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Generalized Maximum Likelihood Estimators for Gamma Distribution: Semi-Bayesian Approach

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ABSTRACT

Generalized maximum likelihood estimators are obtained for the parameters of gamma distribution and for some inequality measures *viz.* Gini index and Theil's entropy measure using different priors. The said estimators using different priors are compared in terms of their relative efficiency using mean square error through simulation study.

1. Introduction

One of the ways in which Bayesian setup can be employed to estimate the parameters and associated inference is the use of semi-Bayesian approach. Semi Bayesian approach (Yoon *et al.* (2010)) consists of finding the generalized maximum likelihood estimators by maximizing the posterior density. The generalized maximum likelihood estimator is defined as the largest mode of the posterior density that maximizes $L(\theta) * \pi(\theta)$ (likelihood $\times$ prior). Bayesian prefers the name maximum aposteriori (MAP) estimator or simply posterior mode to denote the generalized maximum likelihood estimator. The MAP estimator is popular in Bayesian analysis since it is often computationally less demanding then the posterior mean or median. The reason is simple, the posterior

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need not to be fully specified since $\max_{\theta} \pi(\theta|x) = \max_{\theta} f(\theta|x)\pi(\theta)$. Method of maximum likelihood estimator (MLE) or the method of moments (MM) are used to estimate the parameters of the distribution in classical setup, while Bayesian setup uses the generalized maximum likelihood estimators (GMLE's) (Martz and Waller (1982)) as one of the approaches to estimate the parameters.

The main advantage of semi-Bayesian approach or generalized maximum likelihood estimator over the maximum likelihood estimator is the use of prior information in addition to the sample data. A proper Bayesian analysis will always incorporate genuine prior information, which will help to strengthen inferences about the true value of the parameters. The computational part of expressions is simple in semi-Bayesian approach as compared to full-Bayesian. Moreover no loss function is required for estimating the parameters in the case of semi-Bayesian approach, while the choice of loss function is crucial for full-Bayesian setup.

In the context of income inequality and related inequality parameters, the use of generalized maximum likelihood estimators (GMLE's) is yet to be initiated. Some work in the context of semi-Bayesian approach for generalized extreme value distribution using beta prior and weighted exponential family using conjugate prior (Yoon *et al*. (2010), Mohammad (2008)) is already available in literature but not in the inequality setup. Using full- Bayesian approach some estimators of Poverty measure and inequality measures for classical Pareto distribution *viz.* Gini index and Lorenz curve are available in literature (Abdul-Sathar (2005), Bhattacharya *et al.* (1999)) Ertefaie and Parsian (2005)), yet the semi-Bayesian approach using generalized maximum likelihood estimators (GMLE's) in the context of inequality estimation is still awaiting attention. It can make an interesting study to apply semi-Bayesian approach *i.e* generalized maximum likelihood estimators to estimate the parameters and inequality measures in the context of Gamma distribution, another distribution used in the context of income setup.

In the present study, the generalized maximum likelihood estimators (GMLE's) are obtained for the parameters of Gamma distribution and also for two inequality measures - Gini index and Theil's entropy measure in case of Gamma distribution using three different priors *viz*. uniform prior, Jeffreys' prior and conjugate prior. A comparison of GMLE's and MLE's of these parameters and inequality measures is also done to see the relative efficiency of one over the other.

One of the most popular and useful measure of income inequality, the Gini index

(Gini (1912)) is defined as G = 1−2 (area under the Lorenz curve)

$= 1 - 2\int_0^1 L(p)dp, \quad 0 \le p \le 1,$

where $L(p) = \frac{1}{\mu}\int_0^p F^{-1}(t)dt,$ is the equation of the Lorenz curve

and $\mu = \int_0^1 F^{-1}(t)dt,$ is the mean of the distribution.

Other useful measure of inequality based on the notion of entropy in information theory is Theil entropy measure (Theil (1967)). Entropy is a measure of disorder in thermodynamics. More generally we can characterize the "degree of disorder" – known technically as the entropy –by working out the average information content of the system. This is the weighted sum of all the information values for the various events; the weight given to event *i* in this averaging process is simply its probability p_i: In other words we have:

Entropy $= -\sum_{i=1}^{n} p_i \log p_i$ (Cowell (2009))

Let y_i be the fraction of total income earned by the *i*th individual, where *i* varies from 1 to n. The entropy of income shares is defined as

$H(y) = \sum_{i=1}^{n} y_i \log\frac{1}{y_i},$

which is weighted average of the logarithms of the reciprocal of each income share, weights being the respective income shares.

The upper limit of $H(y)$ is $\log n$, which is reached when all individuals earn equal income (the case of perfect equality) and the minimum of $H(y)$ is zero, which represents one individual receiving all the income (the case of perfect inequality). It follows that $H(y)$ may be regarded as a measure of income inequality. Thus, the inequality measure proposed by Theil (1967) is

$\log n - H(y) = \sum_{i=1}^{n} y_i \log n y_i,$

which varies from zero to $\log n$. The upper limit of this measure approaches infinity as n approaches infinity.

It may be convenient to express Theil's entropy measure as

$T = I(y:p) = \sum_{i=1}^{n} y_i \log\frac{y_i}{p_i},$

where p_i is the population share, in this case, is equal to $\frac{1}{n}$.

One can interpret $I(y:p)$ as expected information of the indirect message that transforms prior probabilities $p_1, p_2, \ldots, p_n$ into posterior probabilities $y_1, y_2, \ldots, y_n$.

If x is a continuous random variable, this measure can be expressed as

$T = \int_0^\infty \frac{x}{E(x)} \log\left\{\frac{x}{E(x)}\right\} dF(x)$

which shows that Theil's entropy measure is the arithmetic mean of strictly convex functions of incomes. These measures satisfy Dalton's principle of transfers, that is, any transfer from a rich to a poor person reduces inequality (Kakwani (1980)).

The organization of the paper is as follows: The expressions for Theil's entropy measure and Gini index in case of Gamma distribution are given in Section 2. In Section 3, generalized maximum likelihood estimators (GMLE's) are derived for parameters of Gamma distribution, Theil's entropy measure and Gini index using uniform prior, Jeffreys' prior and conjugate prior. In Section 4, simulation study is carried out to compare the efficiency of the GMLE's approach and a real life example is also given in Section 4.1. The robustness of the hyper-parameters for the conjugate prior is also carried out in Section 4.2 through simulation study, while conclusion is given in section 5.

2. Gini Index and Theil's Entropy Measure in case of Gamma Distribution

A number of alternative probability density functions *viz*. Pareto and lognormal distributions have been proposed as models of income distribution which provide a reasonably close approximation to the true distributions and their parameters are simple to estimate and interpret in an economically meaningful way. The parameters of Pareto and lognormal distributions can be easily estimated and directly related to inequality measures, but sometimes these distributions do not fit the full range of income data. On the other hand, models that have been proposed with improved fit, such as the displaced lognormal (Metcal, 1969) and the beta distribution (Thurow, 1970) have parameters which are more difficult to interpret.

The Gamma density, which was applied to income data by various authors, provides an alternative model that fits the data reasonably well. The distribution of personal income is approximated by a two-parameter Gamma density function and the two parameters may be considered as indicators of scale and inequality, respectively. Using the Gamma density, inequality is shown to decrease when unemployment or inflation decreases, or when the real national product increases. Salem and Mount (1974) fitted the Gamma distribution to personal income data in the United State and empirical results show that the Gamma distribution fits the data better than Log-normal distribution. Hence, in this study, the two parameter Gamma distribution is chosen as the appropriate model for any income distribution, wherein inequality is estimated using Theil's entropy measure and Gini index.

Suppose that random variables $X_1, X_2, \_\_\_, X_n$ are independent and identically distributed from Gamma distribution. The pdf of Gamma distribution is

$$f(x|\alpha,\lambda) = \frac{\lambda^{\alpha}}{\Gamma\alpha} x^{\alpha-1} e^{-\lambda x}, \quad x > 0; \alpha, \lambda > 0$$

$$\sim \text{ Gamma } (\alpha, \lambda),$$

where λ is scale and α is shape parameter.

The expected value is

$E(X) = \frac{\alpha}{\lambda}$ and likelihood function is

$$L(x|\alpha,\lambda) = \frac{\lambda^{n\alpha}}{(\Gamma\alpha)^n} \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\lambda \sum_{i=1}^{n} x_i} .$$

Theil's entropy measure for Gamma distribution is derived as follows

$$\begin{aligned} T &= \int_0^{\infty} \frac{x}{E(x)} log\left\{\frac{x}{E(x)}\right\} dF(x) \\ &= \int_0^{\infty} \frac{\lambda x}{\alpha} log\left(\frac{\lambda x}{\alpha}\right) \frac{\lambda^{\alpha}}{\Gamma\alpha} e^{-\lambda x} x^{\alpha-1} dx \\ &= \psi(\alpha+1) - log\,\alpha \\ &= \psi(\alpha) + \frac{1}{\alpha} - log\,\alpha, \end{aligned} \tag{1}$$

where $\psi(\alpha+1) = \frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)}$ is digamma function $\left(\Gamma'(\alpha) = \frac{d\Gamma(\alpha)}{d\alpha}\right)$,

and $\psi(\alpha+1) = \psi(\alpha) + \frac{1}{\alpha}$

Gini index for Gamma distribution is given by

$$G = \frac{\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+1)}, \text{ (Duangkamon 2008) .} \tag{2}$$

3. GMLE's for Shape Parameter (α), Scale Parameter (λ) and Inequality Measures of Gamma Distribution $(G(\alpha,\lambda))$

In this section, expressions for generalized maximum likelihood estimators (GMLE's) for the shape and scale parameters of Gamma distribution, Theil's entropy measure and Gini index are obtained under the three cases

(i) α is unknown, λ is known.

(ii) α is known, λ is unknown.

(iii) α and λ both are unknown.

Using semi-Bayesian approach, generalized maximum likelihood estimators (GMLE's) for the above three cases are derived using three different priors *viz.*

uniform prior, Jeffreys' prior and conjugate prior. The results are presented in sections 3.1, 3.2 and 3.4 respectively.

3.1 GMLE's using Uniform Prior

In practice the informative priors are not always available, for such situations, the use of non-informative priors is recommended. One of the most widely used non-informative prior is a uniform prior. The uniform prior is defined as

$g_U(\alpha) \propto 1$

Therefore, uniform prior contains no information and generalized maximum likelihood estimators using uniform prior is same as that of maximum likelihood estimators for all the three cases listed above**.** The posterior density in case of uniform prior is same as that of likelihood function, hence maximizing the likelihood function yields the generalized maximum likelihood estimators. So, generalized maximum likelihood estimators and maximum likelihood estimators coincide for uniform prior.

Using these facts, the GMLE's are derived for uniform prior for the three cases as given below.

Case 1. α is unknown, λ is known.

The likelihood function is given by

$L(x|\alpha,\lambda) = \frac{\lambda^{n\alpha}}{(\Gamma\alpha)^n} \prod_{i=1}^{n} x_i^{\alpha-1}\, e^{-\lambda\sum_{i=1}^{n} x_i}.$

The Log-likelihood for Gamma distribution is

$$ln(L(x|\alpha,\lambda)) = n\alpha ln(\lambda) - n ln(\Gamma\alpha) + (\alpha-1)\sum_{i=1}^{n} ln\,(x_i) - \lambda\sum_{i=1}^{n} x_i \qquad (3)$$

By taking partial derivative of $ln(L(x|\alpha,\lambda))$ w.r.t α and equating to zero, likelihood equation is

$\psi\,(\alpha) - ln\,\lambda + \frac{1}{n}\sum_{i=1}^{n} ln\, x_i = 0,$

which can be solved numerically for α ($\hat{\alpha}_U$) using Newton Raphson's method (Raphson (1690)) or Optim function in R-Software (Nash (1990)).

As seen from the section 2, Theil's entropy measure and Gini index depend only upon the shape parameter α. The GMLE for Theil's entropy measure and Gini index may be obtained by substituting $\hat{\alpha}_U$ for α. The expression of GMLE for Theil's entropy measure and Gini index are given below:

$\hat{T}_U = \psi(\hat{\alpha}_U) + \frac{1}{\hat{\alpha}_U} - log(\hat{\alpha}_U),$

$\hat{G}_U = \frac{\Gamma(\hat{\alpha}_U + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\hat{\alpha}_U + 1)}.$

Case 2. α is known, λ is unknown.

Taking partial derivative of $ln(L(x|\alpha,\lambda))$ w.r.t λ and equating to zero, likelihood equation is

$$\frac{n\alpha}{\lambda} - \sum_{i=1}^{n} x_i = 0,$$

GMLE for λ is

$$\hat{\lambda}_U = \frac{\alpha}{\bar{x}} .$$

As, the expressions of Theil's entropy measure and Gini index are independent of λ, so the GMLE's for these two are not obtained in this case.

Case 3. α and λ both are unknown.

Taking partial derivative of $ln(L(x|\alpha,\lambda))$ w.r.t α and λ and equating to zero, lead to likelihood equations,

$$\frac{n\alpha}{\lambda} - \sum_{i=1}^{n} x_i = 0, \tag{4}$$

$$ln(\alpha) - \psi(\alpha) - ln\,\bar{x} - \frac{1}{n}\sum_{i=1}^{n} ln\, x_i = 0, \tag{5}$$

by solving equation (4), GMLE of λ is

$$\hat{\lambda}_U = \frac{\hat{\alpha}}{\bar{x}} .$$

Equation (5) can be solved numerically (Newton–Raphson algorithm or optim function) for α ($\hat{\alpha}_U$).

The expression of GMLE's for Theil's entropy measure and Gini index are obtained by substituting $\hat{\alpha}_U$ for α and are given below:

$$\hat{T}_U = \psi(\hat{\alpha}_U) + \frac{1}{\hat{\alpha}_U} - log(\hat{\alpha}_U),$$

$$\hat{G}_U = \frac{\Gamma(\hat{\alpha}_U + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\hat{\alpha}_U + 1)} .$$

It is interesting to note that maximum likelihood estimators for all the three cases are same as generalized maximum likelihood estimators as pointed earlier.

3.2 GMLE's using Jeffreys' Prior

The Jeffreys' prior for θ is

$$g(\theta) = \sqrt{[I(\theta)]}$$

where $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} ln\,(L(\theta|x))\right]$ is Fisher's information.

Case 1. α is unknown, λ is known.

To compute Jeffreys' prior, take partial derivative of $ln(L(x|\alpha,\lambda))$ (eq. 3) w.r.t α

$$\frac{\partial}{\partial\alpha} ln\big(L(\theta|x)\big) = n ln(\lambda) - n\psi(\alpha) + \sum_{i=1}^{n} ln\,(x_i) \left(\psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right),$$

$$\frac{\partial^2}{\partial\alpha^2} ln\,(L(\theta|x)) = -n\psi'(\alpha) \left(\psi'(\alpha) = \frac{\partial\psi(\alpha)}{\partial\alpha}\right),$$

$I(\alpha) = E(n\psi'(\alpha)) = n\psi'(\alpha)$

Jeffreys' prior for α in case of Gamma distribution is

$g_J(\alpha) \propto \sqrt{I(\alpha)} = \sqrt{n\psi'(\alpha)}$

The posterior density of Gamma distribution is

$$g_J^*(\alpha) \propto \frac{\lambda^{n\alpha}}{(\Gamma\alpha)^n}\left(\prod_{i=1}^n x_i^{\alpha-1}\right)e^{-\lambda\sum_{i=1}^n x_i}\left[n\psi'(\alpha)\right]^{1/2}. \quad (6)$$

Taking partial derivative of $ln\,(g_J^*(\alpha))$ w.r.t α and equating to zero, likelihood equation for GMLE of α is

$$nln(\lambda) - n\frac{\psi\,(\alpha)}{\Gamma\alpha} + \sum_{i=1}^n ln(x_i) - \frac{\psi''(\alpha)}{2\psi'(\alpha)} = 0,$$

which can be solved numerically for α ($\hat{\alpha}_J$).

Using expression of Theil's entropy measure and Gini index as given in equation (1) and (2), the expressions of GMLE for T and G are given below

$$\hat{T}_J = \psi(\hat{\alpha}_J) + \frac{1}{\hat{\alpha}_J} - log\,\hat{\alpha}_J,$$

$$\hat{G}_J = \frac{\Gamma(\hat{\alpha}_J + \frac{1}{2})}{\sqrt{\pi}\Gamma(\hat{\alpha}_J + 1)}.$$

Case 2. α is known, λ is unknown.

To compute Jeffreys' prior for λ, take partial derivative of $ln(L(x|\alpha,\lambda))$ (eq. 3) w.r.t λ

$$\frac{\partial}{\partial\lambda} ln\big(L(\theta|x)\big) = \frac{n\alpha}{\lambda} - \sum_{i=1}^n x_i\,,$$

$$\frac{\partial^2}{\partial\lambda^2} ln\,(L(\theta|x)) = -\frac{n\alpha}{\lambda^2}\,,$$

$$I(\lambda) = -E\left[\frac{\partial^2}{\partial\lambda^2} ln\,(L(\theta|x))\right] = \frac{n\alpha}{\lambda^2}.$$

The prior density for λ is

$$g_J(\lambda) \propto \left(\frac{n\alpha}{\lambda^2}\right)^{1/2}.$$

The posterior density for λ is given by

$$g_J^*(\lambda) \propto \frac{\lambda^{n\alpha-1}(n\alpha)^{1/2}\left(\prod_{i=1}^n x_i\right)^{\alpha-1} e^{-\lambda\sum_{i=1}^n x_i}}{(\Gamma\alpha)^n}. \quad (7)$$

Taking partial derivative of $ln\,(g_J^*(\lambda))$ w.r.t λ and equating to zero, likelihood equation is

$$\frac{n\alpha-1}{\lambda} - \sum_{i=1}^n x_i = 0,$$

GMLE for λ is

$$\hat{\lambda}_J = \frac{n\alpha-1}{\sum_{i=1}^n x_i}.$$

Case 3. α and λ both are unknown.

The Fisher's information matrix for λ and α is given by

$$I(\alpha,\lambda) = -E\begin{bmatrix} \frac{\partial^2}{\partial\alpha^2} ln\,(L(\theta|x)) & \frac{\partial^2}{\partial\alpha\partial\lambda} ln\,(L(\theta|x)) \\ \frac{\partial^2}{\partial\lambda\partial\alpha} ln\,(L(\theta|x)) & \frac{\partial^2}{\partial\lambda^2} ln\,(L(\theta|x)) \end{bmatrix}$$

$$= -E\begin{bmatrix} -n\,\psi'(\alpha) & \frac{n}{\lambda} \\ \frac{n}{\lambda} & \frac{-n\alpha}{\lambda^2} \end{bmatrix}$$

$$= \frac{n^2}{\lambda^2}\left(1 - \alpha\psi'(\alpha)\right)$$

The joint prior is

$$g_J(\alpha,\lambda) \propto [I(\alpha,\lambda)]^{1/2}$$

$$g_J(\alpha,\lambda) = \left(\frac{n}{\lambda}\right)\left(1 - \alpha\psi'(\alpha)\right)^{1/2}.$$

The posterior density is

$$g_J^*(\alpha,\lambda) \propto \frac{n\lambda^{n\alpha-1}e^{-\lambda\sum_{i=1}^{n}x_i}}{(\Gamma\alpha)^n}\left(\prod_{i=1}^{n}x_i^{\alpha-1}\right)(1-\alpha\psi'(\alpha))^{1/2},$$

Taking partial derivative of $ln\,(g_J^*(\alpha,\lambda))$ w.r.t α and λ and equating to zero, likelihood equations are

$$\frac{n\alpha}{\lambda} - \sum_{i=1}^{n}x_i - \frac{1}{\lambda} = 0, \tag{8}$$

$$nln(\lambda) - n\frac{\psi\,(\alpha)}{\Gamma\alpha} + \sum_{i=1}^{n}ln\,x_i - \frac{[\psi'(\alpha)+\alpha\psi''(\alpha)]}{2(1-\alpha\psi'(\alpha))} = 0\,, \tag{9}$$

by solving equation (8), GMLE of λ is

$$\hat{\lambda}_J = \frac{n\alpha-1}{\sum_{i=1}^{n}x_i}.$$

Equation (9) is not in an explicit form and can be solved numerically for α.

The expression of GMLE for Theil's entropy measure and Gini index are given below

$$\hat{T}_J = \psi(\hat{\alpha}_J) + \frac{1}{\hat{\alpha}_J} - log\,\hat{\alpha}_J,$$

$$\hat{G}_J = \frac{\Gamma(\hat{\alpha}_J+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\hat{\alpha}_J+1)}.$$

3.3 GMLE's using Extension of Jeffreys' Prior

Jeffreys' prior is a particular case of extension of Jeffreys' prior proposed by Al-Kutubi *et al*. (2009), defined as

$$g(\theta) = [I(\theta)]^c\,,$$

where c is a positive constant. For $c = 0.5$, it reduces to Jeffreys' prior.

The expressions for GMLE's for α and λ using extension of Jeffreys' prior are obtained with some modifications in Jeffreys' prior and are listed below:

Table 1. GMLE's for Extension of Jeffreys' prior

Cases	posterior density	$\hat{\alpha}$	$\hat{\lambda}$
1	$\frac{\lambda^{n\alpha}}{(\Gamma\alpha)^n}\prod_{i=1}^{n} x_i^{\alpha-1}\, e^{-\lambda\sum_{i=1}^n x_i}[n\psi'(\alpha)]^c$	$nlog\,\lambda$ - $n\psi\,(\alpha)$ + $\sum_{i=1}^n log\, x_i$ – $c\,\frac{\psi''(\alpha)}{\psi'(\alpha)}$ = 0, which can be solved numerically for α.	
2	$\frac{\lambda^{n\alpha-2c}(\prod_{i=1}^n x_i)^{\alpha-1}e^{-\lambda\sum_{i=1}^n x_i}}{(\Gamma\alpha)^n}$		$\frac{n\alpha-2c}{\sum_{i=1}^n x_i}$
3	$\frac{n^{2c}\lambda^{n\alpha-2c}e^{-\lambda\sum_{i=1}^n x_i}}{(\Gamma\alpha)^n}\left(\prod_{i=1}^{n} x_i^{\alpha-1}\right)$ $(1-\alpha\ \psi'(\alpha))^c$	$n\ log\,\lambda$ - $n\frac{\psi\,(\alpha)}{\Gamma\alpha}$ + $\sum_{i=1}^n log\, x_i$ – $\frac{c\,[\psi'(\alpha)+\alpha\,\psi''(\alpha)]}{1-\alpha\ \psi'(\alpha)}$ = 0, which can be solved numerically for α.	$\frac{n\alpha-2c}{\sum_{i=1}^n x_i}$

The generalized maximum likelihood estimators for T and G using extension of Jeffreys' prior can be obtained by substituting the estimates of α *i.e* $\hat{\alpha}$ in the expression 1 and 2.

3.4 GMLE's using Conjugate Prior

The conjugate prior was introduced by Raiffa and Schlaifer (1961). When the prior and posterior distributions are from the same family, *i.e* the form of the posterior density has the same distributional form as the prior distribution, such prior is called conjugate prior.

A conjugate prior is constructed by first factoring the likelihood function into two parts. One factor must be independent of the parameter(s) of interest but may be dependent on the data. The second factor is a function dependent on the parameter(s) of interest and dependent on the data only through the sufficient statistics. The conjugate family is defined to be proportional to this second factor (Raiffa and Schlaifer 1961). Parameters of the conjugate prior distribution are called hyperparameters, to distinguish them from the parameters of the data generating distribution.

Case 1. α is unknown, λ is known.

The natural conjugate prior density for α is

$g_C(\alpha) = \frac{1}{k}\frac{a^{\alpha-1}\lambda^{\alpha c}}{(\Gamma\alpha)^b}$, where $k = \int_0^\infty \frac{a^{\alpha-1}\lambda^{\alpha c}}{(\Gamma\alpha)^b}$ and hyper-parameters $a, b, c > 0$,

$g_C(\alpha) \propto \frac{a^{\alpha-1}\lambda^{\alpha c}}{(\Gamma\alpha)^b}$, $a, b, c > 0$. Therefore, the posterior density is

$$g_C^*(\alpha) \propto \frac{\lambda^{\alpha(n+c)}\left(a \prod_{i=1}^n x_i\right)^{\alpha-1}}{(\Gamma\alpha)^{b+n}} \tag{10}$$

and posterior is of the same family as that of prior density. Taking partial derivative of $ln\,(g_C^*(\alpha))$ w.r.t α and equating to zero, gives likelihood equation as

$$(n+c)\,ln(\lambda) + ln(a) + \sum_{i=1}^n ln(x_i) - \frac{(b+n)\,\psi\,(\alpha)}{\Gamma\alpha} = 0,$$

which can be solved numerically for $\alpha(\hat{\alpha}_C)$ using different algorithms.

The expression of GMLE for Theil's entropy measure and Gini index are given below

$$\hat{T}_C = \psi(\hat{\alpha}_C) + \frac{1}{\hat{\alpha}_C} - log\,\hat{\alpha}_C,$$

$\hat{G}_C = \frac{\Gamma(\hat{\alpha}_C+\frac{1}{2})}{\sqrt{\pi}\Gamma(\hat{\alpha}_c+1)}$, where $\hat{\alpha}_C$ is the GMLE of α.

Case 2. α is known, λ is unknown.

The prior for λ is taken as Gamma distribution with parameters *a* and *b*.

$\lambda\sim$ Gamma (a ,b) $g_C(\lambda) = \frac{b^a}{\Gamma a}\lambda^{a-1}e^{-b\lambda}$, a and b are hyper-parameters.

The posterior density is

$$g_C^*(\lambda) \propto \lambda^{n\alpha+a-1}e^{-(\sum_{i=1}^n x_i+b)\lambda} \sim \text{Gamma}\,(n\alpha + a, (\textstyle\sum_{i=1}^n x_i + b))\,, \tag{11}$$

Taking logarithm of posterior density

$$ln\,(g_C^*(\lambda)) = (n\alpha + a - 1)ln(\lambda) - (\textstyle\sum_{i=1}^n x_i + b)\lambda\,.$$

Taking partial derivative of $ln\,(g_C^*(\lambda))$ w.r.t λ and equating to zero, GMLE for λ is $\hat{\lambda}_C = \frac{n\alpha+a-1}{\sum_{i=1}^n x_i+b}$.

Case 3. α and λ both are unknown.

The joint prior for λ and α with hyper-parameters *p*, *q*, *r*, *s* > 0 is

$g_C(\alpha,\lambda) = \frac{1}{k}\frac{p^{\alpha-1}e^{-\lambda q}}{(\Gamma\alpha)^r\lambda^{-\alpha s}}$, where, $\lambda > 0$, and $k = \int_0^\infty \frac{p^{\alpha-1}e^{-\lambda q}}{(\Gamma\alpha)^r\lambda^{-\alpha s}}$,

$g_C(\alpha,\lambda) \propto \frac{p^{\alpha-1}e^{-\lambda q}}{(\Gamma\alpha)^r\lambda^{-\alpha s}}$. The joint posterior density is

$$g_C^*(\alpha,\lambda) \propto \frac{\lambda^{(n+s)\alpha}}{(\Gamma\alpha)^{n+r}}\left(p \prod_{i=1}^n x_i\right)^{\alpha-1}e^{-\lambda(\sum_{i=1}^n x_i+q)} \tag{12}$$

which actually belongs to the same family as that of prior density,

Taking partial derivative of $ln\,(g_C^*(\alpha,\lambda))$ w.r.t α and λ and equating to zero, gives likelihood equations as

$$\frac{(n+s)\alpha}{\lambda} - (\textstyle\sum_{i=1}^n x_i + q) = 0, \tag{13}$$

$$(n+s)\,ln(\lambda) - \frac{(n+r)\psi(\alpha)}{\Gamma\alpha} + ln\,(p) + ln\,(\textstyle\sum_{i=1}^n x_i) = 0, \tag{14}$$

by solving equation (13), GMLE of λ is

$\hat{\lambda}_C = \frac{(n+s)\alpha}{\sum_{i=1}^{n} x_i + q}$.

Equation (14) can be solved numerically for α using numerical methods.

The expression of GMLE for Theil's entropy measure and Gini index are given below

$\hat{T}_C = \psi(\hat{\alpha}_C) + \frac{1}{\hat{\alpha}_C} - \log \hat{\alpha}_C,$

$\hat{G}_C = \frac{\Gamma(\hat{\alpha}_C + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\hat{\alpha}_C + 1)}$, where $\hat{\alpha}_C$ is the GMLE of α.

4. Simulation Study

In order to compare the efficiency of GMLE's using three different priors in case of Gamma distribution, a simulation study is carried out. As already pointed out in section 3.1, GMLE's obtained using uniform prior is nothing but the MLE's in fact. So, the simulation results also show the relative efficiency of GMLE's using Jeffrey's and conjugate prior with the MLE's (GMLE's using uniform prior). As pointed out in the beginning, no loss function is required in the derivations of GMLE's (semi-Bayesian approach). Hence mean square error (MSE) and absolute bias (Abias) are considered to compare the relative efficiency of all GMLE's using three priors. The absolute bias (Abias) and Mean square error (MSE) are given by

Abias $(\hat{\theta}) = \frac{1}{10000} \sum_{i=1}^{10000} |\hat{\theta} - \theta|$ and MSE $(\hat{\theta}) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\theta} - \theta)^2$

The mean square errors and absolute bias are computed using generated random samples from Gamma distribution for shape and scale parameters and two inequality indices for different combinations of sample sizes $n = (25, 50, 100)$, parameters $(\alpha, \lambda = 2, 4)$ and hyper-parameters $(a = b = c = p = q = r = s = 1, 2)$. As pointed out earlier likelihood equations for α cannot be solved explicitly and is solved by using "optim" function in R-software. The mean square error and absolute bias is computed using three priors *viz.* uniform prior, Jeffreys' prior and conjugate prior for both shape and scale parameters and two inequality indices *viz.* Theil's entropy measure and Gini index of Gamma distribution. The simulation results using R- Software for all the three cases are reported in the following tables (2 - 12).

Case 1. α is unknown, λ is known.

Table 2. Mean square error (MSE) and Absolute bias (Abias) for shape parameter α

$(\boldsymbol{c = 1, \lambda = 2})$

hyperparametrs	n	α	$MSE_{\hat{\alpha}_U}$ ($Abias_{\hat{\alpha}_U}$)	$MSE_{\hat{\alpha}_C}$ ($Abias_{\hat{\alpha}_C}$)	$MSE_{\hat{\alpha}_J}$ ($Abias_{\hat{\alpha}_J}$)
$a = 1$, $b = 1$	25	2	0.06418403 (0.0542141)	0.06133267 (0.0279388)	0.06363331 (0.0346793)
		4	0.14104182 (0.0301541)	0.12730951 (0.0179470)	0.14050672 (0.0202297)
	50	2	0.03042016 (0.00873364)	0.02947641 (0.00774469)	0.03014402 (0.00800178)
		4	0.07152516 (0.0292238)	0.06775730 (0.01409371)	0.07189076 (0.0153408)
	100	2	0.01522910 (0.00343252)	0.01505602 (0.00101337)	0.01529712 (0.0014306)
		4	0.03529425 (0.0144250)	0.03447581 (0.00543124)	0.03535247 (0.0094536)
$a = 2$, $b = 1$	25	2	0.06048871 (0.01451087)	0.05520600 (0.01225102)	0.06017265 (0.01300918)
		4	0.14205475 (0.0222191)	0.13384592 (0.0183987)	0.14116554 (0.0221473)
	50	2	0.03093084 (0.00802244)	0.02984442 (0.0039593)	0.03124154 (0.00485831)
		4	0.07239059 (0.00869420)	0.0128938 (0.00173571)	0.01590853 (0.0019589)
	100	2	0.01529096 (0.00383456)	0.01527967 (0.00186471)	0.01534917 (0.0019288)
		4	0.03565841 (0.00473456)	0.01211419 (0.00118696)	0.01574099 (0.00236627)
$a = 1$, $b = 2$	25	2	0.06293354 (0.0169306)	0.05296014 (0.00549524)	0.06257284 (0.0025905)
		4	0.14145911 (0.04983160)	0.14063406 (0.0109133)	0.15317102 (0.0366728)
	50	2	0.03100455 (0.0091247)	0.02840451 (0.00484491)	0.03131218 (0.0051335)

		4	0.07118786 (0.0122641)	0.06487734 (0.01001062)	0.07145114 (0.0215032)
	100	2	0.01558493 (0.0056057)	0.01492327 (0.00307743)	0.01564236 (0.0044212)
		4	0.03473031 (0.0028579)	0.03464921 (0.0015998)	0.03479410 (0.0010539)
$a = 2$ $b = 2$	25	2	0.14288727 (0.0459310)	0.12808663 (0.01512771)	0.14240707 (0.04430490)
		4	0.14458676 (0.0291888)	0.12937480 (0.01249217)	0.14409513 (0.0228434)
	50	2	0.01567977 (0.00693691)	0.01536318 (0.00225876)	0.01575190 (0.00280074)
		4	0.07019152 (0.0291888)	0.06648917 (0.01249217)	0.07049002 (0.0228434)
	100	2	0.01544886 (0.00476195)	0.01525719 (0.0010154)	0.01563825 (0.0012842)
		4	0.03537756 (0.00640761)	0.03437598 (0.0014633)	0.03545487 (0.00356162)

Table 3. Mean square error (MSE) and Absolute bias (Abias) for shape parameter $\boldsymbol{\alpha}$

$(\boldsymbol{c = 2, \lambda = 2})$

hyperparamet rs	$\boldsymbol{n}$	$\boldsymbol{\alpha}$	$\boldsymbol{MSE_{\hat{\alpha}_U}}$ $\boldsymbol{(Abias_{\hat{\alpha}_U})}$	$\boldsymbol{MSE_{\hat{\alpha}_C}}$ $\boldsymbol{(Abias_{\hat{\alpha}_C})}$	$\boldsymbol{MSE_{\hat{\alpha}_J}}$ $\boldsymbol{(Abias_{\hat{\alpha}_J})}$
	25	2	0.06214046 (0.02050019)	0.06040602 (0.01824008)	0.06124668 (0.0175890)
		4	0.14260906 (0.0194943)	0.13399647 (0.00137728)	0.14222393 (0.00156988)
$a = 1$ $b = 1$	50	2	0.03012016 (0.00832872)	0.02948342 (0.00377020)	0.03044402 (0.00540942)
		4	0.07152516 (0.0087487)	0.06938742 (0.00105134)	0.07189076 (0.0011408)
	100	2	0.01593814 (0.00751692)	0.0129831 (0.00223264)	0.01603877 (0.00265302)
		4	0.03549757 (0.00523361)	0.03495545 (0.00100276)	0.03557904 (0.0010097)
		2	0.06299925	0.06175714	0.06200858

	25		(0.0171499)	(0.012960)	(0.027741)
		4	0.14416055	0.14380690	0.14391145
			(0.021411)	(0.013383)	(0.0148743)
$a = 2$		2	0.03127105	0.02973333	0.03150391
$b = 1$	50		(0.0093011)	(0.0062426)	(0.0085622)
		4	0.07094723	0.0693433	0.07113879
			(0.0115754)	(0.0010818)	(0.00126323)
		2	0.01537239	0.01483332	0.01544987
	100		(0.0092906)	(0.00612986)	(0.0066367)
		4	0.03546286	0.03447831	0.03554416
			(0.00104238)	(0.0009120)	(0.00095135)
		2	0.06409560	0.05913204	0.06349955
	25		(0.0258485)	(0.0213864)	(0.0232202)
		4	0.14146285	0.12836396	0.14119986
			(0.0285829)	(0.0092275)	(0.0207303)
$a = 1$		2	0.03138245	0.03011734	0.03167533
$b = 2$	50		(0.0066876)	(0.0022350)	(0.0030501)
		4	0.07058966	0.06677861	0.07089721
			(0.0225633)	(0.0090899)	(0.013410)
		2	0.01566982	0.01531993	0.01572174
	100		(0.0034292)	(0.0011528)	(0.0014340)
		4	0.03508357	0.03411485	0.03515906
			(0.0101269)	(0.0030401)	(0.0091342)
		2	0.06272001	0.06202941	0.06235411
	25		(0.0584666)	(0.0221854)	(0.0305618)
		4	0.14096877	0.10040838	0.11769224
			(0.0539655)	(0.0446953)	(0.0515845)
$a = 2$		2	0.03122557	0.02978343	0.03150970
$b = 2$	50		(0.0208260)	(0.0175188)	(0.0208072)
		4	0.06895781	0.06303986	0.06924676
			(0.0209033)	(0.0182234)	(0.0195427)
		2	0.01534949	0.01478344	0.01541029
	100		(0.0053181)	(0.0024098)	(0.0045458)
		4	0.03564965	0.03404204	0.03572855
			(0.0089584)	(0.00619829)	(0.0074457)

Table 4.Mean square error (MSE) and Absolute bias (Abias) for Theil's entropy measure (T)

$(\boldsymbol{c} = \mathbf{1}, \boldsymbol{\lambda} = \mathbf{2})$

hyperparametrs	$\boldsymbol{n}$	$\boldsymbol{\alpha}$	$\boldsymbol{MSE_{\hat{T}_U}}$ $\boldsymbol{(Abias_{\hat{T}_U})}$	$\boldsymbol{MSE_{\hat{T}_C}}$ $\boldsymbol{(Abias_{\hat{T}_C})}$	$\boldsymbol{MSE_{\hat{T}_J}}$ $\boldsymbol{(Abias_{\hat{T}_J})}$
$a = 1$, $b = 1$	25	2	0.00069640 (0.0009722)	0.00063819 (0.0007864)	0.00072921 (0.0009083)
		4	0.00011514 (0.0009709)	0.00011402 (0.0002547)	0.00011508 (0.0003523)
	50	2	0.00034478 (0.0007309)	0.00033004 (0.0001545)	0.00033931 (0.0006177)
		4	5.739254e-05 (0.00055915)	5.605151e-05 (0.0001371)	5.687914e-05 (0.0002474)
	100	2	0.00017029 (3.492362e-04)	0.000166493 (1.912076e-04)	0.00016882 (2.639362e-04)
		4	2.942413e-05 (0.0004747)	2.80061e-05 (0.00013683)	2.927382e-05 (0.0002183)
$a = 2$, $b = 1$	25	2	0.00069843 (0.0040326)	0.00064528 (0.0027202)	0.00073380 (0.0031440)
		4	0.00011795 (0.0003680)	0.00010800 (0.00016627)	0.00012118 (0.0002502)
	50	2	0.00034493 (0.0040197)	0.00033125 (0.0022078)	0.00033882 (0.0030161)
		4	5.869849e-05 (2.986916e-04)	5.627267e-05 (6.677038e-05)	5.821697e-05 (1.867352e-04)
	100	2	0.00017201 (0.0003197)	0.00016924 (0.0001286)	0.00017068 (0.0002356)
		4	2.861282e-05 (2.188633e-04)	2.799087e-05 (1.352237e-05)	2.847154e-05 (1.621603e-04)
$a = 1$, $b = 2$	25	2	0.00032070 (0.0042226)	0.00031454 (0.00098942)	0.00031757 (0.00333777)
		4	0.00012036 (0.0039861)	0.00011725 (0.00131588)	0.00012360 (0.00281162)
	50	2	0.00014387 (0.0018379)	0.00011955 (0.0009265)	0.00013754 (0.0012198)
		4	5.765238e-05 (0.0004525)	5.07666e-05 (0.0003618)	5.701034e-05 (0.0004408)

	100	2	0.00010566 (0.0003798)	0.00010264 (0.0002029)	0.00010321 (0.0003557)
		4	2.876418e-05 (0.0002021)	2.355667e-05 (0.0001915)	2.859703e-05 (0.0002073)
$a = 2$ $b = 2$	25	2	0.00070443 (0.00425542)	0.00059728 (0.00214613)	0.00073885 (0.0037139)
		4	0.00012883 (0.0054709)	0.00011776 (0.0047626)	0.00012099 (0.0049107)
	50	2	0.00033888 (0.0008957)	0.00031170 (0.0008462)	0.00033315 (0.0008837)
		4	5.693016e-05 (0.0002986)	4.865644e-05 (0.00024891)	5.638465e-05 (0.0002670)
	100	2	0.00017100 (0.00067654)	0.00016388 (0.00041954)	0.00016952 (0.00067208)
		4	2.895723e-05 (0.00021875)	2.793509e-05 (0.0001333)	2.878516e-05 (0.0002028)

Table 5. Mean square error (MSE) and Absolute bias (Abias) for Theil's entropy measure (T)

$(c = 2, \lambda = 2)$

hyperparametrs	n	α	$MSE_{\hat{T}_U}$ $(Abias_{\hat{T}_U})$	$MSE_{\hat{T}_C}$ $(Abias_{\hat{T}_C})$	$MSE_{\hat{T}_J}$ $(Abias_{\hat{T}_J})$
$a = 1$ $b = 1$	25	2	0.00070941 (0.00496406)	0.00064667 (0.0010922)	0.00069561 (0.0032051)
		4	0.00011820 (0.0042511)	0.00010534 (0.0010041)	0.00011505 (0.00100787)
	50	2	0.00034509 (0.0028314)	0.00033233 (0.0010205)	0.00033915 (0.0019248)
		4	5.918740e-05 (2.851018e-04)	5.665644e-05 (1.767882e-05)	5.860991e-05 (1.735921e-04)
	100	2	0.00016791 (3.479637e-04)	0.00016416 (1.405564e-04)	0.00016625 (2.621703e-04)
		4	2.842008e-05 (1.514747e-04)	2.780795e-05 (1.176248e-05)	2.828595e-05 (1.148428e-04)
$a = 2$ $b = 1$	25	2	0.00074093 (0.00905267)	0.00070078 (0.00125542)	0.00070647 (0.0033713)
		4	0.00012168 (0.0054709)	0.00011148 (0.00166027)	0.00011839 (0.0017317)
	50	2	0.00033746 (0.0089577)	0.00033498 (0.0011860)	0.00033222 (0.0019371)

		4	5.972830e-05 (0.00061886)	5.763264e-05 (0.00014121)	5.904043e-05 (0.0002701)
	100	2	0.00017035 (0.0004564)	0.00016722 (0.0002213)	0.0001686 (0.0005728)
		4	2.955777e-05 (2.344413e-04)	2.924988e-05 (1.096478e-04)	2.942539e-05 (1.777719e-04)
	25	2	0.00070040 (0.0057876)	0.00057388 (0.0027666)	0.00067070 (0.0036824)
		4	0.00012652 (0.0096723)	0.00011700 (0.0038478)	0.00011998 (0.0046074)
$a = 1$ $b = 2$	50	2	0.00034103 (0.0051686)	0.00031368 (0.0011323)	0.00033528 (0.0016355)
		4	5.807123e-05 (0.0035940)	5.080133e-05 (0.0020509)	5.751385e-05 (0.0021726)
	100	2	0.00017229 (0.0001415)	0.00016516 (0.0001004)	0.00017079 (0.0001214)
		4	2.917408e-05 (0.0003752)	2.801346e-05 (0.0001298)	2.901865e-05 (0.00031864)
	25	2	0.00074455 (0.0047678)	0.00061824 (0.00197117)	0.00070844 (0.0025787)
		4	0.00012774 (0.0038409)	0.00010645 (0.0019583)	0.00011069 (0.0026648)
$a = 2$ $b = 2$	50	2	0.00034282 (0.0036467)	0.00032415 (0.0012297)	0.00033776 (0.00170403)
		4	5.939816e-05 (0.0003023)	5.630069e-05 (0.00011279)	5.889368e-05 (0.0001989)
	100	2	0.00017328 (0.0003117)	0.00016680 (0.0001595)	0.00017158 (0.0001969)
		4	2.941019e-05 (0.00017744)	2.863001e-05 (0.00010172)	2.928689e-05 (0.00012088)

Table 6. Mean square error (MSE) and Absolute bias(Abias) for Gini index (G)

$(c = 1, \lambda = 2)$

hyperparametrs	n	α	$MSE_{\hat{G}_U}$ $(Abias_{\hat{G}_U})$	$MSE_{\hat{G}_C}$ $(Abias_{\hat{G}_C})$	$MSE_{\hat{G}_J}$ $(Abias_{\hat{G}_J})$
	25	2	0.000418958 (0.00203949)	0.000369793 (0.00122717)	0.000392415 (0.0017384)
		4	0.000143515 (0.0002285)	0.000136917 (0.0002071)	0.00014158 (0.00021749)
$a = 1$		2	0.000224396	0.000212033	0.000220004

$b = 1$	50		(0.00025279)	(0.00022959)	(0.00024631)
		4	7.375617e-05	7.167248e-05	7.231917e-05
			(0.0002172)	(0.00013929)	(0.0002086)
		2	0.000109173	0.000106093	0.000108048
	100		(2.618634e-05)	(2.162508e-04)	(2.287899e-04)
		4	3.758789e-05	3.725134e-05	3.737037e-05
			(0.00019291)	(0.00010221)	(0.0002039)
		2	0.000411407	0.000388636	0.000405453
	25		(0.00478912)	(0.00200727)	(0.00217142)
		4	0.000149322	0.000128843	0.000131690
			(0.0009232)	(0.00045269)	(0.00076968)
		2	0.000209906	0.000195598	0.000199738
$a = 2$	50		(0.00216992)	(0.00155972)	(0.00180454)
$b = 1$		4	7.594444e-05	7.240619e-05	7.548331e-05
			(6.135186e-04)	(2.102254e-04)	(4.024819e-04)
		2	0.000104015	0.000103568	0.000104407
	100		(0.00083559)	(0.00011361)	(0.00048632)
		4	3.610181e-05	3.533543e-05	3.600406e-05
			(6.009397e-05)	(1.662743e-05)	(2.201623e-05)
		2	0.000430069	0.000361386	0.000423920
	25		(0.0020044)	(0.0019309)	(0.0019369)
		4	0.000144527	0.000138005	0.000142330
			(0.0043767)	(0.00410206)	(0.0040318)
		2	0.000217656	0.000192411	0.000201299
$a = 1$	50		(0.00052952)	(0.00012663)	(0.00031479)
$b = 2$		4	8.645682e-05	7.291410e-05	7.371082e-05
			(0.0025652)	(0.0020012)	(0.0022902)
		2	0.000107239	0.000102361	0.000106468
	100		(0.0001874)	(0.0001027)	(0.0001202)
		4	3.795402e-05	3.653725e-05	3.656425e-05
			(0.0004154)	(0.000181094)	(0.0002015)
		2	0.000428093	0.000367854	0.000418908
	25		(0.0037508)	(0.0019614)	(0.00225632)
		4	0.000149883	0.000147699	0.000148057
			(0.00097078)	(0.000502723)	(0.0004231)
		2	0.000223010	0.000205274	0.000220241
$a = 2$	50		(0.0032459)	(0.00096805)	(0.00118613)
$b = 2$		4	7.384992e-05	7.233343e-05	7.238489e-05

			(0.00033257)	(0.00010191)	(0.0003229)
		2	0.000111796	0.000106581	0.000110837
	100		(0.00059933)	(0.000141855)	(0.00048526)
		4	3.696256e-05	3.628881e-05	3.643114e-05
			(0.000254994)	(0.00019624)	(0.00017231)

Table 7. Mean square error (MSE) and Absolute bias (Abias) for Gini index (G)

$(c = 2, \lambda = 2)$

hyperparametrs	n	α	$MSE_{\hat{G}_U}$ ($Abias_{\hat{G}_U}$)	$MSE_{\hat{G}_C}$ ($Abias_{\hat{G}_C}$)	$MSE_{\hat{G}_J}$ ($Abias_{\hat{G}_J}$)
		2	0.000423040	0.000414852	0.000435372
	25		(0.00452948)	(0.00183635)	(0.00205974)
		4	0.000146345	0.000134904	0.000149304
			(0.00083656)	(0.00018971)	(0.0003888)
$a = 1$		2	0.000216461	0.000209457	0.000221205
$b = 1$	50		(0.0035917)	(0.00117016)	(0.0020109)
		4	7.542785e-05	7.269129e-05	7.583623e-05
			(4.950804e-04)	(1.198840e-04)	(1.736020e-04)
		2	9.773583e-05	9.726318e-05	9.814429e-05
	100		(2.637513e-04)	(1.064745e-04)	(3.559798e-04)
		4	3.642756e-05	3.562784e-05	3.666891e-05
			(9.855256e-05)	(4.855206e-05)	(5.586829e-05)
		2	0.000454948	0.000437198	0.000453591
	25		(0.00377762)	(0.00196144)	(0.00780139)
		4	0.000158657	0.000155929	0.000161559
			(0.00097078)	(0.00032296)	(0.00326172)
$a = 2$	50	2	0.000218838	0.000213109	0.000215558
$b = 1$			(0.0032459)	(0.00118613)	(0.0037508)
		4	7.211042e-05	7.099413e-05	7.291883e-05
			(0.00010191)	(0.0017234)	(0.00042313)
	100	2	0.000104250	0.000102595	0.000105017
			(0.00019624)	(0.000190108)	(0.00059933)
		4	3.387659e-05	3.274837e-05	3.396842e-05
			(8.440393e-05)	(7.816421e-05)	(2.382612e-05)
		2	0.000401632	0.000348834	0.000415525
	25		(0.00273811)	(0.00143444)	(0.0019582)

		4	0.000159900 (0.0078628)	0.000143787 (0.0013971)	0.000153888 (0.00485572)
$a = 1$ $b = 2$	50	2	0.000228786 (0.00094970)	0.000207019 (0.0009590)	0.000223167 (0.00118503)
		4	7.526489e-05 (0.0003678)	7.449506e-05 (0.000248701)	7.507263e-05 (0.00035775)
	100	2	0.000106958 (0.00014940)	0.000103133 (0.00012407)	0.000107827 (0.00027758)
		4	3.799733e-05 (0.00028439)	3.789738e-05 (0.00012421)	3.810490e-05 (0.00013712)
$a = 2$ $b = 2$	25	2	0.000404452 (0.00608318)	0.000379757 (0.00135037)	0.000417178 (0.0027892)
		4	0.000147377 (0.0028846)	0.000129980 (0.0014192)	0.000149650 (0.00193614)
	50	2	0.000214012 (0.00284067)	0.000210260 (0.00100629)	0.000216278 (0.0019636)
		4	7.565985e-05 (0.0021578)	7.065622e-05 (0.00112614)	7.496627e-05 (0.0019124)
	100	2	0.000108840 (0.00013664)	0.000107209 (0.000111170)	0.000108251 (0.00013156)
		4	3.451742e-05 (0.00060203)	3.345793e-05 (0.00012506)	3.465734e-05 (0.00028524)

From the above tables, it can be observed that

(i) GMLE’s using conjugate prior perform better in comparison with uniform and Jeffreys’prior, as it has smaller mean square error and absolute bias.

(ii) GMLE’s using uniform prior, also called MLE’s, are less efficient than the GMLE’s using Jeffreys’ and conjugate prior throughout the study.

Case 2. α is known, λ is unknown.

Table 8. Mean square error (MSE) and Absolute bias (Abias) for λ ($\alpha = 2$)

hyperparametrs	n	λ	$MSE_{\hat{\lambda}_U}$ ($Abias_{\hat{\lambda}_U}$)	$MSE_{\hat{\lambda}_C}$ ($Abias_{\hat{\lambda}_C}$)	$MSE_{\hat{\lambda}_J}$ ($Abias_{\hat{\lambda}_J}$)
	25	2	0.0888325 (0.0454059)	0.0752711 (0.04110801)	0.0836259 (0.0430021)
		4	0.3519132	0.3004139	0.3307999

			(0.0793978)	(0.02337489)	(0.0251900)
$a = 1$		2	0.04285318	0.0394527	0.0415664
$b = 1$	50		(0.0213883)	(0.012442)	(0.0189451)
		4	0.1624370	0.1514480	0.1577988
			(0.049140)	(0.0122576)	(0.0154403)
		2	0.02034622	0.0194980	0.0200195
	100		(0.0091200)	(0.0010963)	(0.0090550)
		4	0.08158882	0.0781396	0.0802950
			(0.0057585)	(0.0019822)	(0.0020926)
		2	0.0876163	0.0759988	0.0828144
	25		(0.092880)	(0.0415030)	(0.0672009)
		4	0.3572877	0.2862946	0.3360544
			(0.0668766)	(0.01491962)	(0.0164608)
		2	0.0427286	0.0396799	0.0413766
$a = 2$	50		(0.059051)	(0.019796)	(0.0200020)
$b = 1$		4	0.1705744	0.1535467	0.1658293
			(0.0366389)	(0.0126707)	(0.014274)
		2	0.0204979	0.0197617	0.0201696
	100		(0.00780408)	(0.0022122)	(0.00233493)
		4	0.0830142	0.0778295	0.0815273
			(0.0038232)	(0.0010124)	(0.0021721)
		2	0.0879409	0.0752911	0.0827192
	25		(0.06056267)	(0.0168926)	(0.0184956)
		4	0.4295680	0.3293867	0.3504354
			(0.0766743)	(0.0485916)	(0.0503464)
		2	0.0425946	0.0393324	0.0413356
$a = 1$	50		(0.0421474)	(0.01147697)	(0.0130446)
$b = 2$		4	0.1667842	0.1503778	0.1619073
			(0.0306103)	(0.0272724)	(0.0296957)
		2	0.0210436	0.0200346	0.0207136
	100		(0.00284955)	(0.00113552)	(0.0012984)
		4	0.0833163	0.0819654	0.0820528
			(0.00285738)	(0.00133702)	(0.0024609)
		2	0.0906052	0.0724147	0.0852468
	25		(0.0817998)	(0.0367234)	(0.0401099)
		4	0.3824357	0.3408608	0.3615431
			(0.0766711)	(0.04336160)	(0.0486223)
		2	0.0416033	0.0374450	0.0404237

$a = 2$	50		(0.0420693)	(0.0175308)	(0.026444)
$b = 2$		4	0.1691232	0.1547684	0.1639433
			(0.052513)	(0.02315031)	(0.0350996)
		2	0.0210193	0.0197985	0.0206779
	100		(0.0062966)	(0.00235209)	(0.0037347)
		4	0.0812283	0.0793568	0.0800375
			(0.0019156)	(0.0011755)	(0.0013908)

From the above tables, it is observed that GMLE's of λ using conjugate prior perform better in comparison with rest of the two priors.

Case 3. α and λ both are unknown.

Table 9. Mean square error (MSE) and Absolute bias (Abias) for α (r = 1, s = 1)

hyperparametrs	n	α	λ	$MSE_{\hat{\alpha}_U}$ ($Abias_{\hat{\alpha}_U}$)	$MSE_{\hat{\alpha}_C}$ ($Abias_{\hat{\alpha}_C}$)	$MSE_{\hat{\alpha}_J}$ ($Abias_{\hat{\alpha}_J}$)
		2	2	0.1879373	0.1056029	0.1529660
				(0.209810)	(0.1371699)	(0.1763893)
		2	4	0.1798746	0.1673763	0.1482970
				(0.1421591)	(0.1049958)	(0.1163730)
	50	4	2	0.8257052	0.7340724	0.6755012
				(0.2499472)	(0.2004313)	(0.217560)
		4	4	0.7997035	0.8743487	0.6524155
				(0.3292497)	(0.2490484)	(0.2671470)
		2	2	0.0785231	0.0823009	0.0714470
p = 1				(0.0743845)	(0.0247901)	(0.0553625)
q = 1		2	4	0.0800406	0.0774563	0.0724783
	100			(0.0514621)	(0.0403180)	(0.0477257)
		4	2	0.3393113	0.3207846	0.3051927
				(0.0908736)	(0.0112097)	(0.0492079)
		4	4	0.3418790	0.3593887	0.3078205
				(0.0491910)	(0.0112098)	(0.0151296)
		2	2	0.1813348	0.2953424	0.1507793
				(0.300480)	(0.1180587)	(0.2790069)
		2	4	0.1934567	0.2636873	0.1588641
	50			(0.342463)	(0.1112242)	(0.2160770)
		4	2	0.7786777	1.5836532	0.6442952

				(0.6842394)	(0.2163924)	(0.239845)
		4	4	0.7750269	1.2949399	0.6468762
				(0.2481014)	(0.2164297)	(0.239476)
		2	2	0.0763077	0.0984587	0.0699213
p = 2				(0.0549750)	(0.0127663)	(0.0162366)
q = 1		2	4	0.0777658	0.0916028	0.0704512
				(0.0542888)	(0.0102920)	(0.0128941)
		4	2	0.3391408	0.4959271	0.3078898
	100			(0.0447891)	(0.0112553)	(0.0321051)
		4	4	0.3520755	0.5948748	0.3175692
				(0.057954)	(0.0123144)	(0.0401118)
		2	2	0.1875511	0.1300424	0.1530627
				(0.572490)	(0.1087776)	(0.374893)
		2	4	0.1810254	0.1129893	0.1479685
	50			(0.3120797)	(0.1602386)	(0.261940)
		4	2	0.7867510	0.4922316	0.6433997
				(0.2480901)	(0.1120405)	(0.158144)
		4	4	0.7894936	0.4846664	0.6498030
				(0.2443093)	(0.217172)	(0.226060)
		2	2	0.0801425	0.0672268	0.0723920
p = 1				(0.0520627)	(0.011948)	(0.034497)
q = 2		2	4	0.0811436	0.0635922	0.0734611
				(0.0807042)	(0.0581508)	(0.024570)
		4	2	0.3424844	0.2720251	0.3095638
	100			(0.0093928)	(0.00497832)	(0.0065556)
		4	4	0.3462080	0.2688108	0.3133854
				(0.00468120)	(0.00112344)	(0.00146301)
		2	2	0.1928328	0.1794689	0.1576012
				(0.0916435)	(0.01058854)	(0.0147781)
		2	4	0.1898563	0.1197932	0.1566421
	50			(0.0714085)	(0.0116603)	(0.016352)
		4	2	0.7874665	0.8597485	0.6469429
				(0.074006)	(0.02340331)	(0.03138827)
		4	4	0.7989785	0.6439628	0.6546058
				(0.0255213)	(0.0125114)	(0.0131631)
		2	2	0.0815296	0.0788559	0.0734825
p = 2				(0.00521757)	(0.00334005)	(0.00483276)
q = 2		2	4	0.0813718	0.0639134	0.0734983

	100			(0.00531253)	(0.0024186)	(0.00398332)
		4	2	0.3540009	0.3716112	0.3196447
				(0.0015841)	(0.0011767)	(0.0012037)
		4	4	0.3399667	0.3053347	0.3069580
				(0.0058885)	(0.0011562)	(0.0014992)

Table 10. Mean square error (MSE) and Absolute bias (Abias) for α (r = 2, s = 2)

hyperparametrs	n	α	λ	$MSE_{\hat{\alpha}_U}$ ($Abias_{\hat{\alpha}_U}$)	$MSE_{\hat{\alpha}_C}$ ($Abias_{\hat{\alpha}_C}$)	$MSE_{\hat{\alpha}_J}$ ($Abias_{\hat{\alpha}_J}$)
		2	2	0.1873156	0.4175363	0.1534149
				(0.585470)	(0.1210893)	(0.3882318)
		2	4	0.1831961	0.5137488	0.1500971
	50			(0.4554439)	(0.1132362)	(0.1532101)
		4	2	0.7433455	1.0852683	0.6118207
				(0.537010)	(0.2361931)	(0.5031730)
		4	4	0.6954682	1.0627882	0.5756114
				(0.3145170)	(0.1151940)	(0.2541601)
		2	2	0.0809656	0.1250673	0.07346619
				(0.079305)	(0.0200756)	(0.0229814)
p = 1		2	4	0.0803587	0.1443074	0.07223236
q = 1	100			(0.0376566)	(0.0174301)	(0.0185489)
		4	2	0.3464558	0.4169364	0.3142051
				(0.0446003)	(0.0193711)	(0.0316857)
		4	4	0.3412565	0.7808418	0.3102686
				(0.06137443)	(0.01678399)	(0.0491552)
		2	2	0.1815976	0.6188912	0.1508759
				(0.5974020)	(0.1657334)	(0.478490)
		2	4	0.1852638	0.7838433	0.1536690
	50			(0.7059961)	(0.1721621)	(0.538891)
		4	2	0.6861428	1.3196376	0.5682270
				(0.3493777)	(0.10104161)	(0.2567321)
		4	4	0.5130599	1.1375845	0.4917533
				(0.3195631)	(0.2106290)	(0.275894)
		2	2	0.0820535	0.1646216	0.0738566
				(0.0511000)	(0.02273784)	(0.0438381)
p = 2		2	4	0.0788440	0.1844032	0.0707349

$q = 1$	100			(0.0542228)	(0.0135380)	(0.0264373)
		4	2	0.3405034	0.6856084	0.3074706
				(0.04811805)	(0.01149821)	(0.0212210)
		4	4	0.3439788	1.3310810	0.3116509
				(0.0853732)	(0.0125767)	(0.083405)
		2	2	0.1891849	0.2277050	0.1541366
				(0.1915268)	(0.1139921)	(0.1821510)
		2	4	0.1857491	0.1624568	0.1515624
	50			(0.1169536)	(0.0999640)	(0.1965712)
		4	2	0.8024543	0.6479233	0.6593188
				(0.2402770)	(0.1490101)	(0.153021)
		4	4	0.7761700	0.9333951	0.6373757
				(0.522391)	(0.344301)	(0.186338)
		2	2	0.0811908	0.0898427	0.0736114
				(0.085037)	(0.0471306)	(0.082354)
		2	4	0.0808324	0.0759970	0.0727842
$p = 1$	100			(0.053246)	(0.0230112)	(0.046766)
$q = 2$		4	2	0.3471838	0.3126186	0.3135654
				(0.0760627)	(0.0117362)	(0.0186612)
		4	4	0.3492291	0.3885574	0.3135601
				(0.0193945)	(0.0115461)	(0.0165711)
		2	2	0.1854008	0.3336778	0.1538725
				(0.3130464))	(0.1118821)	(0.2806211)
		2	4	0.1890242	0.2420139	0.1547746
	50			(0.2117926))	(0.1168713	(0.188601)
		4	2	0.8421083	1.5346173	0.6799039
				(0.6736578)	(0.180389)	(0.2610931)
		4	4	1.0243877	2.8263651	0.8449021
				(0.592533)	(0.3015693)	(0.560873)
		2	2	0.0807348	0.1133007	0.0723926
$p = 2$				(0.0843081)	(0.0178686)	(0.027838)
$q = 2$		2	4	0.0800512	0.0917787	0.0719877
	100			(0.0595390)	(0.0104448)	(0.038048)
		4	2	0.3573885	0.4930335	0.3205471
				(0.0482481)	(0.01121962)	(0.02958202)
		4	4	0.3439061	0.6267642	0.3098705
				(0.04318895)	(0.01136072)	(0.0345651)

Table 11. Mean square error (MSE) and Absolute bias (Abias) for λ (r = 1, s = 1)

hyperparametrs	n	α	λ	$MSE_{\hat{\lambda}_U}$ ($Abias_{\hat{\lambda}_U}$)	$MSE_{\hat{\lambda}_C}$ ($Abias_{\hat{\lambda}_C}$)	$MSE_{\hat{\lambda}_J}$ ($Abias_{\hat{\lambda}_J}$)
		2	2	0.2438073 (0.294514)	0.2622098 (0.1675804	0.1974556 (0.206306)
	50	2	4	0.9573035 (0.1892284)	0.7872535 (0.3023880)	0.7833887 (0.280361)
		4	2	0.2319797 (0.4375706)	0.2206380 (0.1328701)	0.1895886 (0.1362658)
		4	4	0.9207189 (0.7529812)	0.9986289 (0.2712313)	0.7464754 (0.3509580)
p = 1 q = 1		2	2	0.1038003 (0.0857382)	0.1079851 (0.0668334)	0.0937057 (0.0856974)
	100	2	4	0.4167548 (0.083968)	0.3791985 (0.0123239)	0.3753752 (0.0738635)
		4	2	0.0958868 (0.0631341)	0.0939387 (0.0594019)	0.0861132 (0.0628302)
		4	4	0.3940054 (0.0160818)	0.4124239 (0.0121733)	0.3536089 (0.055822)
		2	2	0.2413406 (0.3002836)	0.3704279 (0.1387172)	0.1983410 (0.2343834)
	50	2	4	1.0084882 (0.3808898)	1.1552301 (0.100576)	0.8215033 (0.2617477)
		4	2	0.2182767 (0.3749075)	0.4608687 (0.115818)	0.1793704 (0.148384)
		4	4	0.8852570 (0.6626932)	1.1689512 (0.2307006)	0.7344270 (0.305671)
p = 2 q = 1		2	2	0.1000758 (0.0685178)	0.1251831 (0.0141438)	0.0910767 (0.0248725)
	100	2	4	0.4137709 (0.0684786)	0.4416856 (0.01248268)	0.3726233 (0.01795920)
		4	2	0.0970455 (0.0625707)	0.1440544 (0.0113964)	0.0881314 (0.01780734)
		4	4	0.4001184 (0.0487984)	0.6542099 (0.0132518)	0.3603681 (0.0410910)
		2	2	0.2436393	0.1588317	0.1977369

				(0.681463)	(0.128977)	(0.261759)
		2	4	0.9530660	0.6227100	0.7714546
	50			(0.4990614)	(0.2687823)	(0.3466454)
		4	2	0.2256248	0.1422089	0.1837200
				(0.480425)	(0.1331227)	(0.1423845)
		4	4	0.9014500	0.5559882	0.7410165
				(0.3935741)	(0.2674103)	(0.3326976)
p = 1		2	2	0.1050133	0.0852869	0.0944002
q = 2				(0.0621918)	(0.0117311)	(0.036444)
		2	4	0.4211057	0.3344625	0.3788977
	100			(0.037946)	(0.0135804)	(0.0257559)
		4	2	0.0982981	0.0786982	0.0888569
				(0.0673694)	(0.0198201)	(0.0351626)
		4	4	0.3895332	0.3013235	0.3515890
				(0.01712530)	(0.0105594)	(0.0127384)
		2	2	0.2485275	0.2047751	0.2013915
				(0.839303)	(0.1386637)	(0.2286341)
		2	4	0.9838098	0.5911049	0.8053704
	50			(0.3328659)	(0.2756468)	(0.3006312)
		4	2	0.2259591	0.2447636	0.1846465
				(0.241703)	(0.1271737)	(0.1668525)
		4	4	0.8973528	0.6562926	0.7337126
				(0.648152)	(0.2731153)	(0.459307)
p = 2		2	2	0.1066691	0.0968710	0.0955038
q = 2				(0.0643241)	(0.0157888)	(0.0394547)
		2	4	0.4255818	0.3217175	0.3816446
	100			(0.0685752)	(0.0124822)	(0.0178780)
		4	2	0.1002213	0.1047287	0.0905251
				(0.0838421)	(0.050582)	(0.0642656)
		4	4	0.3867677	0.3304461	0.3484930
				(0.0277031)	(0.0125621)	(0.0181102)

Table 12. Mean square error (MSE) and Absolute bias (Abias) for λ ($r = 2, s = 2$)

hyperparametrs	n	α	λ	$MSE_{\hat{\lambda}_U}$ ($Abias_{\hat{\lambda}_U}$)	$MSE_{\hat{\lambda}_C}$ ($Abias_{\hat{\lambda}_C}$)	$MSE_{\hat{\lambda}_J}$ ($Abias_{\hat{\lambda}_J}$)
		2	2	0.2462955	0.5702160	0.2004627

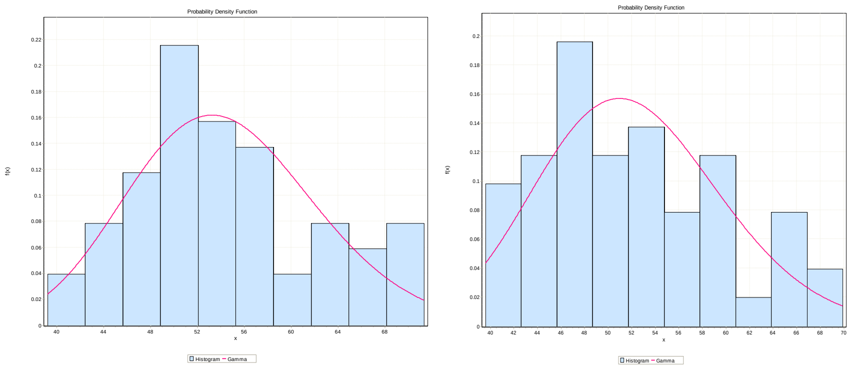
				(0.646863)	(0.1431079)	(0.4613326)
		2	4	0.9825287	2.4409968	0.7994499
	50			(0.9623237)	(0.2743401)	(0.3217504)
		4	2	0.2164602	0.3823574	0.1770025
				(0.3321399)	(0.1281948)	(0.3210854)
		4	4	0.7804277	3.6839652	0.6414612
				(0.533343)	(0.2575144)	(0.4284680)
p = 1		2	2	0.1049133	0.1672921	0.0945329
q = 1				(0.0971136)	(0.0241613)	(0.030223)
		2	4	0.4115840	0.6859298	0.3697357
	100			(0.0938692)	(0.0319882)	(0.036902)
		4	2	0.0979488	0.1314402	0.0887406
				(0.0960954)	(0.0162803)	(0.0261414)
		4	4	0.3869253	0.9198121	0.3508336
				(0.0665016)	(0.01728478)	(0.0440027)
		2	2	0.2400230	0.8133969	0.1978659
				(0.6624754)	(0.1761853)	(0.376291)
		2	4	0.9598626	3.5422137	0.7891772
	50			(0.571785)	(0.2766264)	(0.3599468)
		4	2	0.1685488	0.6652860	0.1466047
				(0.7221818)	(0.1838006)	(0.490684)
		4	4	0.6694569	6.5448094	0.6037156
p = 2				(0.6250921)	(0.2698084)	(0.3298667)
q = 1		2	2	0.1064260	0.2133888	0.0955106
				(0.0612680)	(0.02608105)	(0.0453973)
		2	4	0.4102408	0.8642889	0.3674616
	100			(0.065776)	(0.0125105)	(0.0546374)
		4	2	0.0971325	0.2135286	0.0874480
				(0.0624745)	(0.0123172)	(0.0280189)
		4	4	0.3926804	1.5245054	0.3555776
				(0.09121453)	(0.0133712)	(0.0604002)
		2	2	0.2478707	0.2882623	0.2005027
				(0.039716)	(0.01404725)	(0.0217025)
		2	4	0.9808069	0.6854348	0.7953398
	50			(0.716894)	(0.1267201)	(0.2737879)
		4	2	0.2305396	0.2121525	0.1892411
				(0.323754)	(0.1280854)	(0.1319569)
p = 1		4	4	0.8897764	1.0526962	0.7276668

$q = 2$				(0.193024)	(0.350136)	(0.065967)
		2	2	0.1047517	0.1140550	0.0944315
				(0.0658370)	(0.01035179)	(0.0126824)
		2	4	0.4180933	0.3503591	0.3751284
	100			(0.0255638)	(0.0114154)	(0.0120114)
		4	2	0.0990480	0.0953597	0.0893882
				(0.0654624)	(0.0252533)	(0.046651)
		4	4	0.4014007	0.4428859	0.3588009
				(0.040318)	(0.01233301)	(0.0201595)
		2	2	0.2472908	0.4123845	0.2030402
				(0.4705354)	(0.131214)	(0.3324798)
		2	4	0.9867743	0.9333304	0.8014834
	50			(0.4837012)	(0.2776046)	(0.2912052)
		4	2	0.2454154	0.4899147	0.1977927
				(0.4004408)	(0.129319)	(0.1453752)
$p = 2$		4	4	1.1340047	2.9754980	0.9363118
$q = 2$				(0.711314)	(0.312577)	(0.4642842)
		2	2	0.1045286	0.1401169	0.0933938
				(0.0435476)	(0.01108598)	(0.0205730)
		2	4	0.4134259	0.4008675	0.3698021
	100			(0.01426645)	(0.0104443)	(0.01414679)
		4	2	0.1015562	0.1490712	0.0910121
				(0.0615392)	(0.0175227)	(0.0213944)
		4	4	0.3900035	0.6851213	0.3505940
				(0.051658)	(0.0122162)	(0.04406925)

Even in case of both parameters unknown, the results fall on the similar lines as stated above in case I and case II. The results related to estimates of Gini and Theil's entropy measure are also obtained, but not reported due to space constraint.

4.1 Real Life Example

The median family income data of 50 states of United State are taken from the Current Population Report Series (CPS) published by the US Census Bureau for the four years 2009 to 2012. The first objective is to determine how well the Gamma density fits the distribution of family incomes in the United States. By using the easy fit software, it is seen that data fits well to the Gamma distribution for all the years and graph for the fitting of median family income for first (2009) and last year (2012) are presented in figure 1. The estimated values of parameters $(\alpha\,,\lambda)$ for 2009 and 2012 are (49.18, 0.903) and (47.786, 0.916) respectively.



Year 2009 Year 2012

Fig. 1. Fitting of median family income data

The Kolmogorov Smirnov (KS) test is used to compare the fitting of real data using three income distributions *viz*. Gamma distribution, Pareto distribution and Log-normal distribution for all the years and p-values are reported in table 13.

Table 13. p – Values for different distributions using KS test

Year	Gamma distribution	Pareto distribution	Log-normal distribution
2009	0.1873	0.0885	0.0954
2010	0.0725	0.1245	0.0754
2011	0.2084	0.0779	0.1235
2012	0.2827	0.1546	0.0978

The mean square error of Theil's entropy measure and Gini index for median family income from 2009 to 2012 using GMLE's are presented in tables.

Table 14. Mean square error of Theil's entropy measure (T) of US

Year	Mean square error		
	Uniform prior	**Jeffreys' Prior**	**Conjugate prior**
2009	1.593503e-08	1.534654e-08	2.162987e-09
2010	4.889794e-08	4.784498e-08	1.924202e-08

2011	3.715423e-08	3.624179e-08	1.185206e-08
2012	1.066405e-08	1.018563e-08	3.681847e-9

Table 15. Mean square error of Gini index (G) of US

Year	Mean square error		
	Uniform prior	**Jeffreys' Prior**	**Conjugate prior**
2009	2.321953e-07	2.236445e-07	3.163385e-08
2010	7.094156e-07	6.942152e-07	2.802217e-07
2011	5.352923e-07	5.221511e-07	1.703369e-07
2012	1.555544e-07	1.485918e-07	5.391548e-08

The findings from the analysis of real life example are in accordance with those of simulation study, suggesting the efficiency of conjugate prior in general over the uniform and Jeffreys' prior.

4.2 Robustness of Hyperparameters

The robustness of hyper-parameters following the approach of Sinha (1980), the high $\left(\frac{min}{max}\right)$ index is computed for varying values of hyperparameters and thereby estimating them.

To check the robustness of hyperparameters, simulations are done by taking different values of hyper-parameter and keeping $\alpha = 2$, $\lambda = 2$ and $n = 50$ fixed. Figure 2 shows posterior plot for different combinations of hyper-parameters.

Table 16. GMLE's of shape parameter using conjugate prior

c	*b* \ *a*	1	2	3	4	$\frac{Min}{Max}$
1	1	2.019012	2.0402800	2.0521860	2.0607355	0.979
	2	2.0057613	2.0264535	2.0380231	2.0494601	0.978
	3	2.0463377	2.0112912	2.0230296	2.0313301	0.982
	4	1.9806358	1.9990426	2.0089951	2.0169754	0.981
$\frac{Min}{Max}$		0.980	0.979	0.978	0.978	
	1	2.0352682	2.0614199	2.0724860	2.0819583	0.977

2	2	2.0257092	2.0453244	2.0571359	2.0656695	0.980
	3	2.0134226	2.0331317	2.0442410	2.0533251	0.980
	4	1.9968852	2.0186219	2.0308722	2.0370702	0.980
$\frac{Min}{Max}$		0.981	0.979	0.979	0.978	

Table 17. GMLE's of Theil's entropy measure (T) using conjugate prior

c	***a*** / ***b***	**1**	**2**	**3**	**4**	$\frac{Min}{Max}$
1	1	0.2290811	0.2303707	0.2317854	0.2330564	0.982
	2	0.2271351	0.2280932	0.2299195	0.2309383	0.983
	3	0.2258740	0.2270489	0.2287274	0.2298294	0.982
	4	0.2249702	0.2262591	0.2279185	0.2289292	0.982
$\frac{Min}{Max}$		0.982	0.982	0.982	0.982	
2	1	0.2269689	0.2282154	0.2299678	0.2311021	0.982
	2	0.2249706	0.2261573	0.2276753	0.2285463	0.984
	3	0.2237107	0.2250360	0.2263613	0.2275898	0.982
	4	0.2228899	0.2241288	0.2257275	0.2265750	0.983
$\frac{Min}{Max}$		0.982	0.982	0.981	0.980	

Table 18. GMLE's of Gini index using conjugate prior

c	***a*** / ***b***	**1**	**2**	**3**	**4**	$\frac{Min}{Max}$
1	1	0.37475601	0.37278275	0.37231984	0.37118011	0.990
	2	0.37542666	0.37387949	0.37334815	0.37237085	0.991
	3	0.37714285	0.37534832	0.37448368	0.37330105	0.989
	4	0.37822498	0.37633353	0.37485585	0.37403446	0.988
$\frac{Min}{Max}$		0.990	0.990	0.993	0.992	
2	1	0.37230381	0.37164125	0.37079356	0.36943206	0.992
	2	0.37425980	0.37230368	0.37230368	0.37138140	0.992
	3	0.37520648	0.37409504	0.37274641	0.37147153	0.990
	4	0.37482354	0.37332515	0.37256655	0.37242127	0.993

$\frac{Min}{Max}$	0.992	0.993	0.994	0.991

From the above tables, it is seen that the ratio $\left(\frac{min}{max}\right)$ in case of shape parameter, is close to 1 for different combinations of a, b and c indicating thereby the Bayes estimates are robust with respect to hyper-parameters, selected for simulation study.

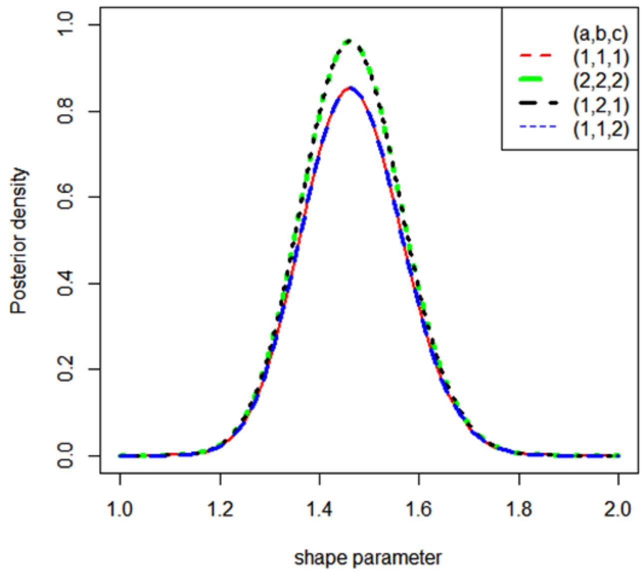


Fig. 2. Posterior density plot for different combinations of hyper-parameters

The plot shows that Bayes estimate and the posterior density are robust to all values of the hyper- parameters.

5. Conclusion

Finally, one can infer that GMLE's using conjugate prior has smaller mean square error and absolute bias as compared to Jeffreys' and uniform prior in case of Gamma distribution in general. When sample size increases mean square error and absolute bias decreases in all cases.

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