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An Alternative Method of Construction of Optima Row Column Designs for Complete Diallel Crosses

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ABSTRACT

A simple method of construction of pair of orthogonal latin squares of order p (= $m\ t\ +1$), where p be a prime integer, m be a prime integer or power of a prime integer, and t > 1 is an integer, respectively, is proposed. By using these pair of orthogonal latin squares, the four series of row-column designs for complete diallel crosses for p> 3 lines, are obtained. Our series 2 and 3 row-column designs are different from Gupta and Choi (1998) and Parsad *et.al* (2005) series 2 and 3 designs while series 1 and 4 designs are similar to their designs. Gupta and Choi (1998) and Parsad *et.al* (2005) constructed the four series of row-column designs for complete diallel crosses for p> 3 lines, by using nested balanced incomplete block designs. Our method is easy in comparison to Gupta and Choi (1998) and Parsad *et.al* (2005) methods, respectively.

1. Introduction

Two Latin squares of the same order are said to be orthogonal to each other, when they are superimposed on one another, every ordered pair of symbols occurs exactly once in the composite square, then the two Latin squares are said to be mutually orthogonal. A set of p-1 Latin squares of order p are called Mutually Orthogonal Latin Squares (MOLS), if they are pair-wise orthogonal. Orthogonal Latin squares are used for the construction of balanced incomplete block designs and square lattice designs. A set of p-1 orthogonal Latin squares of size p can always be constructed if p is a prime integer or power of a prime integer. If $p = 4t + 2, t > 1$, then there exists at least two mutually orthogonal

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Latin squares of order *p*, (see Bose, Shrikhande and Parker, 1960). An exhaustive list of these squares is available, (see Fisher and Yates, 1973)and most extensive treatment of Latin squares can also be found in D'enes and Keedwell (1991).

A diallel cross is a type of mating design used in plant breeding and animal breeding to study the genetic properties and potential of inbred lines or individuals. Let *p* denotes the number of lines and let a cross between lines *i* and *j* be denoted by $i \times j$, where $i<j = 0, 1, \ldots, p$-1 and there are $p(p-1)/2$ possible crosses. Among the four types of diallel discussed by Griffing (1956), method 4 is the most commonly used diallel in plant breeding. This type of diallel crossing includes the genotypes of one set of F_1' ^{s}hybrid that is cross of the type $(i \times j) = (j \times i)$, but neither the parents nor the reciprocals with all possible $v = p(p-1)/2$ crosses. This is sometimes referred to as the modified diallel. We shall refer to it as a complete diallel cross (CDC).

The problem of finding optimal mating designs for complete diallel cross experiments without specific combining ability has been investigated using nested balanced incomplete block designs, triangular partially balanced incomplete block designs, group divisible partially balanced incomplete block designs, circular designs and mutually orthogonal Latin squares by Gupta and Kageyama (1994), Dey and Midha (1996), Mukerjee (1997), Das, Dey and Dean (1998), Parsad, Gupta and Srivastava (1999), Parsad, Gupta and Gupta (2005), and Sharma (2004) and Sharma and Sileshi (2010) respectively. The above literature restricts to universal optimality and combinatorial aspects for complete diallel cross experiments in the one-way elimination of heterogeneity except Gupta and Choi (1998) and Parsad, Gupta and Gupta (2005) who studied this problem in the two-way elimination of heterogeneity set up. These authors constructed row- column designs using nested balanced incomplete block designs and studied their universal optimality by using Kiefer (1975) proposition.

In this paper, first we propose a simple method of construction of pair of orthogonal latin squares of order p (= $mt+1$), where p be a prime integer, m be a prime integer or a power of a prime integer, and $t \geq 1$ is an integer, respectively. Then by using these latin squares we are constructing four series of optimal row-column designs. Our method is simple in comparison to Gupta and Choi (1998) and Parsad *et.al* (2005) methods, respectively and may be considered as an alternative method for constructing row-column designs for CDC method (4).

2. Preliminary

Consider a complete diallel cross experiment involving *p* lines in a row-column design *d* with *k* rows and *b* columns and n = *b k*. Following Parsad *et.al* (2005), the model for the data can be written in the form

$$Y = \mu 1_n + \Delta_1' \gamma + \Delta_2' \beta + \Delta_3' \gamma + \varepsilon \qquad (2.1)$$

Where Y is an n × 1 vector of observed responses, μ is the general mean, g, β and γ are the column vectors of *p* general combining ability (gca) parameters, k row effects and b column effects, respectively. Δ_1', Δ_2', and Δ_3' are the corresponding design matrices of order (n×p), (n×k) and (n×b), respectively and **e** denotes the vector of independent random errors having mean **0** and covariance matrix $\sigma^2 I_n$.

Let $N_{d1} = \mathbf{\Delta}_1 \mathbf{\Delta}_2'$ be the *p* × *k* incidence matrix of lines versus rows and $N_{d2} = \mathbf{\Delta}_1 \mathbf{\Delta}_3'$ be the *p* × *b* incidence matrix of lines versus columns and $\mathbf{\Delta}_2 \mathbf{\Delta}_3' = 1_k 1_b$. Let r_{dl} denote the number of times the l^{th} cross appears in the design, $l = 1, 2, \ldots, v$ and similarly s_{di} denote the number of times the i^{th} line occurs among crosses in the whole design *d*, $i = 1, \ldots p$. Under (2.1), it can be shown that the coefficient matrix of the reduced normal equations for estimating linear functions of the *gca* parameters using a design *d*, is

$$C_d = G_d - \frac{1}{b} N_{d1} N_{d1}' - \frac{1}{k} N_{d2} N_{d2}' + \frac{s_d s_d'}{s_d' 1} \qquad (2.2)$$

where $G_d = \Delta_1 \Delta_1' = (g_{dii'})$, $s_d = (s_{d1}, s_{d2}, \ldots, s_{dp})$, $N_{d1} = (n_{dij})$, n_{dij} is the number of times the line *i* occurs in row j of *d*, $N_{d2} = (n_{di.t})$, $n_{i.t}$ is the number of times the line i occurs in the column t. We shall denote by D (*p, k, b*) the class of all complete diallel cross designs with p lines, k rows and b columns. Let n_{dijt} denote the number of times i^{th} line appears in the j^{th} row and t^{th} column of the design. Then we have the following result see Parsad *et. al* (2005)

Theorem 2.1: For any design $d \,\varepsilon\, D(p, k, b)$, trace (C_d) will be maximum when

(i) $n_{dij.} = \frac{n_{di..}}{k}$, means that the design is orthogonal with respect to the lines versus rows as blocks classification i.e. a row –regular setting with respect to lines.

(ii) $n_{di.t} = x$ or $x + 1$, means that the lines appear either x or $x+1$ times in columns as block classification, where $x = \text{int}\left(\frac{2k}{p}\right)$.

Note that if $2k<p$ then $x = 0$.

Therefore, the value of maximum trace is given by

trace (C_{d*}) = $k^{-1}\, b\, \{ 2\, k\, (k - 1 - 2\, x) + p\, x\, (x + 1) \}$

Now using Proposition 1 of Kiefer (1975) we have the following result:

Theorem 2.2: Let $d^* \varepsilon$ D (p, b, k) be a row – column design satisfying

(i) trace (C_{d*}) = $k^{-1}\, b\, \{ 2\, k\, (k - 1 - 2\, x) + p\, x\, (x + 1) \}$

(ii) C_{d*} is completely symmetric.

Then d^* is universally optimal in D (p, k, b), and in particular minimizes the average variance of the best linear unbiased estimators of all elementary contrasts among the general combining ability effects.

3. Methods of construction of row-column designs

It is known that when p is a prime or power of a prime, it is possible to construct (p-1) orthogonal Latin squares in such a way that they differ in a cyclical interchange of the rows from 2nd to p^{th}. Such squares are taken for the construction of row- column designs for complete diallel cross (CDC). So first we will discuss a simple method of construction pair of orthogonal Latin squares of order p (=$mt+1$). The method is as given below:

Method 1:-One may generate pair of orthogonal latin squares of order p (= 2$m+1$), where m be a prime or power of a prime and t $\geq$ 1, by developing the following initial two rows mod (p). First row may be taken as numbers serially from 0, 1, 2, … , p -2 , p-1 and other row will be the mirror image of first row i.e. p -1, p – 2, p – 3, . . . , 1, 0. We get other (p-1) rows by adding numbers from 1 to (p-1) in turn to the initial rows and recording results 0 in place of p. Thus we get following two Latin squares L_1 and L_2 which are orthogonal latin squares.

L_1

Initial row	0	1	2 . . .	p-2	p -1
	1	2	3 . . .	p -1	0
	2	3	4 . . .	0	1
	. . . .				
. . . .					
	p -1	0	1	p-3	p -2

L_2

Mirror row	p -1	p -2	p -3 . .	1	0
	0	p-1	p -2 . . .	2	1
	1	0	p -1 . . .	3	2
	. . . .				
	. . . .				
	p-2	p-3	p-4	0	p -1

Now superimposing one over other (L_1overL_2**)**, we get following composite Latin square in which each cell contains order pair of integers (i,j) taking value from 0 to p-1. These ordered pair of integers occurs once in a composite square.

Composite Square

Middle Column

$(0,p-1)$	$(1,p-2)$	…	…	$\{(p-1)/2, (p-1)/2\}$	…	…	$(p-2,1)$	$(p-1,0)$
$(1,0)$	$(2,p-1)$	…	…	$\{(p+1)/2,(p+1)2\}$	…	…	$(p-1,2)$	$(0,1)$
$(2,1)$	$(3,0)$	…	…	$\{(p+3)/2,p+3)/2\}$	…	…	$(0, 3)$	$(1,2)$
.								
.								
.								
$(p-1,2p-2)$	$(0,2p-3)$	…	…	$\{3(p-1)/2,3(p-1)/2\}$	…	…	$(2p-3, 0)$	$(2p-2,p-1)$

After seeing the above composite square we find that the middle column contains the pair of elements of the type (i, i), where i = 0,1, . . ., p-1 and other p (p-1) ordered pair are of the type (i, j) , where i, j = 0, 1, . . ., p-1, equally distributed to right and left side of the middle column. When columns represent rows and rows represent columns each p (p-1)/2 ordered pairi.e $v = p$ (p-1)/2 crosses represents a row-column design d Ɛ D (p, k, b) for CDC with the following parameters.

$v = p\,(p\text{-}1)/2 = m\,(2m{+}1)$, $k = (p\text{-}1)/2 = m$, $b = p = 2m{+}1$, and $r = 1$

The method of construction of mutually orthogonal Latin squares and row column designs is illustrated below.

Note**:** The series 1 of Gupta and Choi (1998) can be constructed by using above method. Series 2 and 3 of Gupta and Choi (1998) follow as a particular case of this method when t = 4 and t =6.

Example 3.1:- Let us consider construction of the row-column design for p = 5, when m = 2 and t =2. First we take two initial rows as given below and develop them cyclically mod (p) for the construction of MOLS of order 5.

	L_1						L_2				
Initial row	**0**	**1**	**2**	**3**	**4**	**Mirror row**	**4**	**3**	**2**	**1**	**0**
	1	2	3	4	0		0	4	3	2	1
	2	3	4	0	1		1	0	4	3	2
	3	4	0	1	2		2	1	0	4	3
	4	0	1	2	3		3	2	1	0	4

Now superimposing L_2 over L_1 or L_1 over L_2 we get a following composite square.

Composite Square

		Middle Column		
(0, 4)	(1, 3)	(2, 2)	(3, 1)	(4, 0)
(1, 0)	(2, 4)	(3, 3)	(4, 2)	(0, 1)
(2, 1)	(3, 0)	(4, 4)	(0, 3)	(1, 2)
(3, 2)	(4, 1)	(0, 0)	(1, 4)	(2, 3)
(4, 3)	(0, 2)	(1, 1)	(2, 0)	(3, 4)

10 ordered pairs to the left and10 to the right of middle column and each of the 10 ordered pairs replicated only once in both sides. When k =2 columns represent rows and b =5 rows represent columns then each 10 ordered pair represents a row-column design *d* Ɛ D (*p*, *k*, *b*) for CDC for p = 5 lines, with the following parameters.

$$v = 10,\ k = 2,\ b = 5,\ \text{and } r = 1$$

Method 2: For the construction of series 2 one may generate pair of latin squares of order *p* (= *mt+1*), when t =4 and m be prime or power of a prime, by using the above procedure. Now consider left or right (p-1)/2 columns from the middle column. Now divide left columns or right columns into (p-1)/4 distinct columns. Now these (p-1)/4 distinct columns can be put together in juxtaposition into (p-1)/4 rows and 2p blocks.. Then we get a row –column design *d* with parameter *p* = 4*m+1*, *k* =*m*, *b* =2(4*m+1*) and *r* =1 which is different from Gupta and Choi (1998) series 2 design.

Example 3.2: Let us consider the construction of the row-column design for *p* = 9, where *m* = 4. By constructing pair of orthogonal latin squares of order 9 we divide first four columns into groups of two distinct columns. Now juxtaposing these distinct columns into two rows and 18 blocks yields following row-column design with parameters *p* = 9, *k* =2, *b* = 18 and *r* =1.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}
R_1	(0,8)	(1, 0)	(2, 1)	(3, 2)	(4, 3)	(5, 4)	(6, 5)	(7, 6)	(8, 7)	(2, 6)	(3, 7)	(4, 8),	(5, 0)
R_2	(1, 7)	(2, 8)	(3, 0)	(4, 1)	(5, 2)	(6, 3)	(7, 4)	(8, 5)	(0, 6)	(3, 5)	(4, 6),	(5,7)	(6, 8)

	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}
R_1	(6, 1)	(7, 2)	(8, 3)	(0, 4)	(1, 5)
R_2	(7, 0)	(8, 1)	(0, 2)	(1, 3)	(2, 4)

Method 3:For the construction of series 3 one may generate pair of orthogonal latin squares of order *p* (= *mt+1*), when t =6 and m be prime or power of a prime. By using the procedure given in method 1 and consider left or right (p-1)/2 columns from the middle column we get a row –column design *d* with parameter *p* = 6*m+1*, *k* =*m*, *b* = (6*m+1*)and *r* =1 which is different from Gupta and Choi (1998) series 3 design.

Example 3.3: Let us consider the construction of the row-column design for p = 7, where m = 3. By constructing pair of orthogonal latin squaresof order 7 and using the above procedure and considering left or right columns (p-1)/2 columns yields following row-column design with parameters p = 7, k =3, b = 7 and r =1.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7
R_1	(0,6))	(1,0)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)
R_2	(1,5))	(2,6)	(3,0)	(4,1)	(5,2)	(6,3)	(0,4)
R_3	(2,4)	(3,5)	(4,6)	(5,0)	(6,1)	(0,4)	(1,3)

Method 4:- For the construction of series 4 of Gupta and Choi (1998) one may generate pair of orthogonal latin squares of order p (= $mt+1$), when t =2 and m be prime or power of a prime, by using the procedure given in method 1. Now we take left (p-1)/2 columns from the middle column and arrange them in a one row and mp blocks. Now take right (p-1)/2 columns from the middle column and put them in another row just below the elements of first row. One should take care that the element of the type (i, j) = (j, i) should not occur in the same block. Then we get the row- column design d with parameters with p =2$m+1$, k = 2, b =m (2$m+1$) and r =2.

Example 3.4: Let us consider the construction of the row-column design for p = 7, where m = 3. Constructing a pair of orthogonal latin squares of order 7 by the procedure given in Method 1 and applying the procedure given in method 3, we get following row-column design with parameters p = 7, k =2, b = 21 and r =2.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}
R_1	(0, 6)	(1, 0)	(2, 1)	(3, 2)	(4, 3)	(5, 4)	(6, 5)	(1, 5)	(2, 6)	(3, 0)	(4, 1)	(5, 2)	(6, 3)
R_2	(4, 2)	(5, 3)	(6, 4)	(0, 5)	(1, 6)	(2, 0)	(3, 1)	(6, 0)	(0, 1)	(1, 2)	(2, 3)	(3, 4)	(4, 5)

	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}
R_1	(0, 4)	(2, 4)	(3, 5)	(4, 6)	(5, 0)	(6, 1)	(0, 2)	(1, 3)
R_2	(5, 6)	(5, 1)	(6, 2),	(0, 3)	(1, 4)	(2, 5)	(3, 6)	(4, 0)

4. Optimality

It is not hard to see in d ε D (p, b, k) that the pairwise comparisons among gca parameters are estimated with

$$\text{variance} = 2k\,\sigma^2 / p\,(k-1) \qquad (4.3)$$

$$\text{trace}\,(\mathbf{C}_d) = p\,(p-1)\,(k-1) / k \qquad (4.4)$$

Following theorem 2.2 the trace of information matrix (C_d) is as given below. Since in d ε D (p, b, k), we see that $2k$ is less than p, then $x = 0$. Hence trace (C_d) = 2 b (k-1) =p (p-1) (k-1)/k , where b and k are number of columns and number of rows in the corresponding row-column designs obtained by methods 1, 2, 3 and 4.

Hence we state the following theorem.

Theorem 4.3: Let d ε D (p, b, k), be a row – column design for complete diallel cross constructed by using the mutually orthogonal Latin squares of order$p$ =mt+1, where $m$be a prime integer or a power of a prime integer and $t \geq 1$, respectively. and satisfying

(i) trace (C_d) = p (p-1) (k-1)/k

(ii) C_d is completely symmetric.

Then d is universally optimal in the relevant class of competing design in D (p, b, k) and particularly A- optimal.

5. Conclusion

In this paper we have given a simple method of construction of pair of orthogonal latin squares of order p where p = (mt +1), where m be a prime integer or a power of a prime integer and $t \geq 1$, respectively. By using these pair of orthogonal latin squares we have derived four series of row-column designs. Our two series i.e. series 2 and 3 are different from Gupta and Choi (1998) and Parsad *et.al* (2005) series 2 and 3 designs.. Our method seems simple in comparison to Gupta and Choi (1998) and Parsad *et.al* (2005) methods.

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