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Some Inferences on the Skew Semi Circular Distribution

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ABSTRACT

Skewed distributions have received a great deal of attractions in literature, because some data display some degrees of skewness. Several distributional properties of the skew semi-circular distribution are presented. Some characterizations based on the truncated moments are given.

1. Introduction

Wishart (1928) considered random matrix in connection to the statistical analysis of data. A matrix is called random matrix if the entries of matrix are random variables from any specified distribution. If the distribution of the random variables is Gaussian, then we call the matrix as Gaussian random matrix. Let $\lambda_i, i = 1, 2,..n$ be the eigenvalues of the nxn matrix. The empirical spectral density (ESD) m (λ) is defined as

$$m(\lambda)=\frac{1}{n}\sum_{j=1}^{n}\delta(\lambda-\lambda_j)$$

where δ is the Dirac's delta function. The ESD is a probability measure over the complex plane. If the matrix is Harmitian, then the eigenvalues are real. Wagner (1955) proved that the ESD of a Hermitian nxn Gaussian matrix when normalized by $\frac{1}{\sqrt{n}}$ tends to the semi-circular distribution with

pdf f_{sc} as

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$$f_{sc(x)} = \frac{1}{2\pi}\sqrt{4-x^2}, -2 \le x \le 2 \qquad (1.1)$$

= 0, otherwise.

The pdf given in (1.1) is a semi-circular distribution and we denote it as SCD (-2.2). However we will consider here the standard SCD(-1,1) with pdf f(x) as

$$f(x) = \frac{2}{\pi}\sqrt{1-x^2}, -1 \le x \le 1 \qquad (1.1)$$

= 0, otherwise.

For some basic properties of circular and semi-circular distribution see Fisher (1993), Mardia (1972) and Ahsanullah (2016).

Azzalini (1985) introduced univariate skew normal distribution with pdf $f_{sn}(x)$ as $f_{sn}(x) = 2\phi(x)\Phi(\lambda x), -\infty < x < \infty,$ (1.2)

where λ is any real number with ϕ and Φ denoting the standard normal *pdf* and cumulative distribution function (*cdf*). This technique has been used by several researchers to skew any continuous symmetric distribution. In the univariate case, skew Cauchy distribution was considered by Arnold and Beaver (2000) and for details about other skew univariate and multivariate distributions see Azzalini(2014). We will define the *pdf* of the skew-circular distribution

$f_{sscd}(x)$ as

$$f_{sscd}(x) = \frac{2}{\pi}\sqrt{1-x^2}(1+\lambda x), -1 \le x \le 1 \qquad (1.3)$$

=0, otherwise;

where -1 $\le \lambda \le 1$.

We will denote (1.3) as the pdf of the distribution SSCD (-1,1, λ)

In this paper we will present some basic properties and characterizations of SSDD(-1,1, λ)

2. Main Results

The Fig. 2.1.gives the pdf of SSCD (-1,1, λ) for $= -0.8. -0.4, 0.4$ and 08.

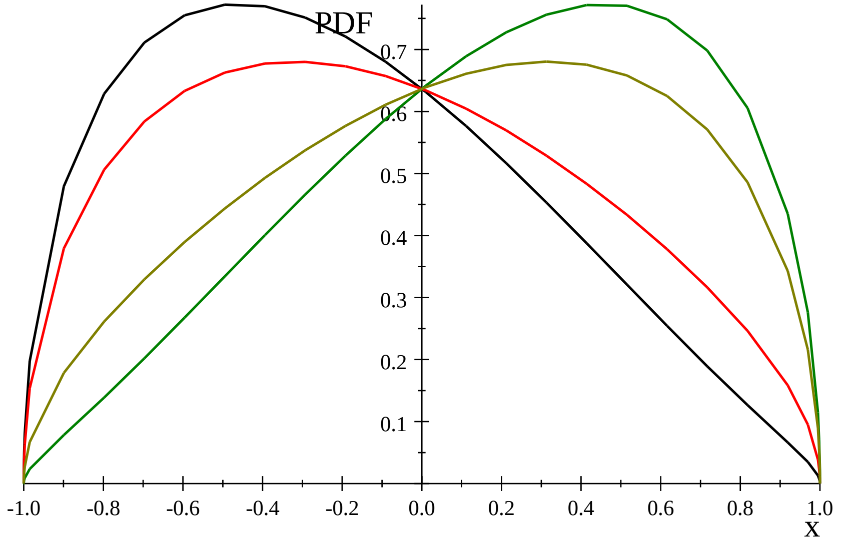

Fig. 2.1: PDFs SSCD(-1.1.-0.8)-Black , SSCD(-1,1,-0,4)- Red, SSCD(-1.1.0.8)-Green and SSCD(-1.1.0.4)-Brown.

The cdf $F_{sscd}(x)$ of SSCD(1,-1, λ) is as follows:

$F_{sscd}(x)$ =$\frac{1}{2}$ +$\frac{1}{\pi}$(arcsin(x)+x$\sqrt{(1-x^2})$- $\frac{2\lambda}{3}(1-x^2)^{\frac{3}{2}}$)

It can be shown that x_p is the pth percentile point for SSCS(1-,1, λ), then $-x_p$ will be 1-p th percentile point of SSCD(-1,1,- λ).

Let $\mu_r^m = E(X^m)$, then

$$\mu_r^{2m} = \int_{1}^{1} \frac{2}{\pi} x^{2m} (\sqrt{1-x^2} (1+\lambda x) dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{\pi} \sin^{2m} \cos^2 \theta (1+\lambda \sin\theta) d\theta$$

$$= \frac{2}{\pi} \mathrm{B}(\frac{2m+1}{2,}, \frac{3}{2}),$$

where B(p,q) = $\int_0^1 x^{p-1}(1-x)^{q-1}$dx.

$$\mu_r^{2m+1} = \int_{1}^{1} \frac{2}{\pi} x^{2m+1} (\sqrt{1-x^2} (1+\lambda x) dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{\pi} \sin^{2m+1} \cos^2 \theta (1+\lambda \sin\theta) d\theta$$

$$= \frac{2\lambda}{\pi} B(\frac{2m+3}{2}, \frac{3}{2}).$$

We will use the following two lemmas to prove the characterizations of

SSCD (-1,1, λ) distribution.

Assumption A

X is an absolutely continuous random variable with cdf F(x), pdf f(x) $\alpha = Sup\{x \mid f(x) > 0\}$ and $\beta = \inf\{x \mid f(x) < 1\}$. We assume E(X) exists and f(x) is differentiable in $\alpha < x < \beta$

Lemma 2.1.

Under the assumption A,if E(X|X$\leq x) = $ g(x)$\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and g(x) is a differentiable function in (α, β), then f(x) = ce $^{\int \frac{x-g'(x)}{g(x)}dx}$,

c is determined by the condition $\int_\alpha^\beta f(x)dx = 1.$

Proof.

$$g(x) = \frac{\int_\alpha^x uf(u)du}{f(x)}.$$

thus

$$\int_\alpha^x uf(u)du = f(x)g(x).$$

Differentiating both sides of the above equation, we obtain

xf(x) = f'(x)g(x) +f(x)g'(x)

On simplification, we get

$$\frac{f'(x)}{f(x)} = \frac{x-g'(x)}{g(x)}.$$

On integrating both sides of the above equation with respect to x, we obtain

f(x) = c e $^{\int \frac{x-g'(x)}{g(x)}dx}$, c is determined by the condition $\int_\alpha^\beta f(x)dx = 1.$

Lemma 2.2

Under the assumption A, if E(X|X$\geq x) = h(x)\, r(x)$, where r(x) = $\frac{f(x)}{1-F(x),}, 0 < F(x) < 1$ *and* h(x) is a differentiable function in (α, β), then

f(x) = ce $^{\int -\frac{x+h'(x)}{h(x)}dx}$,

c is determined by the condition $\int_\alpha^\beta f(x)dx = 1.$

Proof.

h(x) = $\frac{\int_x^\beta uf(u)du}{f(x)}$,

thus

$\int_x^\beta uf(u)du$=f(x)h(x)

Differentiating both sides of the above equation, we obtain

-xf(x) = f'(x)h(x) +f(x)h'(x)

On simplification, e get

$\frac{f'(x)}{f(x)} = -\frac{x+h'(x)}{h(x)}$.

On integrating both sides of the above equation, we obtain

f(x) = c e $^{\int -\frac{x+h'(x)}{h(x)}dx}$, c is determined by the condition $\int_\alpha^\beta f(x)dx = 1.$

Theorem 2.1.

Suppose that X is an absolutely continuous random variable with cdf F(x), pdf f(x) with $-1 = \sup\{x|F(x) > 0\}$ and $1 = \inf\{x|F(x) < 1\}$1. We assume E(X) exists and f(x) is differentiable in -1<x<1 Then E(X|X$\leq x$) = g(x)$\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and

g(x)=$\frac{-\frac{1}{3}(1-x^2)^{\frac{3}{2}} + \frac{\lambda}{32}(2\pi - \sin(4arcsinx) + 4arcsinx)}{(1+\lambda x)\sqrt{(1-x^2)}}$ if and only if

f(x) $= \frac{2}{\pi}(1 + \lambda x)\sqrt{(1 - x^2)}$, -1<x<1

= 0, otherwise,

where -1 $\leq \lambda \leq 1.$

Proof.

Suppose that

f(x) $= \frac{2}{\pi}(1 + \lambda x)\sqrt{1 - x^2}, then$

f(x)g(x) =$\int_{-1}^{x} \frac{2}{\pi}$ u(1+λu)$\sqrt{(1 - u^2)}$ du

$= -\frac{2}{3\pi}(1 - x^2)^{\frac{3}{2}} + \frac{\lambda}{16\pi}(2\pi - \sin(4arcsinx) + 4rcsinx)$

Thus

$$g(x) = \frac{-\frac{1}{3}(1-x^2)^{\frac{3}{2}} + \frac{\lambda}{32}(2\pi - \sin(4arcsinx) + 4rcsinx)}{(1+\lambda x)\sqrt{1-x^2}),}$$

Suppose

$$g(x) = \frac{-\frac{1}{3}(1-x^2)^{\frac{3}{2}} + \frac{\lambda}{32}(2\pi - \sin(4arcsinx) + 4rcsinx)}{(1+\lambda x)\sqrt{1-x^2})},$$

then

$$g'(x) = x - \frac{-\frac{1}{3}(1-x^2)^{\frac{3}{2}} + \frac{\lambda}{32}(2\pi - \sin(4arcsinx) + 4rcsinx)}{(1+\lambda x)\sqrt{1-x^2},} \quad \left(\frac{\lambda}{1-\lambda x} - \frac{x}{1-x^2}\right)$$

$$= x - g(x)\left(\frac{\lambda}{1+\lambda x} - \frac{x}{1-x^2}\right)$$

$$\frac{x - g'(x)}{g(x)} = \left(\frac{\lambda}{1+\lambda x} - \frac{x}{1-x^2}\right)$$

By Lemma 2.1, we have

$$\frac{f'(x)}{f(x)} = \frac{\lambda}{1+\lambda x} - \frac{x}{1-x^2}$$

Integrating both sides of the above equation with respect to x, we obtain

f(x) =c $(1+\lambda x)\sqrt{1-x^2}$.

where c is a constant.

Using the condition $\int_{-1}^{1} f(x)dx = 1,$ we obtain

$$f(x) = f_{sscd}(x) = \frac{2}{\pi}\sqrt{1-x^2}\,(1+\lambda x), -1 \le x \le 1$$

Theorem 2.2.

Suppose that X is an absolutely continuous random variable with cdf F(x) and pdf f(x) with $-1 = \sup\{x|F(x) > 0\}$ and $1 = \inf\{x|F(x) < 1\}$. We assume E(X) exists and f(x) is differentiable in -1<x<1.Then if $E(X|X \geq x) = h(x)r(x), where\ r(x) = \frac{f(x)}{1-F(x)}$ and

$$h(x) = \frac{\frac{\lambda\pi}{8} + \frac{1}{3}(1-x^2)^{\frac{3}{2}} - \frac{\lambda}{32}(2\pi - \sin(4arcsinx) + 4rcsinx)}{(1+\lambda x)\sqrt{1-x^2})}$$ if and only if

f(x) $= \frac{2}{\pi}(1+\lambda x)\sqrt{(1-x^2)}$, -1<x<1

=0, otherwise;

where -1< λ <1.

Proof.

Suppose that

f(x) $=\frac{2}{\pi}(1+\lambda x)\sqrt{1-x^2}, then$

f(x)g(x) $=\int_x^1 \frac{2}{\pi}$ u(1+λu)$\sqrt{(1}-u^2)$ du

= E(x) -g(x)

$=\frac{2}{\pi}\left(\frac{\lambda\pi}{8}+\frac{1}{3}(1-x^2)^{\frac{3}{2}}-\frac{\lambda}{32}(2\pi-\sin(4arcsinx)+4arcsinx)\right)$

Thus

$$h(x)=\frac{\frac{\lambda\pi}{8}+\frac{1}{3}(1-x^2)^{\frac{3}{2}}-\frac{\lambda}{32}(2\pi-\sin(4arcsinx)+4arcsinx)}{(1+\lambda x)\sqrt{(1-x^2)}}$$

Suppose

$$h(x)=\frac{\frac{\lambda\pi}{8}+\frac{1}{3}(1-x^2)^{\frac{3}{2}}-\frac{\lambda}{32}(2\pi-\sin(4arcsinx)+4arcsinx)}{(1+\lambda x)\sqrt{(1-x^2)}} \text{ then}$$

$$h'(x)=-x-\frac{\frac{\lambda\pi}{8}+\frac{1}{3}(1-x^2)^{\frac{3}{2}}-\frac{\lambda}{32}(2\pi-\sin(4arcsinx)+4arcsinx)}{(1+\lambda x)\sqrt{(1-x^2)}}\left(\frac{\lambda}{1-\lambda x}-\frac{x}{1-x^2}\right)$$

$$h'(x)=-x-h(x)\left(\frac{\lambda}{1-\lambda x}-\frac{x}{1-x^2}\right)$$

and

$$-\frac{x+h'(x)}{h(x)}=\frac{\lambda}{1+\lambda x}-\frac{x}{1-x^2}$$

By Lemma 2, 2 we have

$$\frac{f'(x)}{f(x)}=\frac{\lambda}{1+\lambda x}-\frac{x}{1-x^2}$$

Integrating both sides of the above equation with respect to x, we obtain

f(x) =c (1+λx)$\sqrt{1-x^2}$.

where c is a constant.

Using the condition $\int_{-1}^{1} f(x)dx=1,$ we obtain

$$f(x)=f_{sscd}(x)=\frac{2}{\pi}\sqrt{1-x^2}(1+\lambda x), -1\le x\le 1.$$

3. Conclusion

The skew symmetric distributions were developed as skewed alternatives to the central symmetric distributions. Before a particular distribution model is applied to the real world data, it is essential to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. Thus, characterization of a probability distribution plays an important role in statistics and mathematical sciences. The results of the paper will enable the readers to identify semicircular distribution using the properties presented in the paper.

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