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Concomitants of Generalized Order Statistics from Bivariate Burr XII Distribution

Nayabuddin*, Haseeb Athar and Mahmoud Al-Helan

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ABSTRACT

In this paper probability density function for r^{th}, $1 \le r \le n$ and the joint *pdf* of r^{th} and s^{th}, $1 \le r < s \le n$, concomitants of generalized order statistics from bivariate Burr XII distribution are obtained and then used to derived single and product moments. Further, the results are deduced for moments of order statistics and k^{th} upper record values.

1. Introduction

The Burr XII distribution is the another important member of the Burr family of distributions given by Burr (1942). Since its density functions have a wide variety of shapes, this system is useful for approximating histograms, particularly when a simple mathematical structure for the fitted cumulative distribution function (cdf) is required. Rodriguez (1977) shown that the Burr coverage area on a specific plane is occupied by various well known and useful distributions, including the normal, log-normal, gamma, logistic and extreme value type-I distributions. Hence, it represents a good model for modeling failure time data [Zimmer *et al*., (1998)]. Shao (2004) discussed the maximum likelihood estimation for the three-parameter Burr XII distribution. According to Soliman (2005), this distribution covers the curve shape characteristics for a large number of distributions. Wu *et al*. (2007) studied the estimation problems for this distribution on the basis of progressive type II censoring with random removals, where the number of units removed at each failure time has a discrete uniform distribution.

The *pdf* of bivariate Burr XII distribution [Johnson and Kotz, 1972] is given as

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Corresponding author :* Nayabuddin, Faculty of Public Health and Tropical Medicine, Jazan University, KSA. E-mail: nayabstats@gmail.com. Haseeb Athar, Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah, KSA. Mahmoud Al-Helan , Department of Mathematics & Statistics, Faculty of Science, Mutah University, Jordan.

$$f(x,y) = c(c+1)\alpha_1 k_1 x^{k_1-1} \alpha_2 k_2 y^{k_2-1} (1+\alpha_1 x^{k_1} + \alpha_2 y^{k_2})^{-(c+2)},$$

$$x, y > 0,\ k_1, k_2, c, \alpha_1, \alpha_2 > 0 \qquad (1.1)$$

and corresponding df is

$$F(x,y) = 1-(1+\alpha_1 x^{k_1})^{-c} - (1+\alpha_2 y^{k_2})^{-c} + (1+\alpha_1 x^{k_1} + \alpha_2 y^{k_2})^{-c},$$

$$x, y > 0,\ k_1, k_2, c, \alpha_1, \alpha_2 > 0. \qquad (1.2)$$

The conditional pdf of Y given X is

$$f(y \mid x) = \frac{(c+1)\alpha_2 k_2 y^{k_2-1}(1+\alpha_1 x^{k_1})^{(c+1)}}{(1+\alpha_1 x^{k_1} + \alpha_2 y^{k_2})^{(c+2)}},\ y > 0. \qquad (1.3)$$

The marginal pdf of X is

$$f(x) = \frac{c\,\alpha_1 k_1 x^{k_1-1}}{(1+\alpha_1 x^{k_1})^{(c+1)}},\ x > 0 \qquad (1.4)$$

and the marginal df of X is

$$F(x) = 1-(1+\alpha_1 x^{k_1})^{-c},\ x > 0. \qquad (1.5)$$

The concept of generalized order statistics (gos) have been introduced and extensively studied by Kamps(1995).

Let $n \ge 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}, k \ge 1$ be the parameters such that the random variables $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k), \ldots, X(n,n,\tilde{m},k)$ are said to be *gos* from an absolutely continuous population with df $F()$ and pdf $f()$, if their joint density function is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n) \qquad (1.6)$$

on the cone $F^{-1}(0) < x_1 \le x_2 \le \ldots \le x_n < F^{-1}(1)$.

If $m_i = 0, i = 1,2,\ldots,n-1,\ k = 1$, we obtain the joint pdf of the order statistics and for $m_i = -1, k \in N$, we get the joint pdf of k^{th} record values.

For the case $m_i = m, i = 1,2,\ldots,n-1$, the pdf of $X(r,n,m,k)$ is

$$f_{r,n,m,k}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x)[g_m(F(x))]^{r-1} \tag{1.7}$$

and joint *pdf* of $X(s,n,m,k)$ and $X(r,n,m,k),\ 1 \le r < s \le n,$ is

$$f_{r,s,n,m,k}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^{m} f(x)[g_m(F(x))]^{r-1}$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(y),\ \alpha \le x < y \le \beta \tag{1.8}$$

where,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i\ ,\ \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1} & ,\ m \ne -1 \\ -\log(1-x) & ,\ m = -1 \end{cases}$$

and $g_m(x) = h_m(x) - h_m(0),\ x \in (0,1).$

Let $(X_i, Y_i), i = 1,2,...,n$ be n pairs of independent random variables from some bivariate population with distribution function $F(x,y)$. If we arrange the $X-$ variates in ascending order as $X(1,n,m,k) \le X(2,n,m,k) \le ... \le X(n,n,m,k)$ then $Y-$ variates paired (not necessarily in asescending order) with these generalized ordered statistics are called the concomitants of generalized order statistics and are denoted by $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, ..., Y_{[n,n,m,k]}$. The *pdf* of $Y_{[r,n,m,k]}$, the r^{th} concomitant generalized order statistics, is given as

$$g_{[r,n,m,k]}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y \mid x)\ f_{r,n,m,k}(x)\,dx \tag{1.9}$$

The joint *pdf* of r^{th} and s^{th} concomitants, $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ is respectively given by

$$g_{[r,s,n,m,k]}(y_1,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} f_{Y|X}(y_1 \mid x_1)\ f_{Y|X}(y_2 \mid x_2)\ f_{r,s,n,m,k}(x_1,x_2)\,dx_1\,dx_2 \tag{1.10}$$

In the literature, there are many papers which deals with concomitants of order random variables. Das *et al.* (2012) carried out the comparative study on concomitant of order statistics and record statistics for weighted inverse Gaussian distribution. Further, Tahmasebi and Behboodian (2012) obtained the Shannon's entropy for the concomitants of generalized order statstics in Farlie-Gumbel-Morgenstern family. An excellent review on concomitants of order statistics is given Bhattacharya (1984) and David and Nagaraja (1998). For some recent theoretical developments in the theory of concomitants of order statistics one can see, Veena and Thomas (2008, 2017) and Thomas and Veena (2011) . Do and

Hall (1991) obtained distribution theory using concomitants of order statistics with application to Monte Carlo simulation for the bootstrap. O'Connell and David (1976) studied order statistics and their concomitants in some double sampling situations. Tsukibayashi (1998) obtained the joint distribution and moments of an extreme of the dependent variable and the concomitant of an extreme of the independent variable. Balasubramanian and Beg (1996, 1997, 1998) studied the concomitants for bivariate exponential distribution of Marshall-Olkin, Morgesnstern type bivariate exponential distribution and Gumbel's bivariate exponential distribution and gave the recurrence relation between single and product moment of order statistics. Begum and Khan (1997, 1998, 2000) studied the concomitants for Gumbel's bivariate Weibull distribution, bivariate Burr distribution, Marshall and Olkin bivariate Weibull distribution and gave single and product moment of order statistics. Ahsanullah and Beg (2006) studied the concomitants for Gumbel's bivariate exponential distribution and derived the recurrence relation between single and product moment of generalized order statistics.

2. Probability Density Function

For the bivariate Burr XII distribution as given in (1.1), using (1.3), (1.4), (1.5) and (1.7) in (1.9), the *pdf* of $r-th$ concomitant of *gos* $Y_{[r,n,m,k]}$ is given as:

$$g_{[r,n,m,k]}(y) = \frac{\alpha_2 k_2\, C_{r-1}}{(r-1)!(m+1)^{r-1}}\, c\,(c+1)\, y^{k_2-1} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i}$$
$$\times \int_0^\infty \frac{\alpha_1 k_1\, x^{k_1-1}}{(1+\alpha_1 x^{k_1} + \alpha_2 y^{k_2})^{c+2}} \frac{1}{(1+\alpha_1 x^{k_1})^{c(\gamma_{r-i}-1)}}\, dx. \tag{2.1}$$

Letting $t = \alpha_1 x^{k_1}$, then the R.H.S. of (2.1) reduces to

$$= \frac{\alpha_2 k_2\, C_{r-1}}{(r-1)!(m+1)^{r-1}}\, c\,(c+1)\, y^{k_2-1} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i}$$
$$\times \int_0^\infty (1+t+\alpha_2 y^{k_2})^{-(c+2)} (1+t)^{-c(\gamma_{r-i}-1)}\, dt\,. \tag{2.2}$$

Note that [Erd'elyi *et al*., 1954].

$$\int_0^{\infty} x^{\nu-1}\,(a+x)^{-\mu}\,(x+y)^{-\rho}\,dx$$

$$= \frac{\Gamma\nu\,\Gamma(\mu-\nu+\rho)}{\Gamma(\mu+\rho)\,a^{\mu}}\,y^{(\nu-\rho)}\;{}_3F_2\left[\begin{matrix}\mu, \quad \nu \\ \\ \mu+\rho\end{matrix}\;;\;1-\frac{y}{a}\right] \tag{2.3}$$

$$[\,|\arg a|<\Pi,\, Re\,\nu>0,\;|\arg y|<\Pi,\;Re\,\rho>Re\,(\nu-\mu)]$$

Using relation (2.3) in (2.2),we get

$$g_{[r,n,m,k]}(y)=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\,c\,(c+1)\frac{(\alpha_2 k_2\,y^{k_2-1})}{(1+\alpha_2 y^{k_2})^{(c+1)}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}$$

$$\times\frac{1}{(c\gamma_{r-i}+1)}\;{}_2F_1\left[\begin{matrix}(c\,\gamma_{r-i}-c),\;\;1 \\ \\ (c\,\gamma_{r-i}+2)\end{matrix}\;;\;-\alpha_2 y^{k_2}\right], \tag{2.4}$$

where,

$${}_2F_1\left[\begin{matrix}a\,,\;b \\ \\ c\end{matrix}\;;-z\right]=\sum_{p=0}^{\infty}\frac{(a)_p(b)_p}{(c)_p}\frac{(-z)^p}{p!} \tag{2.5}$$

is conditionally convergent for $|z|=1,\ z\neq-1$ if $-1<\mathrm{Re}(\omega)\leq 0,$ [Prudnikov *et al.*, 1986].

3. Moment of Single Concomitant

In this section, we derive the moments of $Y_{[r,n,m,k]}$ for bivariate Burr XII distribution by using the results of the previous section. In view of (2.4), the moments of $Y_{[r,n,m,k]}$ is

$$E(Y^{(a)}_{[r,n,m,k]})=\int y^a\,g_{[r,n,m,k]}(y)\,dy. \tag{3.1}$$

$$=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\,c\,(c+1)\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{1}{c\gamma_{r-i}+1}$$

$$\times\int_0^{\infty}y^a\,\frac{(\alpha_2 k_2 y^{k_2-1})}{(1+\alpha_2 y^{k_2})^{(c+1)}}\;{}_2F_1\left[\begin{matrix}(c\,\gamma_{r-i}-c),\quad 1 \\ \\ (c\,\gamma_{r-i}+2)\end{matrix}\;;-\alpha_2 y^{k_2}\right]dy. \tag{3.2}$$

In view of (2.5), we have

$$E(Y_{[r,n,m,k]}^{(a)}) = \frac{c\,(c+1)C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1}(-1)^i \binom{r-1}{i} \frac{1}{c\gamma_{r-i}+1} \sum_{p=0}^{\infty} \frac{(c\gamma_{r-i}-c)_p\,(1)_p}{(c\gamma_{r-i}+2)\,p!}$$
$$\times \int_0^\infty y^a \frac{(\alpha_2 k_2 y^{k_2-1})}{(1+\alpha_2 y^{k_2})^{(c+1)}} (-\alpha_2 y^{k_2})^p \; dy. \quad (3.3)$$

Now letting $t = 1+\alpha_2 y^{k_2}$ in (3.3), we get

$$E(Y_{[r,n,m,k]}^{(a)}) = \frac{1}{(\alpha_2)^{\frac{a}{k_2}}} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} c\,(c+1) \sum_{i=0}^{r-1}(-1)^i \binom{r-1}{i} \frac{1}{c\gamma_{r-i}+1}$$

$$\times \sum_{p=0}^{\infty} \frac{(c\gamma_{r-i}-c)_p\,(1)_p\,(-1)^p}{(c\gamma_{r-i}+2)\,p!} \int_1^\infty t^{-(c+1)}\,(1-t)^{(p+\frac{a}{k_2})}\,dt. \quad (3.4)$$

Note that [Erd'elyi *et al.*, 1954].

$$\int_y^\infty x^{-\lambda}\,(x-y)^{\mu-1}\,dx = \frac{\Gamma(\lambda-\mu)\,\Gamma\mu}{\Gamma\lambda}\,y^{(\mu-\lambda)},\; 0 < Re\,\mu < Re < \lambda. \quad (3.5)$$

Now applying relation (3.5) in (3.4), we get

$$E(Y_{[r,n,m,k]}^{(a)}) = \frac{1}{(\alpha_2)^{\frac{a}{k_2}}} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} c\,(c+1) \sum_{i=0}^{r-1}(-1)^i \binom{r-1}{i} \frac{1}{c\gamma_{r-i}+1}$$

$$\times \sum_{p=0}^{\infty} \frac{(c\gamma_{r-i}-c)_p\,(1)_p\,(-1)^p}{(c\gamma_{r-i}+2)_p\;p!}\; \frac{\Gamma(c-\frac{a}{k_2}-p)\,\Gamma(p+1+\frac{a}{k_2})}{\Gamma(c+1)}. \quad (3.6)$$

Noting that [Srivastava and Karlsson, 1985].

$$(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n},\; n = 1, 2, 3, \ldots.,\; \lambda \neq 0, \pm 1, \pm 2, \ldots \quad (3.7)$$

On using (3.7) in (3.6), we get after simplification

$$E(Y_{[r,n,m,k]}^{(a)}) = \frac{1}{(\alpha_2)^{\frac{a}{k_2}}} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} c\,(c+1) \sum_{i=0}^{r-1}(-1)^i \binom{r-1}{i} \frac{1}{c\gamma_{r-i}+1}$$

$$\times \frac{\Gamma(c-\frac{a}{k_2})\,\Gamma(1+\frac{a}{k_2})}{\Gamma(c+1)}\ {}_3F_2\left[\begin{matrix}(c\gamma_{r-i}-c), & 1, & (1+\frac{a}{k_2}) \\ & & ;\,1 \\ (c\gamma_{r-i}+2), & (1-c+\frac{a}{k_2}) & \end{matrix}\right]. \tag{3.8}$$

Note that [Prudnikov *et al;* 1986].

$${}_3F_2\left[\begin{matrix}-N, & 1, & a \\ & & ,1 \\ b, & a-m & \end{matrix}\right] = \frac{(b-1)(a-m-1)}{(N+b-1)(a-1)}\ {}_3F_2\left[\begin{matrix}-m, & 1, & 2-b \\ & & ,1 \\ 2-b-N, & 2-a & \end{matrix}\right]$$

$$[m=1,2,\ldots,\ l=-N,-N-1,-N-1,\ldots,]. \tag{3.9}$$

Applying (3.9) in (3.8), we have

$$E(Y_{[r,n,m,k]}^{(a)}) = \frac{c}{(\alpha_2)^{\frac{a}{k_2}}}\,\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\,\frac{(c-\frac{a}{k_2})\Gamma(c-\frac{a}{k_2})}{\Gamma(c+1)}\,\frac{\Gamma(1+\frac{a}{k_2})}{(\frac{a}{k_2})}$$

$$\times \sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\ {}_2F_1\left[\begin{matrix}1, & (-c\gamma_{r-i}) \\ & ;\,1 \\ (1-\frac{a}{k_2}) & \end{matrix}\right]. \tag{3.10}$$

Noting that [Prudnikov *et al;* 1986].

$${}_2F_1\left[\begin{matrix}a, & b \\ & ;\,1 \\ c & \end{matrix}\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)},\ Re\,(c-a-b)>0,\ c\neq,0,-1,-2,\ldots. \tag{3.11}$$

Now in view of (3.11), (3.10) becomes

$$E(Y_{[r,n,m,k]}^{(a)}) = \frac{1}{(\alpha_2)^{\frac{a}{k_2}}}\,\frac{C_{r-1}}{(r-1)!(m+1)^{r}}\,\frac{\Gamma(c+1-\frac{a}{k_2})\,\Gamma(1+\frac{a}{k_2})}{\Gamma(c+1)}$$

$$\times \sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}B\left(\frac{k}{m+1}+(n-r)+i-\frac{a}{c\,k_2\,(m+1)}\ ,\ 1\right). \tag{3.12}$$

For real positive k, c and a positive integer b.

$$\sum_{a=0}^{b}(-1)^a\binom{b}{a}B(a+k,\ c) = B(k,\ c+b) \tag{3.13}$$

Applying (3.13) in (3.12), we get after simplification

$$E(Y_{[r,n,m,k]}^{(a)}) = \frac{1}{(\alpha_2)^{\frac{a}{k_2}}} \frac{\Gamma(c+1-\frac{a}{k_2})\ \Gamma(1+\frac{a}{k_2})}{c\,\Gamma(c)} \frac{1}{\prod_{i=1}^{r}(1-\frac{a}{c\,k_2\,\gamma_i})}. \tag{3.14}$$

Remark 3.1: Set $k_2 = 1$ in (3.14), to get moments of concomitant of generalized order statistics from bivariate Lomax distribution as,

$$E(Y_{[r,n,m,k]}^{(a)}) = \frac{1}{(\alpha_2)^{a}} \frac{\Gamma(c+1-a)\ \Gamma(1+a)}{c\,\Gamma(c)} \frac{1}{\prod_{i=1}^{r}(1-\frac{a}{c\ \gamma_i})}$$

as obtained by Nayabuddin (2013).

Remark 3.2: Set $m = 0$, $k = 1$ in (3.14), to get moments of concomitant of order statistics from bivariate Burr XII distribution as,

$$E(Y_{[r\,:\,n]}^{(a)}) = \frac{1}{(\alpha_2)^{\frac{a}{k_2}}} \frac{\Gamma(c+1-\frac{a}{k_2})\ \Gamma(1+\frac{a}{k_2})}{c\,\Gamma(c)} \frac{1}{\prod_{i=1}^{r}(1-\frac{a}{c\,k_2\,(n-i+1)})}.$$

Remark 3.3: For $m \to -1$ in (3.14), we get moment of concomitant of $k-th$ upper record value from bivariate Burr XII distribution as,

$$E(Y_{[U(r)]}^{k})^{a} = \frac{1}{(\alpha_2)^{\frac{a}{k_2}}} \frac{\Gamma(c+1-\frac{a}{k_2})\ \Gamma(1+\frac{a}{k_2})}{c\,\Gamma(c)} \frac{1}{(1-\frac{a}{c\,k\,k_2})^{r}}.$$

4. Joint Probability Density Function

For the bivariate Burr XII distribution as given in (1.1), using (1.3), (1.4), (1.5) and (1.8) in (1.10), the joint pdf of r^{th} and s^{th} concomitants of *gos* $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ for $m \neq -1$ is given as

$$g_{[r,s,n,m,k]}(y_1, y_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} c^2\ (c+1)^2\ (\alpha_1 k_1)\,(\alpha_2 k_2)^2$$

$$\times\, y_1^{k_2-1}\ y_2^{k_2-1} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i}\binom{s-r-1}{j}$$

$$\times \int_0^\infty \frac{x_2^{k_1-1}}{(1+\alpha_1\ x_2^{k_1})^{c(\gamma_{s-j}-1)}} \frac{1}{(1+\alpha_1\ x_2^{k_1}+\alpha_2\ y_2^{k_2})^{c+2}}\, I(x_2, y_1)\, dx_2 \tag{4.1}$$

where,

$$I(x_2,y_1)=\int_0^{x_2}\frac{\alpha_1 k_1\, x_1^{k_1-1}}{(1+\alpha_1\, x_1^{k_1})^{c(s-r+i-j)(m+1)-c}}\frac{1}{(1+\alpha_1\, x_1^{k_1}+\alpha_2\, y_1^{k_2})^{c+2}}\,dx_1. \qquad (4.2)$$

If we put $t_1=(1+\alpha_1\, x_1^{k_1})$, then the R.H.S. of (4.2) reduce to

$$I(x_2,y_1)=(\lambda)^{-\beta}\int_1^{1+\alpha_1 x_2^{k_1}} t_1^{-\alpha}\,(1+\frac{t_1}{\lambda})^{-\beta}\,dt_1, \qquad (4.3)$$

where $\alpha=c(s-r+i-j)(m+1)-c$, $\beta=(c+2)$ and $\lambda=\alpha_2\, y_1^{k_2}$.

Note that [Prudnikov *et al;* 1986]

$$(1+z)^{-l}=\sum_{k=0}^{\infty}\frac{(-1)^k\,(l)_k\,z^k}{k!}. \qquad (4.4)$$

Using (4.4) in (4.3), and after simplification, we get

$$I(x_2,y_1)=(\lambda)^{-\beta}\sum_{p=0}^{\infty}(-1)^p\,\frac{(\beta)_p\,(\frac{1}{\lambda})^p}{p!}\,\frac{1}{(-\alpha+p+1)}[(1+\alpha_1\, x_2^{k_1})^{-(\alpha-p-1)}-1] \qquad (4.5)$$

Now putting the value of $I(x_2,y_1)$ from (4.5) in (4.1), we obtained

$$g_{[r,s,n,m,k]}(y_1,y_2)=\frac{c^2\,(c+1)^2\,(\alpha_1 k_1)\,(\alpha_2 k_2)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\,y_1^{k_2-1}\,y_2^{k_2-1}$$

$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}(\lambda)^{-\beta}\sum_{p=0}^{\infty}\frac{(-1)^p}{(-\alpha+p+1)}\,\frac{(\beta)_p\,(\frac{1}{\lambda})^p}{p!}$$

$$\times\int_0^{\infty}\frac{\alpha_1 k_1 x_2^{k_1-1}}{(1+\alpha_1\, x_2^{k_1})^{(c\gamma_{s-j}-c)}}\,\frac{1}{(1+\alpha_1\, x_2^{k_1}+\alpha_2\, y_2^{k_2})^{c+2}}[(1+\alpha_1\, x_2^{k_1})^{-(\alpha-p-1)}-1]dx_1 \qquad (4.6)$$

Setting $t_2=(1+\alpha_1\, x_2^{k_1})$ in (4.5), and using relation (4.4), we get

$$g_{[r,s,n,m,k]}(y_1,y_2)=\frac{c^2\,(c+1)^2\,\,(\alpha_2 k_2)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times\,y_1^{k_2-1}\,y_2^{k_2-1}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$

$$\times(\lambda)^{-\beta}(\delta)^{-\beta}\sum_{p=0}^{\infty}\frac{(\beta)_p\,(-\frac{1}{\lambda})^p}{p!}\sum_{l=0}^{\infty}\frac{(\beta)_l\,(-\frac{1}{\delta})^l}{l!}\frac{1}{(1-\theta+l)(2-\theta-\alpha+l+p)}, \tag{4.7}$$

where $\delta=\alpha_2\,y_2^{k_2}$ and $\theta=c\gamma_{s-j}-c$.

Set $d=1-\theta$ and $g=2-\theta-\alpha$ in (4.7), to get

$$g_{[r,s,n,m,k]}(y_1,y_2)=\frac{c^2\,(c+1)^2\,(\alpha_2 k_2)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$

$$\times\, y_1^{k_2-1}\,y_2^{k_2-1}(\lambda)^{-\beta}(\delta)^{-\beta}\sum_{p=0}^{\infty}\frac{(\beta)_p\,(-\frac{1}{\lambda})^p}{(g+p+l)\,p!}\sum_{l=0}^{\infty}\frac{(\beta)_l\,(-\frac{1}{\delta})^l}{(d+l)\,l!} \tag{4.8}$$

After subsituting the value of λ and δ in (4.8), we get

$$=\frac{c^2\,(c+1)^2\,(\alpha_2 k_2)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}y_1^{k_2-1}$$

$$\times\, y_2^{k_2-1}(\alpha_2 y_2^{k_2})^{-\beta}(\alpha_2 y_1^{k_2})^{-\beta}\sum_{l=0}^{\infty}\frac{(\beta)_l\,(\frac{-1}{\alpha_2 y_2^{k_2}})^l}{(d+l)\,l!}\sum_{p=0}^{\infty}\frac{(\beta)_p\,(\frac{-1}{\alpha_2 y_1^{k_2}})^p}{(g+p+l)\,p!}. \tag{4.9}$$

Noting that [Srivastava and Karlsson, 1985].

$$(\lambda+m)=\frac{\lambda\,(\lambda+1)_m}{(\lambda)_m} \tag{4.10}$$

$$(\lambda+m+n)=\frac{\lambda\,(\lambda+1)_{m+n}}{(\lambda)_{m+n}}. \tag{4.11}$$

Using relation (4.10) and (4.11) in (4.9), it becomes

$$g_{[r,s,n,m,k]}(y_1, y_2) = \frac{c^2\,(c+1)^2\, C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i}\binom{s-r-1}{j} \frac{1}{dg}\, \frac{(\alpha_2 k_2\, y_1^{k_2-1})}{(\alpha_2\, y_1^{k_2})^{\beta}}\, \frac{(\alpha_2 k_2\, y_2^{k_2-1})}{(\alpha_2\, y_2^{k_2})^{\beta}}\, F^{1:1;0}_{1:2;1}$$

$$\times \left[\begin{matrix} (g), & (\beta), & (d), & (\beta) \\ & & & \\ (g+1), & (d+1) & & \end{matrix} \; ; \frac{-1}{\alpha_2 y_2^{k_2}}, \frac{-1}{\alpha_2 y_1^{k_2}}\right], \tag{4.12}$$

where,

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix} (a_p), & (b_q), & (c_k) \\ & & \\ (\alpha_l), & (\beta_m), & (\gamma_n) \end{matrix} \; ; x, y\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s} \prod_{j=1}^{q}(b_j)_r \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s} \prod_{j=1}^{m}(\beta_j)_r \prod_{j=1}^{n}(\gamma_j)_s} \frac{x^r}{r!}\, \frac{y^s}{s!} \tag{4.13}$$

is known as Kampé de Fériet’s series (Srivastava and Karlsson, 1985).

5. Product Moment Of Two Concomitants

The product Moments of two concomitants $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ is given by

$$E(Y^{(a)}_{[r,n,m,k]} Y^{(b)}_{[s,n,m,k]}) = \int_0^{\infty}\int_0^{\infty} y_1^a\; y_2^b\;\; g_{[r,s,n,m,k]}(y_1, y_2)\, dy_1\, dy_2. \tag{5.1}$$

In view of (4.12) and (5.1), we have

$$E(Y^{(a)}_{[r,n,m,k]} Y^{(b)}_{[s,n,m,k]}) = A \int_0^{\infty}\int_0^{\infty} y_1^a\; y_2^b\, \frac{(\alpha_2 k_2\, y_1^{k_2-1})}{(\alpha_2\, y_1^{k_2})^{\beta}}\, \frac{(\alpha_2 k_2\, y_2^{k_2-1})}{(\alpha_2\, y_2^{k_2})^{\beta}}$$

$$\times F^{1:1;0}_{1:2;1}\left[\begin{matrix} (g), & (\beta), & (d), & (\beta) \\ & & & \\ (g+1), & (d+1) & & \end{matrix} \; ; \frac{-1}{\alpha_2 y_2^{k_2}}, \frac{-1}{\alpha_2 y_1^{k_2}}\right] dy_1\, dy_2, \tag{5.2}$$

where,

$$A=\frac{c^2\,(c+1)^2\,C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\frac{1}{dg} \tag{5.3}$$

$$E(Y_{[r,n,m,k]}^{(a)}Y_{[s,n,m,k]}^{(b)})=A\int_0^\infty\int_0^\infty y_1^a\;y_2^b\;\frac{(\alpha_2 k_2\;y_1^{k_2-1})}{(\alpha_2\;y_1^{k_2})^{\beta}}\frac{(\alpha_2 k_2\;y_2^{k_2-1})}{(\alpha_2\;y_2^{k_2})^{\beta}}$$

$$\times\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\frac{(g)_{l+p}}{(g+1)_{l+p}}\frac{(\beta)_l\,(d)_l\,(\beta)_p}{(d+1)_l}\frac{\left(\frac{-1}{\alpha_2 y_1^{k_2}}\right)^p}{p!}\frac{\left(\frac{-1}{\alpha_2 y_2^{k_2}}\right)^l}{l!}dy_1\,dy_2. \tag{5.4}$$

Note that [Srivastava and Karlsson, 1985]

$$(\lambda)_{m+n}=(\lambda)_m\,(\lambda+m)_n. \tag{5.5}$$

On applying (5.5) in (5.4), we get

$$E(Y_{[r,n,m,k]}^{(a)}Y_{[s,n,m,k]}^{(b)})=A\int_0^\infty y_2^b\;\frac{(\alpha_2 k_2\;y_2^{k_2-1})}{(\alpha_2\;y_2^{k_2})^{\beta}}\sum_{l=0}^{\infty}\frac{(g)_l}{(g+1)_l}\frac{(\beta)_l\,(d)_l}{(d+1)_l}\frac{\left(\frac{-1}{\alpha_2 y_2^{k_2}}\right)^l}{l!}$$

$$\times\left\{\int_0^\infty y_1^a\;\frac{(\alpha_2 k_2\;y_1^{k_2-1})}{(\alpha_2\;y_1^{k_2})^{\beta}}\sum_{p=0}^{\infty}\frac{(\beta)_p\,(g+l)_p}{(g+1+l)_p}\frac{\left(\frac{-1}{\alpha_2 y_1^{k_2}}\right)^p}{p!}dy_1\right\}dy_2, \tag{5.6}$$

$$=A\int_0^\infty y_2^b\;\frac{(\alpha_2 k_2\;y_2^{k_2-1})}{(\alpha_2\;y_2^{k_2})^{\beta}}\sum_{l=0}^{\infty}\frac{(g)_l}{(g+1)_l}\frac{(\beta)_l\,(d)_l}{(d+1)_l}\frac{\left(\frac{-1}{\alpha_2 y_2^{k_2}}\right)^l}{l!}$$

$$\times\int_0^\infty y_1^a\;\frac{(\alpha_2 k_2\;y_1^{k_2-1})}{(\alpha_2\;y_1^{k_2})^{\beta}}\,{}_2F_1\left[\begin{matrix}(\beta), & (g+l) \\ & ;\frac{-1}{\alpha_2\;y_1^{k_2}} \\ (g+l+1) & \end{matrix}\right]dy_1\,dy_2. \tag{5.7}$$

Now letting $t_1 = \dfrac{1}{\alpha_2\, y_1^{k_2}}$ in (5.7), we get

$$= \frac{A}{(\alpha_2)^{\frac{a}{k_2}}} \int_0^\infty y_2^b \, \frac{(\alpha_2 k_2\, y_2^{k_2-1})}{(\alpha_2\, y_2^{k_2})^\beta} \sum_{l=0}^{\infty} \frac{(g)_l}{(g+1)_l} \, \frac{(\beta)_l\,(d)_l}{(d+1)_l} \, \frac{\left(\dfrac{-1}{\alpha_2 y_2^{k_2}}\right)^l}{l!}$$

$$\times \left\{ \int_0^\infty t_1^{(\beta - \frac{a}{k_2} - 1) - 1} \, {}_2F_1\left[\begin{matrix} (\beta), & (g+l) \\ & & ;-t_1 \\ (g+l+1) & \end{matrix}\right] dt_1 \right\} dy_2. \tag{5.8}$$

Note that [Prudnikov *et al.*, 1986]

$$\int_0^\infty x^{\alpha-1} \, {}_2F_1\left[\begin{matrix} a, & b \\ & & ;-\omega x \\ c & \end{matrix}\right] dx = \omega^{-\alpha} \, \frac{\Gamma(c)\,\Gamma(\alpha)\,\Gamma(a-\alpha)\,\Gamma(b-\alpha)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c-\alpha)},$$

$$[\,0 < Re\,\alpha < Re\,a,\ \ Re\,b\,;\ \ |\,arg\,\omega| < \pi\,]. \tag{5.9}$$

Using relation (5.9) in (5.8), we have

$$E(Y_{[r,n,m,k]}^{(a)} Y_{[s,n,m,k]}^{(b)}) = \frac{A}{(\alpha_2)^{\frac{a}{k_2}}} \, \frac{(g+l)\Gamma(\beta - \frac{a}{k_2} - 1)\Gamma(\frac{a}{k_2}+1)}{\Gamma(\beta)\,(g+l+1-\beta+\frac{a}{k_2})}$$

$$\times \int_0^\infty y_2^b \, \frac{(\alpha_2 k_2\, y_2^{k_2-1})}{(\alpha_2\, y_2^{k_2})^\beta} \sum_{l=0}^{\infty} \frac{(g)_l}{(g+1)_l} \, \frac{(\beta)_l\,(d)_l}{(d+1)_l} \, \frac{\left(\dfrac{-1}{\alpha_2 y_2^{k_2}}\right)^l}{l!} \; dy_2. \tag{5.10}$$

Using relation (4.10) in (5.10), we have

$$= \frac{A}{(\alpha_2)^{\frac{a}{k_2}}} \, \frac{g}{(g+1-\beta+\frac{a}{k_2})} \, \frac{\Gamma(\beta - \frac{a}{k_2} - 1)\Gamma(\dfrac{a}{k_2}+1)}{\Gamma(\beta)} \int_0^\infty y_2^b \, \frac{(\alpha_2 k_2\, y_2^{k_2-1})}{(\alpha_2\, y_2^{k_2})^\beta}$$

$$\times \, {}_3F_2\left[\begin{matrix} (d), & (g+1-\beta+\frac{a}{k_2}), & (\beta) \\ & & & ;-\dfrac{1}{\alpha_2 y_2^{k_2}} \\ (d+1), & (g+2-\beta+\frac{a}{k_2}) & \end{matrix}\right] dy_2. \tag{5.11}$$

Letting $t_2 = \dfrac{1}{\alpha_2\, y_2^{k_2}}$ in (5.11), we get

$$= \frac{A}{(\alpha_2)^{\frac{a+b}{k_2}}} \frac{g}{(g+1-\beta+\frac{a}{k_2})} \frac{\Gamma(\beta-\frac{a}{k_2}-1)\Gamma(\frac{a}{k_2}+1)}{\Gamma(\beta)} \int_0^\infty t_2^{(\beta-\frac{b}{k_2}-1)-1}$$

$$\times\, {}_3F_2\left[\begin{matrix}(d), & (g+1-\beta+\frac{a}{k_2}),\ (\beta) \\ (d+1), & (g+2-\beta+\frac{a}{k_2})\end{matrix};-t_2\right] dt_2. \tag{5.12}$$

Note that [Prudnikov *et al.*, 1986]

$$\int_0^\infty x^{s-1}\ {}_3F_2\left[\begin{matrix}a_1, & a_2, & a_3 \\ b_1, & b_2 & \end{matrix};-x\right] dx$$

$$= \frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(s)\,\Gamma(a_1-s)\,\Gamma(a_2-s)\,\Gamma(a_3-s)}{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(a_3)\,\Gamma(b_1-s)\,\Gamma(b_2-s)}$$

$$[\,0 < Re\, s < Re\, a_j\,;\quad j=1,\ 2,\ 3]. \tag{5.13}$$

Using relation (5.13) in (5.12), we have

$$= \frac{A}{(\alpha_2)^{\frac{a+b}{k_2}}} \frac{\Gamma(\beta-\frac{a}{k_2}-1)\Gamma(\frac{a}{k_2}+1)}{\Gamma(\beta)} \frac{\Gamma(\beta-\frac{b}{k_2}-1)\Gamma(\frac{b}{k_2}+1)}{\Gamma(\beta)}$$

$$\times \frac{d\,g}{(d-\beta+\frac{b}{k_2}+1)\,(g-2\beta+\frac{a}{k_2}+\frac{b}{k_2}+2)}. \tag{5.14}$$

Now putting the value of A, d, g and β in (5.14), we get

$$= \frac{1}{(\alpha_2)^{\frac{a+b}{k_2}}} \frac{c^2\,(c+1)^2\, C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \frac{\Gamma(c-\frac{a}{k_2}+1)\,\Gamma(\frac{a}{k_2}+1)}{\Gamma(c+2)} \frac{\Gamma(c-\frac{b}{k_2}+1)}{\Gamma(c+2)}$$

$$\times\Gamma(\tfrac{b}{k_2}+1)\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\left(\frac{1}{c\{k+(n-s+j)(m+1)\}-\frac{b}{k_2}}\right)$$

$$\times\left(\frac{1}{c\{k+(n-r+i)(m+1)\}-\frac{a}{k_2}-\frac{b}{k_2}}\right), \tag{5.15}$$

$$=\frac{1}{(\alpha_2)^{\frac{a+b}{k_2}}}\frac{(c+1)^2\,C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\frac{\Gamma(c-\frac{a}{k_2}+1)\,\Gamma(\frac{a}{k_2}+1)}{\Gamma(c+2)}\frac{\Gamma(c-\frac{b}{k_2}+1)}{\Gamma(c+2)}$$

$$\times\Gamma(\tfrac{b}{k_2}+1)\sum_{i=0}^{r-1}(-1)^{i}\binom{r-1}{i}B\left(\frac{k}{m+1}+(n-r+i)-\frac{a+b}{ck_2(m+1)}\ ,\ 1\right)$$

$$\times\sum_{j=0}^{s-r-1}(-1)^{j}\binom{s-r-1}{j}B\left(\frac{k}{m+1}+(n-s+j)-\frac{b}{ck_2(m+1)}\ ,\ 1\right). \tag{5.16}$$

Using relation (3.13) in (5.16), we get

$$=\frac{1}{(\alpha_2)^{\frac{a+b}{k_2}}}\frac{(c+1)^2\,C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\frac{\Gamma(c-\frac{a}{k_2}+1)\,\Gamma(\frac{a}{k_2}+1)}{\Gamma(c+2)}\frac{\Gamma(c-\frac{b}{k_2}+1)\,\Gamma(\frac{b}{k_2}+1)}{\Gamma(c+2)}$$

$$\times B\left(\frac{k}{m+1}+(n-r)-\frac{a+b}{ck_2(m+1)}\ ,\ r\right)B\left(\frac{k}{m+1}+(n-s)-\frac{b}{ck_2(m+1)}\ ,\ s-r\right).$$

Which after simplification, gives

$$E(Y^{(a)}_{[r,n,m,k]}Y^{(b)}_{[s,n,m,k]})=\frac{1}{(\alpha_2)^{\frac{a+b}{k_2}}}\frac{\Gamma(c-\frac{a}{k_2}+1)\,\Gamma(\frac{a}{k_2}+1)}{\Gamma(c+1)}\frac{\Gamma(c-\frac{b}{k_2}+1)\,\Gamma(\frac{b}{k_2}+1)}{\Gamma(c+1)}$$

$$\times\frac{1}{\prod_{i=1}^{r}(1-\frac{a+b}{c\,k_2\gamma_i})\prod_{j=r+1}^{s}(1-\frac{b}{c\,k_2\gamma_i})}. \tag{5.17}$$

Remark 5.1: Set $k_2=1$ in (5.17), to get product moments of concomitants of generalized order statistics from bivariate Lomax distribution.

$$E(Y^{(a)}_{[r,n,m,k]}Y^{(b)}_{[s,n,m,k]})=\frac{1}{(\alpha_2)^{a+b}}\frac{\Gamma(c-a+1)\,\Gamma(a+1)}{\Gamma(c+1)}\frac{\Gamma(c-b+1)\,\Gamma(b+1)}{\Gamma(c+1)}$$

$$\times\frac{1}{\prod_{i=1}^{r}(1-\frac{a+b}{c\,\gamma_i})\prod_{j=r+1}^{s}(1-\frac{b}{c\,\gamma_i})}$$

as obtained by Nayabuddin (2013).

Remark 5.2 : Set $m=0,\ k=1$ in (5.17), to get product moments of concomitants of order statistics from bivariate Burr XII distribution as,

$$E(Y_{[r:n]}^{(a)}Y_{[s:n]}^{(b)}) = \frac{1}{(\alpha_2)^{\frac{a+b}{k_2}}}\frac{n!}{(n-s)!}\frac{\Gamma(c-\frac{a}{k_2}+1)\Gamma(\frac{a}{k_2}+1)}{\Gamma(c+1)}\frac{\Gamma(c-\frac{b}{k_2}+1)\Gamma(\frac{b}{k_2}+1)}{\Gamma(c+1)}$$
$$\times\frac{\Gamma(n-r+1-\frac{a+b}{ck_2})}{\Gamma(n+1-\frac{a+b}{ck_2})}\frac{\Gamma(n-s+1-\frac{b}{ck_2})}{\Gamma(n-r+1-\frac{b}{ck_2})},$$

as obtained by Begum and Khan, (1998).

Remark 5.3 : Set $m\to -1,$ in (5.17), we get product moment of concomitants of $k-th$ upper record value from bivariate Burr XII distribution as;

$$E[(Y_{[U(r)]}^{k})^{a}.(Y_{[U(s)]}^{k})^{b}] = \frac{1}{(\alpha_2)^{\frac{a+b}{k_2}}}\frac{\Gamma(c-\frac{a}{k_2}+1)\Gamma(\frac{a}{k_2}+1)}{\Gamma(c+1)}\frac{\Gamma(c-\frac{b}{k_2}+1)\Gamma(\frac{b}{k_2}+1)}{\Gamma(c+1)}$$
$$\times\frac{1}{(1-\frac{a+b}{ckk_2})^{r}(1-\frac{b}{ckk_2})^{(s-r)}}.$$

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