X(X^T V^{-1} X)^{-1} X^T V^{-1} Y$ and X is of rank p. Variance components models, mixed effects ANOVA models, and linear models for longitudinal data are all special cases of model (2.1). When observation are missing in planned experiments, like incomplete block design standard linear mixed model approach can be used to estimate the parameters and required hypothesis could be tested in the presence of random effects. The ML and REML approach to repeated measurements and the traditional treatment effects are based on the covariance models using the linear mixed model, paying special attention to complete data. An important advantage of the linear mixed model approach is that completeness of the data on the dependent variable does not complicate the analysis.

The ML approach will first obtain the values of β, σ_γ^2 and σ_ϵ^2 that maximize the likelihood function over the parameter space. The advantages of ML approach are avoidance of negative estimates for the variance components. Its disadvantages are; (a) numerically intensive (b) resulting estimators are not unbiased, (c) solving the likelihood equations requires an iterative process which may or may not converge. Even when it converges, it may converge to a local maxima rather than a global maximum, (d) tends to underestimate the variance components and (e) distributional properties are not known except asymptotically.

REML estimators of the variance components are found by maximizing that part of the likelihood function that is invariant to fixed effects in the model. The likelihood function from Y depends only on the variance components; and the REML estimates of the variance components are those values of σ_γ^2 and σ_ϵ^2 that maximize the restricted likelihood function. The advantages in the case of REML are: (a) less numerically intensive than the ML method, (b) REML estimates and the ANOVA estimates agree when the data are balanced and all Method of Moment estimates of the variance component are non-negative, and (c) REML estimates tend to be less biased than the ML estimates. However the REML too has disadvantages like ML such as, distributional properties of these estimators are not known, except asymptotically.

3. Selection of Covariance Structure

The selection of the most appropriate covariance structure is important in the analysis of longitudinal data. The estimates of the parameters of the longitudinal model largely depends on the covariance structure. The covariance estimation is also of interest by itself (Fitzmaurice et al. 2004). Several authors have contributed to the study of covariance structures.

The process of analyzing data in the linear mixed model usually begins with the choice of G and R, often referred as the covariance structure specification. Commenting on the importance of this decision, Littell et al. (2000) noted that using incorrect covariance structures could risk obtaining invalid estimators and the relevant inference. Littell et al. (2000) suggested that the first thing to do when choosing a covariance structure in repeated measures studies is to compute the Heterogeneous Compound Symmetry (CSH) covariance matrix and compare it to the estimates obtained using other heterogeneous covariance structures namely, Unstructured (UN), First Order Banded Unstructured (UN(1)), Second Order Banded Unstructured (UN(2)), Third Order Banded Unstructured (UN(3)), First Order Banded Unstructured Correlations (UNR(1)), Second Order Banded Unstructured Correlations(UNR(2)), First Order Heterogeneous Auto Regressive (ARH(1)), Heterogeneous Toeplitz (TOEPH), Heterogeneous Toeplitz With One Banded (TOEPH(1)), Heterogeneous Toeplitz With Two Banded (TOEPH(2)), Heterogeneous Toeplitz with Three Banded (TOEPH(3)), First Order-Factor Analytic (FA(1)), Hyunh-Feldt (HF) and notations of the above structures are given in the Table(1). PROC MIXED in SAS provides a very flexible environment in which model can be many of repeated measures data. Correlation among measurements made on same subject or experiment unit can be modeled using random effect and through the specification of a covariance structure. PROC MIXED provides a useful covariance structures for modeling both time and space, including that of discrete & continuous increments.

In this paper, we have studied the selection of appropriate covariance structure of the data of TLC trail experiment. Radhakrishnan and Manoharan (2012) have studied the general linear regression model using REML approach of selecting different homogenous and heterogeneous with banded, non banded covariance structures and considered three information criteria namely AIC, AICC and BIC. The analysis based on simulation confirmed that Unstructured covariance structure is most appropriate among the different covariance structures. Gazel (2012) have studied the problem of covariance structure relating to the data weights of animal, in which they have adapted the Random Intercept and Slop Model (RISM), the covariance structures consider by them are given in two groups of covariance structures namely, heterogeneous and homogeneous. The heterogeneous covariance structure group members are UN, CSH, ARH(1), TOEPH and ANTE(1). The homogeneous covariance structure group members are CS, TOEP and AR(1). The covariance structures are based on the analysis of weights of animal data and it is concluded that the heterogeneous models are the better performers. Kincaid (2012) have practically described linear mixed models with various covariance structure, and concluded that random effect models is a better fit in the evaluation of covariance structure and the analyses were worked in PROC MIXED documentation in the SAS software. Hussein and Marshadi (2007) studied simulation data set using twelve covariance structure and model selection using six information criteria, and concluded that CAIC and BIC are better performers in these covariance model selection study. Keselman et al. (1999) have investigated using the AIC and BIC for various conditions (e.g., covariance heterogeneity, non-normality, unequal group sizes) and concluded AIC is a better fit in the repeated measures data. Eyduran and Akbas (2010) have compared the performance of univariate and multivariate approaches by using nine covariance structures using five information criteria and concluded multivariate approach is a better fit in the covariance structure study. Ohlson and Koski (2009) studied banded matrices with unequal elements except that certain covariance are zero. This banded structure is a special case of the structure considered by Chaudhuri et al. (2007). The basic idea is that widely separated observations in time or space often appear to be uncorrelated. Therefore, it is reasonable to work with a banded covariance structure, where all covariance more than ‘n’ steps apart equal zero, a so called n-dependent structure. However, in general it was observed that more the number of covariance parameters in the model the lower the efficiency values in the information criteria. However for the first order banded structure UN(1), UNR(1) and TOEPH(1) have the same covariance structure.

Table (1) Matrix notation of different covariance models

$$UN = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_4^2 \end{bmatrix} \qquad UN(1) = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \sigma_3^2 & \\ & & & \sigma_4^2 \end{bmatrix} \qquad UN(2) = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & & \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \\ & \sigma_{32} & \sigma_3^2 & \sigma_{34} \\ & & \sigma_{43} & \sigma_4^2 \end{bmatrix}$$

$$TOEPH(1) = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \sigma_3^2 & \\ & & & \sigma_4^2 \end{bmatrix}$$

$$UN(3)=\begin{bmatrix}\sigma_1^2 & \sigma_{12} & \sigma_{13} & \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \sigma_{24}\\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \sigma_{34}\\ & \sigma_{42} & \sigma_{43} & \sigma_4^2\end{bmatrix} \qquad UNR(1)=\begin{bmatrix}\sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \sigma_3^2 & \\ & & & \sigma_4^2\end{bmatrix}$$

$$UNR(2)=\begin{bmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho_{12} & & \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 & \sigma_2\sigma_3\rho_{23} & \\ & \sigma_3\sigma_2\rho_{32} & \sigma_3^2 & \sigma_3\sigma_4\rho_{34}\\ & & \sigma_4\sigma_3\rho_{43} & \sigma_4^2\end{bmatrix}$$

$$CSH=\begin{bmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho & \sigma_1\sigma_3\rho & \sigma_1\sigma_4\rho\\ \sigma_2\sigma_1\rho & \sigma_2^2 & \sigma_2\sigma_3\rho & \sigma_2\sigma_4\rho\\ \sigma_3\sigma_1\rho & \sigma_3\sigma_2\rho^2 & \sigma_3^2 & \sigma_3\sigma_4\rho\\ \sigma_4\sigma_1\rho & \sigma_4\sigma_2\rho & \sigma_4\sigma_3\rho & \sigma_4^2\end{bmatrix} \qquad ARH(1)=\begin{bmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho & \sigma_1\sigma_3\rho^2 & \sigma_1\sigma_4\rho^3\\ \sigma_2\sigma_1\rho & \sigma_2^2 & \sigma_2\sigma_3\rho & \sigma_2\sigma_4\rho^2\\ \sigma_3\sigma_1\rho^2 & \sigma_3\sigma_2\rho^2 & \sigma_3^2 & \sigma_3\sigma_4\rho\\ \sigma_4\sigma_1\rho^3 & \sigma_4\sigma_2\rho^2 & \sigma_4\sigma_3\rho & \sigma_4^2\end{bmatrix}$$

$$TOEPH=\begin{bmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho_1 & \sigma_1\sigma_3\rho_2 & \sigma_1\sigma_3\rho_3\\ \sigma_2\sigma_1\rho_1 & \sigma_2^2 & \sigma_2\sigma_3\rho_1 & \sigma_2\sigma_4\rho_2\\ \sigma_3\sigma_1\rho_2 & \sigma_3\sigma_2\rho_1 & \sigma_3^2 & \sigma_3\sigma_4\rho_1\\ \sigma_4\sigma_1\rho_3 & \sigma_4\sigma_2\rho_2 & \sigma_4\sigma_3\rho_1 & \sigma_4^2\end{bmatrix}$$

$$TOEPH(2)=\begin{bmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho_1 & & \\ \sigma_2\sigma_1\rho_1 & \sigma_2^2 & \sigma_2\sigma_3\rho_1 & \\ & \sigma_3\sigma_2\rho_1 & \sigma_3^2 & \sigma_3\sigma_4\rho_1\\ & & \sigma_4\sigma_3\rho_1 & \sigma_4^2\end{bmatrix} \qquad TOEPH(3)=\begin{bmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho_1 & \sigma_1\sigma_3\rho_2 & \sigma_1\sigma_3\rho_3\\ \sigma_2\sigma_1\rho_1 & \sigma_2^2 & \sigma_2\sigma_3\rho_1 & \sigma_2\sigma_4\rho_2\\ \sigma_3\sigma_1\rho_2 & \sigma_3\sigma_2\rho_1 & \sigma_3^2 & \sigma_3\sigma_4\rho_1\\ \sigma_4\sigma_1\rho_3 & \sigma_4\sigma_2\rho_2 & \sigma_4\sigma_3\rho_1 & \sigma_4^2\end{bmatrix}$$

$$FA(1)=\begin{bmatrix}\lambda_1^2+\rho_1 & \lambda_1\lambda_2 & \lambda_1\lambda_3 & \lambda_1\lambda_4\\ \lambda_2\lambda_1 & \lambda_2^2+\rho_2 & \lambda_2\lambda_3 & \lambda_1\lambda_2\\ \lambda_3\lambda_1 & \lambda_3\lambda_2 & \lambda_3^2+\rho_3 & \lambda_3\lambda_4\\ \lambda_4\lambda_1 & \lambda_4\lambda_2 & \lambda_4\lambda_3 & \lambda_4^2+\rho_4\end{bmatrix}$$

$$HF=\begin{bmatrix}\sigma_1^2 & \frac{\sigma_1^2-\sigma_2^2}{2}-\lambda & \frac{\sigma_1^2-\sigma_3^2}{2}-\lambda & \frac{\sigma_1^2-\sigma_4^2}{2}-\lambda\\ \frac{\sigma_2^2-\sigma_1^2}{2}-\lambda & \sigma_2^2 & \frac{\sigma_2^2-\sigma_3^2}{2}-\lambda & \frac{\sigma_2^2-\sigma_4^2}{2}-\lambda\\ \frac{\sigma_3^2-\sigma_1^2}{2}-\lambda & \frac{\sigma_3^2-\sigma_2^2}{2}-\lambda & \sigma_3^2 & \frac{\sigma_3^2-\sigma_4^2}{2}-\lambda\\ \frac{\sigma_4^2-\sigma_1^2}{2}-\lambda & \frac{\sigma_4^2-\sigma_2^2}{2}-\lambda & \frac{\sigma_4^2-\sigma_3^2}{2}-\lambda & \sigma_4^2\end{bmatrix}$$

Model selection involves the choice of an appropriate model among a set of candidate models and is used when there is no particular clear choice among many different models. Model selection tools are a useful set of techniques for screening through many different covariance models. The choice among models can be made by comparing the ML for each of the covariance pattern models. The more complex the model is (more parameters) a better fit and a higher likelihood function value is obtained and information criteria are widely used in model selection.

4. Information Criteria

There are two statistics based on the likelihood that make allowance for the number of covariance parameters fitted and can be used to compare two models which fit the same fixed effects. The log-likelihood value of REML or ML, total number of cases 'n' (or total of case weights if used) and 'k' the numbers of model parameters, provide the basis for various information criteria such as:

(1) Akaike Information Criteria (AIC) : $AIC = 2K - 2\log(l)$

(2) Corrected Akaike Information Criteria (AICC : $AICC = -2\log(l) + \frac{2k \times n}{(n-k-1)}$

(3) Bayesian Information Criteria (BIC) : $BIC = -2\log(l) + k \times \log(n)$

(4) Consistent Akaike Information Criteria(CAIC) : $CAIC = -2\log(l) + k(\log n + 1)$

(5) Hannan-Quinn Information Criteria (HQIC): $HQIC = -2\log(l) + 2\log(\log(n))k$

(6) Average Information Criteria (AVIC): $AVIC = AIC + AIC_C + HQIC + BIC + CAIC$

For REML, the value of 'n' is chosen to be total number of cases minus number of fixed effects parameters and 'k' is the number of covariance parameters and for ML, the value of 'n' is total number of cases and 'k' is the number of fixed effects parameters plus number of covariance parameters.

The AIC was developed by Akaike in (1973), AICC was developed by Hurvich and Tsai (1989), BIC by Schwarz (1978) and HQIC was proposed by Hannan and Quinn (1979) proved to be useful in the case of autoregressive models.

As k, the dimension (number of parameters) of the candidate model, increases in comparison to n, the sample size, AIC becomes a strongly negatively biased estimate of the information and the bias AICC. The correction is of particular interest and use when the sample size is small, or when the number of fitted parameters is moderate to large fraction of the sample size. For linear mixed model, the corrected method, AICC, is exactly unbiased. The BIC is a criterion of model selection among a finite set of models. It is based, in part, on the likelihood function and it is closely related to the AIC. When fitting models, it is possible to increase the likelihood by adding parameters, but doing so may result in over fitting. Both BIC and AIC resolve this problem by introducing a penalty term for the number of parameters in the model; the penalty term is larger in BIC than in AIC. The HQIC contain both of AIC and BIC and proved to be useful in the determination of the order of an Auto regression. Burnham and Anderson (2002) noted that HQIC, like BIC, but unlike AIC, is not an estimator of Kullback–Leibler divergence but Claeskens and Hjort (2008) observed that HQIC is not asymptotically efficient. The comparative studies of information criteria are the powerful tool that can be used to find out the better fit between various covariance models described above.

Pankaj and Shukla (2011) discussed the longitudinal data for linear mixed model setting using REML approach for covariance model selection based on six information criteria. Gazel (2012) has proposed the linear mixed model for repeated measure data to draw comparison between ML and REML approaches of covariance model selection criteria. Radhakrishnan and Monoharan (2012) have discussed longitudinal data for linear regression model settings using REML approach of covariance model selection. In this study heterogeneous covariance structures and model selection information criteria are analyzed in SAS software.

The heterogeneous models are divided into two types, namely,

(1) Heterogeneous for non-zero correlation models structures;

UN, CSH, ARH(1), TOEPH, FA(1) and HF

(2) Heterogeneous for zero correlation models structures;

UN (1), UNR (1) and TOEPH (1)

Generally Heterogeneous models have unequal variances on main diagonal and separate covariances as off diagonal, variances are estimated for each treatment time and covariances estimated for each pair of treatment times and UN model is the most complex of the heterogeneous model. In this paper, all the covariance structures on main diagonals are unequal.

The likelihood ratio (LR) test can also be used to compare models which fit the same fixed effects and whose covariance patterns are nested. Nesting is when the covariance pattern in the simpler model can be obtained by restricting some of the parameters in the more complex model. An example of this would be as follows:

$$CSH = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho & \sigma_1\sigma_3\rho & \sigma_1\sigma_4\rho \\ \sigma_2\sigma_1\rho & \sigma_2^2 & \sigma_2\sigma_3\rho & \sigma_2\sigma_4\rho \\ \sigma_3\sigma_1\rho & \sigma_3\sigma_2\rho^2 & \sigma_3^2 & \sigma_3\sigma_4\rho \\ \sigma_4\sigma_1\rho & \sigma_4\sigma_2\rho & \sigma_4\sigma_3\rho & \sigma_4^2 \end{bmatrix}$$

$$TOEPH = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_1 & \sigma_1\sigma_3\rho_2 & \sigma_1\sigma_3\rho_3 \\ \sigma_2\sigma_1\rho_1 & \sigma_2^2 & \sigma_2\sigma_3\rho_1 & \sigma_2\sigma_4\rho_2 \\ \sigma_3\sigma_1\rho_2 & \sigma_3\sigma_2\rho_1 & \sigma_3^2 & \sigma_3\sigma_4\rho_1 \\ \sigma_4\sigma_1\rho_3 & \sigma_4\sigma_2\rho_2 & \sigma_4\sigma_3\rho_1 & \sigma_4^2 \end{bmatrix}$$

The CSH model is nested within TOEPH model, since the CSH model holds, then the TOEPH model must necessarily hold, with $\rho_1 = \rho_2 = \rho_3$.

In many ways, the LR test is conceptually the easiest of the test considered in this section using covariance model. The estimate obtained by ML, $l(CM)_{IC_{full}}$ is the value of the likelihood function for the maximum of the unconstrained model and $l(CM)_{IC_{\min}}$ is the value when the constraints are imposed. The LR test is obtained by taking the difference computed as:

$$\Delta_{IC_{CM}} = 2(l(CM)_{IC_{full}} - l(CM)_{IC_{\min}}) \sim \chi^2_{\mathrm{df}}$$

and the LR difference are used in comparison of the model selection criteria with df as number of covariance parameters.

This test is always positive (or zero) since the likelihood of the unconstrained model is at least as high as that of the constrained model. The LR statistic is distributed asymptotically as a Chi-Squared distribution with (n-1) degrees of freedom equal to the number of constraints. The LR test described above can be used to compare nested covariance models. Two covariance pattern models are said to be nested when one (the minimum) model is a special case of the other (full i^{th}) model. For a pair of nested models, a LR test statistic can be constructed that compares the "full i^{th}" and "minimum" model.

5. Analysis of TLC Data

Fitzmaurice et al. (2004) have analyzed Treatment of Lead-Exposed Children (TLC) data collected by Rogan et al. (2001) and the same data set is considered for the present analysis. Exposure to lead can produce cognitive impairment, especially among young children and infants. The TLC data considers a placebo-controlled, randomized trial of Succimer conducted in 80 children. Children received up to three 26-day courses of Succimer or placebo and were followed for 3 years. The blood lead levels at baseline, week 1, week 4, and week 6 are measured for each child and a sample of ten children is presented in the following table. Most of the researchers have analyzed the complete TLC data, but practical consideration provides an opportunity to relook at the same data with missing data analysis. Firstly the study consist of choosing 85% of them randomly and the remaining 15% are considered to be missing responses. Similarly we create a data set with 20%, 25% and 30% of missing values. The bootstrap of missing longitudinal data analyses are considered using the mixed model approach of ML and REML with all possible covariance structures described in the earlier section.

TLC Data

ID	Group	Week0	Week1	Week4	Week6
1	P	31.9	27.9	27.3	34.2
2	-	29.6	15.8	23.7	-
3	S	21.5	6.5	7.1	16
4	P	26.2	-	25.3	24.8
5	S	21.8	12	-	19.2
6	S	23	4.2	4	16.2
7	S	22.2	11.5	9.5	14.5
8	-	20.5	21.1	-	21.1
9	S	25	3.9	12.8	12.7
10	P	33.3	26.2	-	-

TLC data of missing observations are analyzed through the pairwise deletion technique. The MAR procedure is used in this data set, since observations are missing on the independent longitudinal outcome variables. By deletion of a missing data case in total, there will be a severe loss of information from the point of analysis. The SAS codes for the analysis of performance based on information criteria using different covariance structures are given in the APPENDIX-I.

The present study considers the analysis of TLC data based on the general linear fixed effect model which is a special case of linear mixed model given in equation (**1**) with $Z = 0$ and $R = \sigma^2 I_n$. The heterogeneous covariance models with maximum number of estimated parameters, namely UN, FA(1), CSH and TOEPH and HF were found to be the good performers as compared to all the covariance structures. In particular, UN and CSH covariance structures have outperformed in these entire heterogeneous models. The results are presented in Table (2), Table (3), Table (4), Table (5) as given in APPENDIX-II shows the performance of the covariance structures based on the estimation methods, ML and REML. Goodness of fit result indicates REML as the best covariance parameter estimation approach. It is true for a single sample data study but cannot be seen directly in the bootstrap samples. The AIC, AICC, HQIC, BIC, CAIC, AVIC chosen criteria showed a tendency to rank the Heterogeneous covariance models as the best model group as UN, CSH, FA(1) and HF.

The values of AIC and AICC criteria based on ML and REML estimation of parameters strongly recommend for these four covariance structures and in particular UN covariance model, because the other three covariance models of the likelihood differences are smaller (p-value) than UN covariance model and their information criteria. The computed values of HQIC, BIC, CAIC and AVIC criteria based on ML and REML estimation of parameters strongly recommend CSH covariance structure, also these four information criteria have given higher likelihood values compared to the values for AIC and AICC. In these heterogeneous covariance groups of model selection criteria has shown the banded covariance models are the weakest models in the heterogeneous covariance model group namely, UN(1), UN(2), UN(3), UNR(1), UNR(2), TOEPH(1), TOEPH(2) and TOEPH(3). More critically, the estimates of all fourteen covariance structure are very similar and the differences are unlikely to affect the results of the primary analyses.

Now for each of the fourteen covariance structure corresponding to each of the six information criteria, the values of Likelihood Ratio (LR) test statistics are obtained. The difference in the values of LR test statistics and values of the probability (p-value) are given in Table (2-5). This has been done for each of the four bootstrap sample data with difference percentages. For example in the case of 15% bootstrap missing data set, the difference in the values of the LR test statistic between the two covariance structure CSH and UN with respect to the six information criteria are given below.

$$\Delta_{AIC_{CSH}} = 2(.5*1651.7 - .5*1644.2) = 7.5 = \chi_5^2\,(p = 0.7349)$$

$$\Delta_{AICC_{CSH}} = 2(.5*1653.2 - .5*1647) = 6.2 = \chi_5^2\,(p = 0.2872)$$

$$\Delta_{HQIC_{UN}} = 2(.5*1669.2 - .5*1661.3) = 7.9 = \chi_5^2\,(p = 0.6386)$$

$$\Delta_{BIC_{UN}} = 2(.5*1687 - .5*1682.7) = 4.3 = \chi_5^2\,(p = 0.9328)$$

$$\Delta_{CAIC_{UN}} = 2(.5*1705 - .5*1695.7) = 9.3 = \chi_5^2\,(p = 0.5039)$$

$$\Delta_{AVIC_{UN}} = 2(.5*1670 - .5*1668) = 1.7 = \chi_5^2\,(p = 0.9981)$$

Each one of the above difference of the LR statistics is compared to a chi-squared distribution with 5 degrees of freedom. The value of the first difference (7.5) and the p-value (0.7349) show that $\Delta_{AIC_{CSH}}$ of CSH covariance structure does not provide an adequate fit, when compared to the UN covariance structure model. The UN covariance structure (p =1.00) provides an adequate fit based on the values given in Tables (2-5). In a similar manner the values of the second difference (6.2) and their p value (p = 0.2872) showed that $\Delta_{AICC_{CSH}}$ of CSH covariance structure does not provide an adequate fit and UN covariance structure (p =1.00) provides an adequate fit to the various bootstrap sample sizes.

The value of the third difference (7.9) and the p-value (0.6386) show that $\Delta_{HQIC_{UN}}$ of UN covariance structure does not provide an adequate fit, when compared to the CSH covariance structure model. The CSH covariance structure (p =1.00) provides an adequate fit based on the values given in Tables (2-5). In a similar manner the values of the four, fifth and sixth differences (4.3, 9.3 and 1.7) and their p values (p=0.9328, 0.5039 and 0.9981) showed that, $\Delta_{BIC_{UN}}$, $\Delta_{CAIC_{UN}}$ and $\Delta_{AVIC_{UN}}$ of UN covariance structure does not provide an adequate fit and CSH (p =1.00) covariance structure provides an adequate fit to the various percentages of bootstrap sample sizes. In the tables (2-5) values of p =1.000 can be seen directly for the other p values are not seen explicitly since this values corresponds to the results of the four bootstrap sample sizes.

In the above first two difference of the LR test statistics, $\Delta_{AIC_{CSH}}$ and $\Delta_{AICC_{CSH}}$ indicate that the UN Covariance structure is a significant improvement over the simpler CSH covariance model. However, $\Delta_{HQIC_{UN}}$, $\Delta_{BIC_{UN}}$, $\Delta_{CAIC_{UN}}$ and $\Delta_{AVIC_{UN}}$ indicate that the CSH covariance structure is a significant improvement over the simpler UN Covariance model. In those heterogeneous model with p values (1.00) indicate most appropriate model fit (refer to the Table (2-5) in Appendix-II). The comparison of the nested and non-nested cases in the heterogeneous models of the observed χ^2 value at the appropriate degrees of freedom is approximately significant as indicated by the high p-values (1.00).

The bootstrap study as carried out has no unanimity and this could be seen selecting a covariance structure based on information criteria for various sizes of bootstrap missing data samples. For example, the values of AIC, corresponding to ML and REML in Table (2) show that UN is an adequate fit in 80% and 80%. The CSH is an adequate fit in 10% and 13.3%. The TOEPH is an adequate fit in 10% and 6.6% and The FA(1) is an adequate fit in 6.6% and 6.6%. In a similar manner for each of the other five information criteria, the percentage of bootstrap sample providing adequate fit (1.00) to the corresponding models can be obtained. The values of the Tables (2-5) provide details about the percentage giving the adequate fit with respect to the corresponding model for bootstrap sample of sizes 15%, 20%, 25% and 30%. The results were similar in nature when the percentages of missing observations are increased from 15% to 30%.

We note that, in most of the bootstrap missing data sets whenever UN model is find to provide adequate fit, FA(1) model also gives an adequate fit to the data.

Also, whenever CSH model is find to provide adequate fit, HF model also gives an adequate fit in most of bootstrap samples. Generally UN and CSH covariance model performance are good in information criteria, as observed in the range of 55% to 80% of results and FA(1) and HF covariance structures are contained in 20 % to 45% of results.

The estimates of all fourteen covariance structures are very similar and the differences are unlikely to affect the results of the primary analyses. Therefore, statistical analysis justify in using the more complex covariance pattern of the UN, CSH, FA(1) and HF covariance model are significantly better fit in comparison to other covariance models.

6. Conclusion

In this study of bootstrap method, the linear mixed model was applied on TLC missing data of longitudinal study. The results show that performance of REML approach seems to be more efficient than compared to ML approach.

The inference based on different bootstrap sample data set of various percentages are not similar but different. The overall result revealed that heterogeneous covariance models are the best model group in modeling the covariance structures as in UN, CSH, FA(1), HF and TOEPH particularly the UN and CSH, structures outperform in all classes of information criteria.

The model selection criteria play an important role in selecting the appropriate covariance structures. The study has shown that though well known criteria namely, AIC and AICC suggest UN covariance structure as the most appropriate and better than CSH, FA(1) and HF models. The HQIC, CAIC, BIC and AVIC strongly suggest CSH covariance structure as the most appropriate and better than UN, FA(1) and HF models. The banded covariance models are weakest among the heterogeneous covariance structures namely UN(1), UN(2), UN(3), UNR(1), UNR(2), TOEPH(1), TOEPH(2) and TOEPH(3). The results show that, the information criteria AIC and AICC indicate that the model with UN covariance structure is an appropriate choice, while the information criteria HQIC, BIC, CAIC and AVIC show the selection of the model with CSH covariance structure as most appropriate. In conclusion we observe that, the analysis of various sizes of MAR bootstrap sample sets from the longitudinal data based on linear mixed model have shown that UN and CSH covariance structure to be more appropriate in the context of fourteen covariance structures.

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Appendix-I

Sas Codes In Covariance Structures Analysis

```
%macro mixmiss;
%LET rept = 5;
%LET setname = TLC20MISS;
%do i = 1 %to &rept;
```

```
data TLC;
    set &setname (where = (samp = &i));
    y=l0; time=0; output;
    y=l1; time=1; output;
    y=l4; time=4; output;
    y=l6; time=6; output;
   drop l0 l1 l4 l6;
    run;
              proc print data = tlc(obs=1) NOOBS;
    var samp;
    run;
     proc mixed noclprint=10 order=data data=tlc;
    class id group time;
    model y = group time group*time/s chisq;
    repeated time/Type=UN subject=id r;
    run;
    %end;
    %mend;
    %mixmiss;
```

{ TYPE=UN, UN(1), UN(2), UN(3), UNR(1), UNR(2), CSH, ARH (1), TOEPH, TOEPH(1),
TOEPH(2), TOEPH(3), FA(1), HF }

Appendix-II

Table (2) % of most appropriate model for 14 covariance structures corresponding to 15 % Missing data (p<1.00)

S.NO	Covariance Structure	Estimation method	AIC (%) $(LR=\Delta_{AIC}=\chi^2_{df})$	AICC (%) $(LR=\Delta_{AIC_C}=\chi^2_{df})$	HQIC (%) $(LR=\Delta_{HQIC}=\chi^2_{df})$	BIC (%) $(LR=\Delta_{BIC}=\chi^2_{df})$	CAIC (%) $(LR=\Delta_{CAIC}=\chi^2_{df})$	AVIC (%) $(LR=\Delta_{AVIC}=\chi^2_{df})$
1	UN	ML	80% (1.00)	70% (1.00)	33.3% 1.00)	0% (1.00)	0% (1.00)	16.6%(1.00)
		REML	80% (1.00)	73.3%(1.00)	36.6%(1.00)	0%(1.00)	3.3%(1.00)	23.3%(1.00)
2	UN(1)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
3	UN(2)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
4	UN(3)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
5	UNR(1)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)

6	UNR(2)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
7	CSH	ML	10%(1.00)	20%(1.00)	46.6%(1.00)	70%(1.00)	73.3%(1.00)	46.6%(1.00)
		REML	13.3%(1.00)	16.6%(1.00)	43.3%(1.00)	70%(1.00)	73.3%(1.00)	43.3%(1.00)
8	ARH(1)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	1.00 (3.3%)	1.00 (0%)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	1.00 (3.3%)	1.00 (0%)
9	TOEPH	ML	10%(1.00)	6.6%(1.00)	6.6%(1.00)	3.3%(1.00)	0%(1.00)	6.6%(1.00)
		REML	6.6%(1.00)	6.6%(1.00)	6.6%(1.00)	3.3%(1.00)	0%(1.00)	6.6%(1.00)
10	TOEPH(1)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
11	TOEPH(2)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
12	TOEPH(3)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
13	FA(1)	ML	6.6%(1.00)	3.3%(1.00)	16.6%(1.00)	10%(1.00)	3.3%(1.00)	16.6%(1.00)
		REML	6.6%(1.00)	3.3%(1.00)	10%(1.00)	6.6%(1.00)	0%(1.00)	16.6%(1.00)
14	HF	ML	10%(1.00)	16.6%(1.00)	16.6%(1.00)	20%(1.00)	20%(1.00)	16.6%(1.00)
		REML	1.00(10%)	13.3%(1.00)	16.6%(1.00)	20%(1.00)	20%(1.00)	16.6%(1.00)

Table (3) % of most appropriate model for 14 covariance structures corresponding to 20 % Missing data (p<1.00)

S.NO	Covariance Structure	Estimation method	AIC (%) $(LR=\Delta_{AIC}=\chi^2_{df})$	AICC (%) $(LR=\Delta_{AIC_C}=\chi^2_{df})$	HQIC (%) $(LR=\Delta_{HQIC}=\chi^2_{df})$	BIC (%) $(LR=\Delta_{BIC}=\chi^2_{df})$	CAIC (%) $(LR=\Delta_{CAIC}=\chi^2_{df})$	AVIC (%) $(LR=\Delta_{AVIC}=\chi^2_{df})$
1	UN	ML	63.3%(1.00)	46.6%(1.00)	26.6%(1.00)	6.6%(1.00)	3.3%(1.00)	16.6%(1.00)
		REML	63.3%(1.00)	50%(1.00)	23.3%(1.00)	6.6%(1.00)	3.3%(1.00)	16.6%(1.00)
2	UN(1)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
3	UN(2)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
4	UN(3)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
5	UNR(1)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
6	UNR(2)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
7	CSH	ML	13.3%(1.00)	23.3%(1.00)	30%(1.00)	46.6%(1.00)	60%(1.00)	36.6%(1.00)
		REML	16.6%(1.00)	20%(1.00)	33.3%(1.00)	43.3%(1.00)	60%(1.00)	36.6%(1.00)
8	ARH(1)	ML	0%(1.00)	0% (1.00)	0% (1.00)	3.3%1.00)	3.3%1.00)	3.3%1.00)
		REML	0%(1.00)	0% (1.00)	0% (1.00)	3.3%1.00)	3.3%1.00)	3.3%1.00)
9	TOEPH	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
10	TOEPH(1)	ML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)

		REML	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)	0% (1.00)
11	TOEPH(2)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
12	TOEPH(3)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
13	FA(1)	ML REML	26.6%(1.00) 26.6%(1.00)	23.3%(1.00) 23.3%(1.00)	13.3%(1.00) 16.6%(1.00)	16.6%(1.00) 13.3%(1.00)	6.6%(1.00) 6.6%(1.00)	16.6%(1.00) 16.6%(1.00)
14	HF	ML REML	13.3%(1.00) 1.00 (13.3%)	20%(1.00) 16.6%(1.00)	26.6%(1.00) 26.6%(1.00)	30%(1.00) 33.3%(1.00)	33.3%(1.00) 33.3%(1.00)	30%(1.00) 30%(1.00)

Table (4) % of most appropriate model for 14 covariance structures corresponding to 25 % Missing data (p<1.00)

S.NO	Covariance Structure	Estimation method	AIC (%) $(LR=\Delta_{AIC}=\chi^2_{d}$	AICC (%) $(LR=\Delta_{AIC_C}=\chi$	HQIC (%) $(LR=\Delta_{HQIC}=\chi$	BIC (%) $(LR=\Delta_{BIC}=\chi^2_{d}$	CAIC (%) $(LR=\Delta_{CAIC}=\chi$	AVIC (%) $(LR=\Delta_{AVIC}=\chi^2_{df})$
1	UN	ML REML	70%(1.00) 66.6%(1.00)	63.3%(1.00) 63.3%(1.00)	30%(1.00) 30%(1.00)	6.6%(1.00) 3.3%(1.00)	0%(1.00) 0%(1.00)	30%(1.00) 30%(1.00)
2	UN(1)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
3	UN(2)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
4	UN(3)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
5	UNR(1)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
6	UNR(2)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
7	CSH	ML REML	13.3%(1.00) 13.3%(1.00)	13.3%(1.00) 13.3%(1.00)	40%(1.00) 36.6%(1.00)	53.3%(1.00) 60%(1.00)	60%(1.00) 63.3%(1.00)	33.3%(1.00) 43.3%(1.00)
8	ARH(1)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	3.3%(1.00) 3.3%(1.00)	3.3%(1.00) 3.3%(1.00)	3.3%(1.00) 3.3%(1.00)	3.3%(1.00) 3.3%(1.00)
9	TOEPH	ML REML	0%(1.00) 3.3%(1.00)	0%(1.00) 3.3%(1.00)	3.3%1.00) 0%1.00)	0%(1.00) 3.3%(1.00)	3.3%(1.00) 3.3%(1.00)	3.3%(1.00) 3.3%(1.00)
10	TOEPH(1)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
11	TOEPH(2)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
12	TOEPH(3)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
13	FA(1)	ML	20%(1.00)	20%(1.00)	16.6%(1.00)	13.3%(1.00)	6.6%(1.00)	16.6%(1.00)

		REML	26.6%(1.00)	26.6%(1.00)	16.6%(1.00)	10%(1.00)	3.3%(1.00)	16.6%(1.00)
14	HF	ML REML	3.3%(1.00) 0%(1.00)	3.3%(1.00) 0%(1.00)	16.6%(1.00) 20%(1.00)	26.6%(1.00) 23.3%(1.00)	30%(1.00) 30%(1.00)	20%(1.00) 20%(1.00)

Table (5) % of most appropriate model for 14 covariance structures corresponding to 30 % Missing data (p<1.00)

S.NO	Covariance Structure	Estimation method	AIC (%) $(LR=\Delta_{AIC}=\chi^2_{df})$	AICC (%) $(LR=\Delta_{AIC_C}=\chi^2_{df})$	HQIC (%) $(LR=\Delta_{HQIC}=\chi^2_{df})$	BIC (%) $(LR=\Delta_{BIC}=\chi^2_{df})$	CAIC (%) $(LR=\Delta_{CAIC}=\chi^2_{df})$	AVIC (%) $(LR=\Delta_{AVIC}=\chi^2_{df})$
1	UN	ML REML	70%(1.00) 63.3%(1.00)	63.3%(1.00) 60%(1.00)	30%(1.00) 26.6%(1.00)	6.6%(1.00) 0%(1.00)	0% (1.00) 0% (1.00)	13.3%(1.00) 13.3%(1.00)
2	UN(1)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
3	UN(2)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
4	UN(3)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
5	UNR(1)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
6	UNR(2)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
7	CSH	ML REML	13.3%(1.00) 13.3%(1.00)	13.3%(1.00) 16.6%(1.00)	40%(1.00) 46.6%(1.00)	53.3%(1.00) 60%(1.00)	60%(1.00) 60%(1.00)	33.3%(1.00) 46.6%(1.00)
8	ARH(1)	ML REML	0%(1.00) 0%(1.00)	0%(1.00) 0%(1.00)	3.3%(1.00) 3.3%(1.00)	3.3%(1.00) 3.3%(1.00)	3.3%(1.00) 3.3%(1.00)	3.3%(1.00) 3.3%(1.00)
9	TOEPH	ML REML	0%(1.00) 3.3%(1.00)	0%(1.00) 3.3%(1.00)	3.3%(1.00) 0%(1.00)	0% (1.00) 0% (1.00)	3.3%(1.00) 0%(1.00)	3.3%(1.00) 0%(1.00)
10	TOEPH(1)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
11	TOEPH(2)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
12	TOEPH(3)	ML REML	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)	0% (1.00) 0% (1.00)
13	FA(1)	ML REML	20%(1.00) 26.6%(1.00)	20%(1.00) 30%(1.00)	16.6%(1.00) 26.6%(1.00)	13.3%(1.00) 20%(1.00)	6.6%(1.00) 13.3%(1.00)	16.6%(1.00) 20%(1.00)
14	HF	ML REML	3.3%(1.00) 6.6%(1.00)	3.3%(1.00) 6.6%(1.00)	16.6%(1.00) 16.6%(1.00)	26.6%(1.00) 23.3%(1.00)	30%(1.00) 26.6%(1.00)	20%(1.00) 20%(1.00)