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Recurrence Relations For The Concomitants Of Order Statistics From Extended Farlie-Gumbel-Morgenstern Distributions

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ABSTRACT

In this paper, we derive some recurrence relations for the density functions of concomitants of order statistics of an arbitrary size arising from extended Farlie-Gumbel-Morgenstern bivariate distributions. These relations are used to obtain some recurrence relations for the single moments as well as for the moment generating functions of the concomitants of order statistics from the extended family of distributions.

1. INTRODUCTION

Johnson and Kotz (1977) introduced a family of bivariate distributions, which is a generalization of the Farlie-Gumbel-Morgenstern (FGM) family of bivariate distributions. This family is known as extended Farlie-Gumbel-Morgenstern (EFGM) distributions, which is characterized by specified marginal cumulative distribution functions (cdf's) $F_X(x)$ and $F_Y(y)$ and association parameters α_1 and α_2. The bivariate cdf of the random variable (X, Y) associated with EFGM distributions is given by

$$H(x,y) = F_X(x)\ F_Y(y)\ \{1 + \alpha_1\ \bar{F}_X(x)\ \bar{F}_Y(y) + \alpha_2\, F_X(x)\, F_Y(y)\ \bar{F}_X(x)\ \bar{F}_Y(y)\},$$

$$|\ \alpha_1\,| \leq 1;\ -\alpha_1 - 1 \leq \alpha_2 \leq \frac{1}{2}[3 - \alpha_1 + (9 - 6\alpha_1 - 3\alpha_1^{2})^{1/2}], \quad (1.1)$$

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where $\overline{F}_U(u) = 1 - F_U(u)$. When $\alpha_2 = 0$, EFGM distributions reduce to the FGM distributions. If $\alpha_1 = 0 = \alpha_2$, then the random variables X and Y are independent. The bivariate probability density function (pdf) corresponding to (1.1) is given by

$$h(x,y) = f_X(x)\, f_Y(y)\, \{1 + \alpha_1[2F_X(x)-1][2F_Y(y)-1] + \alpha_2[3F_X(x)-2][3F_Y(y)-2]F_X(x)F_Y(y)\}, \quad (1.2)$$

where $f_X(.)$ and $f_Y(.)$ are the pdf's corresponding to the marginal cdf's $F_X(.)$ and $F_Y(.)$ respectively. Also, the conditional pdf of Y given $X = x$, denoted by $h_{Y/X}(y/x)$, is obtained from (1.2) as

$$h_{Y/X}(y/x) = f_Y(y)\{1 + \alpha_1[2F_X(x) - 1][2F_Y(y)-1] + \alpha_2[3F_X(x)-2][3F_Y(y)-2]F_X(x)F_Y(y)\}. \quad (1.3)$$

For arbitrary absolutely continuous distributions given by $F_X(.)$ and $F_Y(.)$, Huang and Kotz (1984) have shown that the maximal correlation correlation between the component random variables of EFGM distributions is 0.5072, which is higher than $1/3$, the maximal correlation coefficient corresponding to a distribution belonging to the FGM family. In addition to the above property of accommodating larger range of correlation coefficient between the marginal random variables X and Y and possessing an association parameter space of dimension 2 by EFGM distributions, there is more modelling flexibility for this family in the sense of picking this model to a large variety of data sets occurring in real life situations when compared with the classical FGM family of distributions.

There is a growing interest shown among users of bivariate models which are capable of accommodating larger amount of correlation and possessing higher degree of flexibility in terms of the choice of parameters so as to portray a closer model to the unknown population distribution of their interest than the traditionally available models. Moreover, the theory and computation techniques of evaluation of the moments of concomitants of order statistics either directly or by means of recurrence relations have to be developed in order to estimate some parameters of interest in the model. The well-known ranked set sampling, in which ranking is made by the observations of the auxiliary variable X, also requires the means and variances of the concomitants of order statistics for the estimation purpose. The above mentioned requirements motivate the authors to study more on the properties of EFGM bivariate distributions.

The concept of concomitants of order statistics was first introduced by David (1973). Let (X_i, Y_i), i = 1.2,..., n be a random sample drawn from an arbitrary bivariate distribution with cdf F(x, y) and pdf f(x, y). If the X observations in the sample are ordered as $X_{1:n}, X_{2:n},\ldots, X_{n:n}$, then the accompanying Y observation in an ordered pair with X observation equal to $X_{r:n}$ is called concomitant of the r^{th} order statistic, $X_{r:n}$ and is denoted by $Y_{[r:n]}$. For a review of basic results on concomitant of order statistics, one may refer to David (1981) and David and Nagaraja (1998).

The concomitants of order statistics are found to have wide applications in many fields of study such as selection procedures and ocean engineering. There are some recurrence relations for the pdf's and moments of concomitants of order statistics arising from the FGM family of bivariate distributions (see, Balasubramanian and Beg (1997); Scaria and Nair (1999); Bairamov et.al. (2001); Beg and Ahsanullah (2008)).

Philip (2011) has developed the distribution theory of concomitants of order statistics from EFGM bivariate distributions. The estimation by ranked set sampling of some parameters of interest in the EFGM bivariate logistic distribution has been studied by Philip and Thomas (2015). However, no results on the recurrence relations for the concomitants of order statistics arising from EFGM distributions seem to have been derived in the available literature. Hence, in this paper, we derive some recurrence relations on the pdf's of concomitants of order statistics arising from EFGM distributions and derive some recurrence relations for the moments of concomitants as well. In section 2, simplified expressions for the pdf and the single moments of the concomitants in terms of the order statistics arising from EFGM distributions are obtained. Some recurrence relations for the pdf's of the concomitants are obtained in section 3. In this section, we have proved a general theorem on expectation of some continuous function of concomitants of order statistics and thereby deduced some recurrence relations for the moments and moment generating functions (mgf's) of the concomitants arising from the EFGM distributions.

2. SIMPLIFIED EXPRESSIONS FOR THE DENSITY FUNCTIONS AND MOMENTS OF THE CONCOMITANTS

Let (X_i, Y_i), $i = 1, 2,\ldots,n$ be a random sample drawn from EFGM distributions with cdf (1.1) and $Y_{[r:n]}$, the concomitant of the r^{th} order statistic $X_{r:n}$, $r = 1, 2,\ldots,n$. Then pdf of $Y_{[r:n]}$, for $1 \le r \le n$, is given by

$$h_{[r:n]}(y) = \int_{-\infty}^{\infty} h_{Y/X}(y/x) f_{r:n}(x)dx \, , \tag{2.1}$$

where $f_{r:n}(x) = r\binom{n}{r}\left[F_X(x)\right]^{r-1}\left[1-F_X(x)\right]^{n-r} f_X(x)$ is the pdf of $X_{r:n}$.

By using the expression for the conditional pdf of EFGM distributions given by (1.3) in (2.1), we get

$$h_{[r:n]}(y) = f_Y(y) \left\{ \int_{-\infty}^{\infty} f_{r:n}(x)dx + \alpha_1\left[2F_Y(y)-1\right]\int_{-\infty}^{\infty}\left[2F_X(x)-1\right] f_{r:n}(x)dx \right.$$

$$\left. +\alpha_2\left[3F_Y(y)-2\right]F_Y(y)\int_{-\infty}^{\infty}\left[3F_X(x)-2\right]F_X(x) f_{r:n}(x)dx \right\}.$$

On evaluating the integrals on the right side of the above equation and simplifying, we obtain the pdf of $Y_{[r:n]}$ as

$$h_{[r:n]}(y) = f_Y(y)$$

$$\left\{1-\frac{\alpha_1(n-2r+1)}{(n+1)}\left[2F_Y(y)-1\right]-\frac{\alpha_2(2n-3r+1)r}{(n+1)(n+2)}\left[3F_Y(y)-2\right]F_Y(y)\right\}. \tag{2.2}$$

In particular, $h_{[1:1]}(y) = f_Y(y)$.

If we write $Y_{r:n}$ to denote the rth order statistic of a random sample of size n arising from the marginal distribution of Y with pdf denoted by $g_{r:n}(y)$, then $2F_Y(y)\, f_Y(y)$ and $3[F_Y(y)]^2 f_Y(y)$ are the expressions for the pdfs $g_{2:2}(y)$ and $g_{3:3}(y)$ of the statistics $Y_{2:2}$ and $Y_{3:3}$ respectively in particular, $g_{1:1}(y) = f_Y(y)$. Hence we can express the pdf of the rth concomitant $Y_{[r:n]}$, for $n \geq 2$ and $r = 1, 2, \ldots, n$, given by (2.2), in terms of the pdfs of the order statistics as

$$h_{[r:n]}(y) = g_{1:1}(y) - \frac{\alpha_1(n-2r+1)}{(n+1)}[g_{2:2}(y) - g_{1:1}(y)]$$

$$-\frac{\alpha_2(2n-3r+1)r}{(n+1)(n+2)}\left[g_{3:3}(y)-g_{2:2}(y)\right]. \tag{2.3}$$

Thus, $g_{1:1}(y)$ and pdf's of two largest order statistics in samples of sizes 2 and 3 only are required for finding the pdf of any concomitant of order statistics in a sample of arbitrary size from EFGM distributions.

Note 2.1. Since $\sum_{r=1}^{n}(n-2r+1)=0=\sum_{r=1}^{n}(2n-3r+1)r$, we obtain the following identity from (2.3):

$$\sum_{r=1}^{n} h_{[r:n]}(y)=n\,\mathrm{g}_{1:1}(y) \tag{2.4}$$

Note 2.2. When n is odd, second term on the right side of (2.3) vanishes for r = $\frac{n+1}{2}$. In this case, the pdf $h_{[r:n]}$ given by (2.3) reduces to

$$h_{\left[\frac{n+1}{2}:n\right]}(y)=\mathrm{g}_{1:1}(y-\frac{\alpha_2(n-1)}{4(n+2)}\left[g_{3:3}(y)-g_{2:2}(y)\right]. \tag{2.5}$$

Note 2.3. When n is expressible in the form n = 3m+l for m = 1, 2,…, then the third term on the right side of (2.3) vanishes for r = $\frac{2n+1}{3}$. In this case, the pdf $h_{[r:n]}$ given by (2.3) becomes

$$h_{\left[\frac{2n+1}{3}:n\right]}(y)=\mathrm{g}_{1:1}(y)+\frac{\alpha_1(n-1)}{3(n+1)}\left[g_{2:2}(y)-\mathrm{g}_{1:1}(y)\right]. \tag{2.6}$$

Remark 2.1. If we put $\alpha_2 = 0$ in (2.3), it reduces to the expression for the pdf of the rth concomitant from FGM distributions as

$$h_{[r:n]}(y)=\mathrm{g}_{1:1}(y)-\frac{\alpha_1(n-2r+1)}{(n+1)}\left[g_{2:2}(y)-\mathrm{g}_{1:1}(y)\right], \tag{2.7}$$

which is a better simplified form of the expression (2.9) given in Scaria and Nair (1999). The expression (2.7) depends on the pdfs of $Y_{1:1}$ and $Y_{2:2}$ only, whereas the expression (2.9) given in Scaria and Nair (1999) depends on the distribution of $Y_{1:2}$ as well, in addition to the marginal distributions of $Y_{1:1}$ and $Y_{2:2}$. Thus, our result (2.7) is a better simplified expression for the pdf of the concomitant of the rth order statistic of a random sample of size n arising from FGM distributions.

Let us write for k = 1,2,… and for n ≥ 2, 1≤ r ≤ n, $\mu_{[r:n]}^{(k)}=E(Y_{[r:n]}^{(k)}),\mu_{r:n}^{(k)}=E\left(Y_{r:n}^{(k)}\right)$.

In particular, $\mu_{[1:1]}^{(k)} = E\left(Y_{[1:1]}^{k}\right)$ and $\mu_{1:1}^{(k)} = E\left(Y_{1:1}^{k}\right)$. Then from (2.3), we get the following expression for the k[th] single moments of the r[th] concomitant from EFGM distributions:

$$\mu_{[r:n]}^{(k)} = \mu_{1:1}^{(k)} - \frac{\alpha_1(n-2r+1)}{(n+1)}\left[\mu_{2:2}^{(k)} - \mu_{1:1}^{(k)}\right] - \frac{\alpha_2(2n-3r+1)r}{(n+1)(n+2)}\left[\mu_{3:3}^{(k)} - \mu_{2:2}^{(k)}\right], \tag{2.8}$$

provided the moments exist. It follows from (2.8) that $\mu_{1:1}^{(k)}$, $\mu_{2:2}^{(k)}$ and $\mu_{3:3}^{(k)}$ alone are required for determining $\mu_{[r:n]}^{(k)}$, r = 1,2,…, n for any n.

Note 2.4. From identity given by (2.4), it follows that

$$\sum_{r=1}^{n} \mu_{[r:n]}^{(k)} = n \ \mu_{1:1}^{(k)} . \tag{2.9}$$

The identity given by (2.9) on the single moments of the concomitants of order statistics is helpful in checking the accuracy in the computed values of the moments of the concomitants.

Note 2.5. The expressions for k[th] single moments corresponding to the pdf's given in (2.5) and (2.6) are given respectively by

$$\mu_{\left[\frac{n+1}{2}:n\right]}^{(k)} = \mu_{1:1}^{(k)} - \frac{\alpha_2(n-1)}{4(n+2)}\left[\mu_{3:3}^{(k)} - \mu_{2:2}^{(k)}\right]$$

and

$$\mu_{\left[\frac{2n+1}{3}:n\right]}^{(k)} = \mu_{1:1}^{(k)} + \frac{\alpha_1(n-1)}{3(n+1)}\left[\mu_{2:2}^{(k)} - \mu_{1:1}^{(k)}\right].$$

In general, if $\psi(y)$ is some continuous function of y, then we have from (2.3)

$$E[\psi(Y_{[r:n]})] = E[\psi(Y_{1:1})] - \frac{\alpha_1(n-2r+1)}{(n+1)}\left\{E\left[\psi(Y_{2:2})\right] - E\left[\psi(Y_{1:1})\right]\right\} - \frac{\alpha_2(2n-3r+1)r}{(n+1)(n+2)}\left\{E\left[\psi(Y_{3:3})\right] - E\left[\psi(Y_{2:2})\right]\right\}, \tag{2.10}$$

provided the expectations exist.

3. RECURRENCE RELATIONS FOR THE DENSITY FUNCTIONS AND SINGLE MOMENTS OF CONCOMITANTS

Let (X_i, Y_i), i = 1,2,…, n be a random sample of size n arising from EFGM distributions and $Y_{[r:n]}$, $r = 1,2,.., n$ be the concomitants of the order statistics. Write $h_{[r:n]}(y)$ for the pdf of r^th^ concomitant $Y_{[r:n]}$ and $g_{r:n}(y)$ for the pdf of the r^th^ order statistic $Y_{r:n}$ in a random sample arising from the marginal distribution of Y. Then we have the following relations between pdf's of the concomitants.

Relation 3.1. For $n \geq 2$ and $1 \leq r \leq n-1$, $h_{[r:n]}(y)=h_{[r:n-1]}(y)$

$$-\alpha_1 \frac{2r}{n(n+1)}\left[g_{2:2}(y)-g_{1:1}(y)\right]+\alpha_2 \frac{2(n-3r-1)r}{n(n+1)(n+2)}\left[g_{3:3}(y)-g_{2:2}(y)\right]\cdot \tag{3.1}$$

This relation follows by applying the expression (2.3) for the density function of concomitants to the difference $h_{[r:n]}(y)$- $h_{[r:n-1]}(y)$ between pdf's of the concomitants $Y_{[r:n]}$ and $Y_{[r:n-1]}$ and on simplification.

Relation 3.2. For $n \geq 2$ and $2 \leq r \leq n$, $h_{[r:n]}(y)=h_{[r-1:n]}(y)$

$$+\alpha_1 \frac{2}{(n+1)}\left[g_{2:2}(y)-g_{1:1}(y)\right]+\alpha_2 \frac{2(3r-n-2)}{(n+1)(n+2)}\left[g_{3:3}(y)-g_{2:2}(y)\right]. \tag{3.2}$$

The above relation follows if we apply (2.3) to the difference $h_{[r:n]}(y)$- $h_{[r-1:n]}(y)$.

Remark 3.1. The result (3.1) expresses the density function of the r^th^ concomitant in a sample of size n in terms of the r^th^ concomitant in a sample of size n-l, marginal density functions of $Y_{1:1}$,$Y_{2:2}$ and $Y_{3:3}$. Similarly, result (3.2) expresses the density function of the r^th^ concomitant in a sample of size n in terms of the (r - l)^th^ concomitant in a sample of size n and marginal density functions of $Y_{1:1}$, $Y_{2:2}$ and $Y_{3:3}$.

Remark 3.2. If we put $\alpha_2 = 0$ in (3.1) and (3.2), we get the recurrence relations (45) and (46) respectively of Beg and Ahsanullah (2008) for the concomitants of order statistics from FGM distributions.

A relation connecting two pdf's of concomitants of symmetrized order statistics arising from a sample of size n is given below.

Relation 3.3. For $n \geq 2$ and $1 \leq r \leq n$,

$$h_{[n-r+1:n]}(y) + h_{[r:n]}(y) = 2g_{1:1}(y) +$$

$$\frac{\alpha_2[n^2+6r^2-6nr-6r+3n+2]}{(n+1)(n+2)}\left[g_{3:3}(y)-g_{2:2}(y)\right]. \tag{3.3}$$

The above relation follows on adding the pdf obtained by putting $r = n\text{-}r\text{+}1$ in the pdf (2.3) to the pdf (2.3) itself and simplifying the resulting expression.

Now, we prove the following theorem on recurrence relations connecting $E[\psi(Y_{[r:n]})]$ and $E[\psi(Y_{[r-1:n]}]$, where $\psi(.)$ is some continuous function and deduce some recurrence relations for moments and mgfs of the concomitants from EFGM distributions as corollary results.

Theorem 3.1. Let (X_i, Y_i), i = 1. 2,...,n be a random sample of size n, where $n \geq 2$ arising from EFGM distributions with marginal pdf $f_Y(y)$ of Y and let $Y_{[r:n]}$, $r = 1,2,\ldots ,n$ be the concomitants of the order statistics $X_{r:n}$. If $\psi(y)$ is some continuous function of y, then we have the following.

For $1\leq r \leq n\text{-}1$, $$E[\psi(Y_{[r:n]})] = E[\psi(Y_{[r:n-1]})] - \alpha_1 \frac{2r}{n(n+1)}\left\{E\left[[\psi(Y_{2:2})\right] - E[\psi(Y_{1:1})]\right\} + \alpha_2 \frac{2(n-3r-1)r}{n(n+1)(n+2)}\left\{E[\psi(Y_{3:3})] - E[\psi(Y_{2:2})]\right\} \tag{3.4}$$

and for 2≤ r ≤ n,

$$E[\psi(Y_{[r:n]})] = E[\psi(Y_{[r-1:n]})] + \alpha_1 \frac{2}{(n+1)}\left\{E[\psi(Y_{2:2})] - E[\psi(Y_{1:1})]\right\} + \alpha_2 \frac{2(3r-n-2)}{(n+1)(n+2)}\left\{E[\psi(Y_{3:3})] - E[\psi(Y_{2:2})]\right\}, \tag{3.5}$$

provided the expectations exist.

The proof of theorem 3.1 follows easily from (2.10).

Remark 3.3. If we put $\alpha_2 = 0$ in (3.5) we obtain relation (56) given by Beg and Ahsanullah (2008).

Corollary 3.1. For n ≥ 2 and 1 ≤ r ≤ (n - 1),

$$\mu_{[r:n]}^{(k)} = \mu_{[r:n-1]}^{(k)} - \alpha_1 \frac{2r}{n(n+1)}\left[\mu_{2:2}^{(k)} - \mu_{1:1}^{(k)}\right] + \alpha_2 \frac{2(n-3r-1)r}{n(n+1)(n+2)}\left[\mu_{3:3}^{(k)} - \mu_{2:2}^{(k)}\right] \tag{3.6}$$

and for n ≥2 and 2≤ r ≤ n,

$$\mu_{[r:n]}^{(k)} = \mu_{[r-1:n]}^{(k)} + \alpha_1 \frac{2}{(n+1)} \left[\mu_{2:2}^{(k)} - \mu_{1:1}^{(k)} \right] + \alpha_2 \frac{2(3r-n-2)}{(n+1)(n+2)} \left[\mu_{3:3}^{(k)} - \mu_{2:2}^{(k)} \right], \tag{3.7}$$

provided the moments exist.

These results follow from (3.4) and (3.5) if we put $\psi(y) = y^k$.

A relation connecting two k^{th} single moments of concomitants of symmetrized order statistics arising from a sample of size n from EFGM distributions is as follows.

Relation 3.4. For $n \geq 2$ and $1 \leq r \leq n$,

$$\mu_{[n-r+1:n]}^{(k)} + \mu_{[r:n]}^{(k)} = 2\mu_{1:1}^{(k)} + \frac{\alpha_2 \left[n^2 + 6r^2 - 6nr - 6r + 3n + 2 \right]}{(n+1)(n+2)} \left\{ \mu_{3:3}^{(k)} - \mu_{2:2}^{(k)} \right\}. \tag{3.8}$$

The relation given by (3.8) follows easily from the relation given by (3.3).

If we write $M_{[r:n]}(t)$ to denote the mgf of $Y_{[r:n]}$ and for r = 1,2,3, $M_{r:r}(t)$ to denote the mgf of $Y_{r:r}$, we have the following corollary.

Corollary 3.2. For $n \geq 2$ and $1 \leq r \leq (n - 1)$,

$$M_{[r:n]}(t) = M_{[r:n-1]}(t)$$
$$-\alpha_1 \frac{2r}{n(n+1)} [M_{2:2}(t) - M_{1:1}(t)] + \alpha_2 \frac{2(n-3r-1)r}{n(n+1)(n+2)} [M_{3:3}(t) - M_{2:2}(t)]$$

and for $n \geq 2$ and $2 \leq r \leq n$, $M_{[r:n]}(t) = M_{[r-1:n]}(t)$

$$+\alpha_1 \frac{2r}{(n+1)} [M_{2:2}(t) - M_{1:1}(t)] + \alpha_2 \frac{2(3r-n-2)}{(n+1)(n+2)} [M_{3:3}(t) - M_{2:2}(t)],$$

provided the mgf's exist.

The results (3.9) and (3.10) follow from (3.4) and (3.5) if we put $\psi(y) = e^{ty}$.

As an illustration, for an EFGM bivariate exponential distribution with marginal distributions each as exponential with mean 1, means and variances of the concomitants of all order statistics in a sample of size *n* are tabulated for *n* = 2(1) 8 and for each of the pairs (α_1,α_2) : (- 0.5, 0.5), (0.25, 0.50), (1.0, 0.5). These are presented in Table 1.

Table 1: Means and Variances of Concomitants of order statistics from EFGM bivariate exponential distribution for α_2= 0.5 when α_1= -0.50, 0.25, 1.00 and for *n*= 2(1)8.

n	r	α_1= - 0.50		α_1= 0.25		α_1= 1.00	
		Mean	Variance	Mean	Variance	Mean	Variance
2	1	1.0555560	1.0304780	0.9305556	0.9089506	0.8055556	0.7561728
	2	0.9444444	0.9564043	1.0694440	1.0848770	1.1944440	1.1820990
3	1	1.0916670	1.0547920	0.9041667	0.8758854	0.7166667	0.6266667
	2	0.9833333	0.9719444	0.9833333	0.9719444	0.9833333	0.9719444
	3	0.9250000	0.9433333	1.1125000	1.1378650	1.3000000	1.2620830
4	1	1.1166670	1.0733330	0.8916667	0.8614583	0.6666667	0.5483333
	2	1.0166670	0.9916667	0.9416667	0.9172917	0.8666667	0.8316667
	3	0.9500000	0.9475000	1.0250000	1.0243750	1.1000000	1.0900000
	4	0.9166667	0.9391667	1.1416670	1.1735420	1.3666670	1.3066670
5	1	1.1349210	1.0876160	0.8849206	0.8544816	0.6349206	0.4963467
	2	1.0436510	1.0103300	0.9186508	0.8880582	0.7936508	0.7345364
	3	0.9761905	0.9597506	0.9761905	0.9597506	0.9761905	0.9597506
	4	0.9325397	0.9365867	1.0575400	1.0658030	1.1825400	1.1637690
	5	0.9126984	0.9381456	1.1626980	1.1990580	1.4126980	1.3349710
6	1	1.1488100	1.0988520	0.8809524	0.8509247	0.6130952	0.4595026
	2	1.0654760	1.0267150	0.9047619	0.8712621	0.7440476	0.6641511
	3	1.0000000	0.9742772	0.9464286	0.9207058	0.8928571	0.8613946
	4	0.9523810	0.9428146	1.0059520	0.9976616	1.0595240	1.0467690
	5	0.9226190	0.9316893	1.0833330	1.0986220	1.2440480	1.2138960
	6	0.9107143	0.9383503	1.1785710	1.2181650	1.4464290	1.3544860
7	1	1.1597220	1.1078800	0.8784722	0.8490910	0.5972222	0.4320988
	2	1.0833330	1.0407990	0.8958333	0.8611111	0.7083333	0.6111111
	3	1.0208330	0.9887153	0.9270833	0.8959418	0.8333333	0.7855903
	4	0.9722222	0.9529321	0.9722222	0.9529321	0.9722222	0.9529321
	5	0.9375000	0.9335938	1.0312500	1.0302730	1.1250000	1.1093750
	6	0.9166667	0.9296875	1.1041670	1.1250000	1.2916670	1.2500000
	7	0.9097222	0.9390432	1.1909720	1.2329890	1.4722220	1.3687310

8	1	1.1685190	1.1152740	0.8768519	0.8481833	0.5851852	0.4109534
	2	1.0981480	1.0528670	0.8898148	0.8547574	0.6814815	0.5698422
	3	1.0388890	1.0021910	0.9138889	0.8796219	0.7888889	0.7258025
	4	0.9907407	0.9644204	0.9490741	0.9225609	0.9074074	0.8772291
	5	0.9537037	0.9399863	0.9953704	0.9826175	1.0370370	1.0217760
	6	0.9277778	0.9285802	1.0527780	1.0580940	1.1777780	1.1563580
	7	0.9129630	0.9291529	1.1212960	1.1465530	1.3296300	1.2771470
	8	0.9092593	0.9399143	1.2009260	1.2448140	1.4925930	1.3795750

4. PRODUCT MOMENTS BETWEEN CONCOMITANTS

The joint pdf of $Y_{[r:n]}$ and $Y_{[s:n]}$, $1 \le r < s \le n$, the concomitants of rth and sth order statistics of a random sample of size n arising from EFGM distributions is obtained by (see, Philip, 2011),

$$h_{[r,s:n]}(y_1, y_2) = g_{1:1}(y_1)\, g_{1:1}(y_2) - \alpha_1 \frac{(n-2r+1)}{(n+1)}[g_{2:2}(y_1) - g_{1:1}(y_1)]g_{1:1}(y_2)$$

$$-\alpha_1 \frac{(n-2s+1)}{(n+1)}[g_{2:2}(y_2) - g_{1:1}(y_2)]g_{1:1}(y_1) - \alpha_2 \frac{(2n-3r+1)r}{(n+1)(n+2)}[g_{3:3}(y_1) - g_{2:2}(y_1)]g_{1:1}(y_2)$$

$$-\alpha_2 \frac{(2n-3s+1)s}{(n+1)(n+2)}[g_{3:3}(y_2) - g_{2:2}(y_2)]g_{1:1}(y_1)$$

$$+\alpha_1^2 \left\{ \frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)} \right\} [g_{2:2}(y_1) - g_{1:1}(y_1)][g_{2:2}(y_2) - g_{1:1}(y_2)]$$

$$+\alpha_1\alpha_2 \left\{ \frac{-2r}{n+1} + \frac{[3r(r+1)+4r(n-s+1)]}{(n+1)(n+2)} - \frac{6r(r+1)(n-s+1)}{(n+1)(n+2)(n+3)} \right\}$$

$$\times [g_{3:3}(y_1) - g_{2:2}(y_1)][g_{2:2}(y_2) - g_{1:1}(y_2)]$$

$$+\alpha_1\alpha_2 \left\{ \frac{-s}{n+1} + \frac{[2r(s+1)+3s(n-s+1)]}{(n+1)(n+2)} - \frac{6r(s+1)(n-s+1)}{(n+1)(n+2)(n+3)} \right\}$$

$$\times [g_{2:2}(y_1) - g_{1:1}(y_1)][g_{3:3}(y_2) - g_{2:2}(y_2)]$$

$$+\alpha_2^{\,2}\left\{-\frac{2r(s+1)}{(n+1)(n+2)}+\frac{[3r(r+1)(s+2)+6r(s+1)(n-s+1)]}{(n+1)(n+2)(n+3)}-\frac{9r(r+1)(s+2)(n-s+1)}{(n+1)(n+2)(n+3)(n+4)}\right\}$$
$$\times\ [g_{3:3}(y_1)-g_{2:2}(y_1)][g_{3:3}(y_2)-g_{2:2}(y_2)]\,. \tag{4.1}$$

Using the joint pdf (4.1), the product moments $E(Y_{[r:n]}^{l_1}\times Y_{[s:n]}^{l_2})$, for l_1, $l_2 > 0$,denoted by $\mu_{[r,s:n]}^{(l_1,l_2)}$ are given by

$$\mu_{[r,s:n]}^{(l_1,l_2)}=\mu_{1:1}^{(l_1)}\mu_{1:1}^{(l_2)}$$
$$-\frac{\alpha_1(n-2r+1)}{(n+1)}\left[\mu_{2:2}^{(l_1)}-\mu_{1:1}^{(l_1)}\right]\mu_{1:1}^{(l_2)}-\frac{\alpha_1(n-2s+1)}{(n+1)}\left[\mu_{2:2}^{(l_2)}-\mu_{1:1}^{(l_2)}\right]\mu_{1:1}^{(l_1)}$$
$$-\frac{\alpha_2(2n-3r+1)r}{(n+1)(n+2)}\left[\mu_{3:3}^{(l_1)}-\mu_{2:2}^{(l_1)}\right]\mu_{1:1}^{(l_2)}-\frac{\alpha_2(2n-3s+1)s}{(n+1)(n+2)}\left[\mu_{3:3}^{(l_2)}-\mu_{2:2}^{(l_2)}\right]\mu_{1:1}^{(l_1)}$$
$$+\alpha_1^2\left\{\frac{n-2s+1}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\right\}\left(\mu_{2:2}^{(l_1)}-\mu_{1:1}^{(l_1)}\right)\left(\mu_{2:2}^{(l_2)}-\mu_{1:1}^{(l_2)}\right)$$
$$+\alpha_1\,\alpha_2\left\{-\frac{2r}{n+1}+\frac{\left[3r(r+1)+4r(n-s+1)\right]}{(n+1)(n+2)}-\frac{6r(r+1)(n-s+1)}{(n+1)(n+2)(n+3)}\right\}$$
$$\times\left(\mu_{2:2}^{(l_2)}-\mu_{1:1}^{(l_2)}\right)\left(\mu_{3:3}^{(l_1)}-\mu_{2:2}^{(l_1)}\right)$$
$$+\alpha_1\,\alpha_2\left\{-\frac{s}{n+1}+\frac{\left[2r(s+1)+3s(n-s+1)\right]}{(n+1)(n+2)}-\frac{6r(s+1)(n-s+1)}{(n+1)(n+2)(n+3)}\right\}$$
$$\times\left(\mu_{2:2}^{(l_1)}-\mu_{1:1}^{(l_1)}\right)\left(\mu_{3:3}^{(l_2)}-\mu_{2:2}^{(l_2)}\right)$$
$$+\alpha_2^2\left\{-\frac{2r(s+1)}{(n+1)(n+2)}+\frac{\left[3r(r+1)(s+2)+6r(s+1)(n-s+1)\right]}{(n+1)(n+2)(n+3)}\right.$$
$$\left.-\frac{9r(r+1)(s+2)+(n-s+1)}{(n+1)(n+2)(n+3)(n+4)}\right\}\left(\mu_{3:3}^{(l_1)}-\mu_{2:2}^{(l_1)}\right)\left(\mu_{3:3}^{(l_2)}-\mu_{2:2}^{(l_2)}\right). \tag{4.2}$$

The joint mgf of $Y_{[r:n]}$ and $Y_{[s:n]}$, $E\left(e^{t_1Y_{[r:n]}+t_2Y_{[s:n]}}\right)$, denoted by $M_{[r,s:n]}(t_1, t_2)$, is given by

$$M_{[r,s:n]}(t_1, t_2) = M_{1:1}(t_1)\, M_{1:1}(t_2) \; -\frac{\alpha_1(n-2r+1)}{(n+1)}\left[M_{2:2}(t_1)-M_{1:1}(t_1)\right]M_{1:1}(t_2)$$

$$-\frac{\alpha_1(n-2s+1)}{(n+1)}\left[M_{2:2}(t_2)-M_{1:1}(t_2)\right] M_{1:1}(t_1)$$

$$-\frac{\alpha_2(2n-3r+1)r}{(n+1)(n+2)}\left[M_{3:3}(t_1)-M_{2:2}(t_1)\right]M_{1:1}(t_2)$$

$$-\frac{\alpha_2(2n-3s+1)s}{(n+1)(n+2)}\left[M_{3:3}(t_2)-M_{2:2}(t_2)\right]M_{1:1}(t_1)$$

$$+\alpha_1^2\left\{\frac{n-2s+1}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\right\}\left[M_{2:2}(t_1)-M_{1:1}(t_1)\right]\left[M_{2:2}(t_2)-M_{1:1}(t_2)\right]$$

$$+\alpha_1\,\alpha_2\left\{-\frac{2r}{n+1}+\frac{[3r(r+1)+4r(n-s+1)]}{(n+1)(n+2)}-\frac{6r(r+1)(n-s+1)}{(n+1)(n+2)(n+3)}\right\}$$

$$\times\left[M_{3:3}(t_1)-M_{2:2}(t_1)\right]\left[M_{2:2}(t_2)-M_{1:1}(t_2)\right]$$

$$+\alpha_1\,\alpha_2\left\{-\frac{s}{n+1}+\frac{[2r(s+1)+3s(n-s+1)]}{(n+1)(n+2)}-\frac{6r(s+1)(n-s+1)}{(n+1)(n+2)(n+3)}\right\}$$

$$\times\left[M_{2:2}(t_1)-M_{1:1}(t_1)\right]\left[M_{3:3}(t_2)-M_{2:2}(t_2)\right]$$

$$+\alpha_2^2\left\{-\frac{2r(s+1)}{(n+1)(n+2)}+\frac{[3r(r+1)(s+2)+6r(s+1)(n-s+1)]}{(n+1)(n+2)(n+3)}\right.$$

$$\left.-\frac{9r(r+1)(s+2)(n-s+1)}{(n+1)(n+2)(n+3)(n+4)}\right\}$$

$$\times\left[M_{3:3}(t_1)-M_{2:2}(t_1)\right]\left[M_{3:3}(t_2)-M_{2:2}(t_2)\right]. \quad (4.3)$$

Remark 4.1. If we put $\alpha_2 = 0$ in (4.2) and (4.3), we obtain the results for order statistics corresponding two (89) and (92) given in Beg and Ahsanullah (2008) for FGM distributions.

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