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On A Mixture Of Standard Normal And Two-Piece Skew Normal Distributions

C. Satheesh Kumar[1] and M.R. Anusree[2]

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ABSTRACT

In this paper, we consider a new class of two-piece skew normal distribution as a mixture of the standard normal distribution and the two-piece skew normal distribution of Kim (Statistics, 2005). We discuss some of the important aspects of this new class of distributions by obtaining explicit expressions for its distribution function, characteristic function, reliability measures, conditions for plurimodality etc. Further, the estimation of the parameters of this class of distribution is attempted and illustrated with the help of certain real life data sets.

1. INTRODUCTION

There has been a renewed interest in the development of asymmetric version of normal distribution during the last three decades. For details see Azzalini (1985), Kumar and Anusree (2011, 2014) or Genton (2004). Kim (2005) introduced and studied a two-piece version of the skew normal distribution through the following probability density function (p.d.f.), for $\lambda \in R = (-\infty, \infty)$, $x \in R$,

$$g_k(x;\lambda) = C_\lambda f(x) F(\lambda | x |), \qquad (1.1)$$

with

$$C_\lambda = 2\pi\left[\pi + 2\tan^{-1}(\lambda)\right]^{-1} \qquad (1.2)$$

where f(.) and F(.) are respectively the p.d.f. and cumulative distribution function (c.d.f.) of a standard normal variate. The distribution of the random variable X

[1]*Address for Correspondence:* C. Satheesh Kumar, Department of Statistics, University of Kerala, Trivandrum 695 581, India E-mail: drcsatheeshkumar@gmail.com and M.R. Anusree[2],Department of Operations, Rajagiri Business School, Kochi, India.

with p.d.f.(1.1) hereafter we denoted as $TSND(\lambda)$. The main drawback of the $TSND(\lambda)$ in the practical point of view is that there exists a symmetric behavior as determined by the sign of λ on either side of the origin. To overcome this difficulty, through this paper we develop a more wide class of distribution as a convex mixture of the standard normal distribution and the $TSND(\lambda)$, and we termed this class of distributions as"the two-piece mixture skew normal distribution (TMSND)".

The paper is organized as follows. In section 2, we present the definition of TMSND and discuss some of its important properties. In section 3, we obtain certain conditions for the existence of pluromodality, which highlights the importance of the $TSND(\lambda)$ in practical point of view. In section 4, we obtain expression for certain reliability measures and a location-scale extension of the TMSND is considered in section 5. Further, the parameters of the extended TMSND are estimated by the method of maximum likelihood in section 6 and a numerical illustration is given in section 7, with the help of certain real life data sets.

We need the following shorter notations in the sequel. For any reals a, b and k such that $bx + k > 0$

$$\xi_k(a;b) = \int_a^{\infty}\int_0^{bx+k} \frac{e^{-\frac{x^2+y^2}{2}}}{2\pi} dydx. \tag{1.3}$$

2. DEFINITION AND PROPERTIES

Here, first we define a wide class of two-piece skew normal distribution as a convex mixture of the standard normal distribution and the two-piece skew normal distribution of Kim (2005) and derive some of its important properties.

Definition 2.1 *A random variable Z is said to follow a two-piece mixture skew normal distribution with parameters* $\lambda \in R = (-\infty, \infty)$, $\alpha \in [0,1]$ *if its p.d.f. h(z;* λ, α *) is of the following form, in which* $C_{\lambda,\alpha} = (1-\alpha)C_\lambda$ *, with* C_λ *defined in (1.2). For z* $\in R$ $h(z;\lambda,\alpha) = f(z)[\alpha + C_{\lambda,\alpha}F(\lambda|z|)]$ (2.1)

Note that the function given in (2.1) is a proper p.d.f. since,

$$= \alpha + \frac{C_{\lambda,\alpha}}{C_\lambda}\left[\int_{-\infty}^{0} C_\lambda f(z)F(-\lambda z)dz + \int_0^{\infty} C_\lambda f(z)F(\lambda z)dz\right]$$

=1,

in the light of (1.1) and $C_{\lambda,\alpha} = (1-\alpha)C_{\lambda}$.

The distribution of a random variable Z with p.d.f.(2.1) we denoted as $TMSND(\lambda,\alpha)$. Also it can be noted that (2.1) can be viewed as the convex mixture of the *p.d.f.'s of the* standard normal distribution and $TMSND(\lambda,\alpha)$. Clearly, $TMSND(\lambda,1)$ or $TMSND(0,\alpha)$ is the standard normal distribution and $TMSND(\lambda,0)$ is the skew normal distribution of Kim (2005). Further, if $\lambda \to -\infty$, $TMSND(\lambda,2)$ reduces to the standard half normal distribution.

For some particular choices of λ and α, the p.d.f. given in (2.1) of TMSND is plotted as shown in Figure 1.

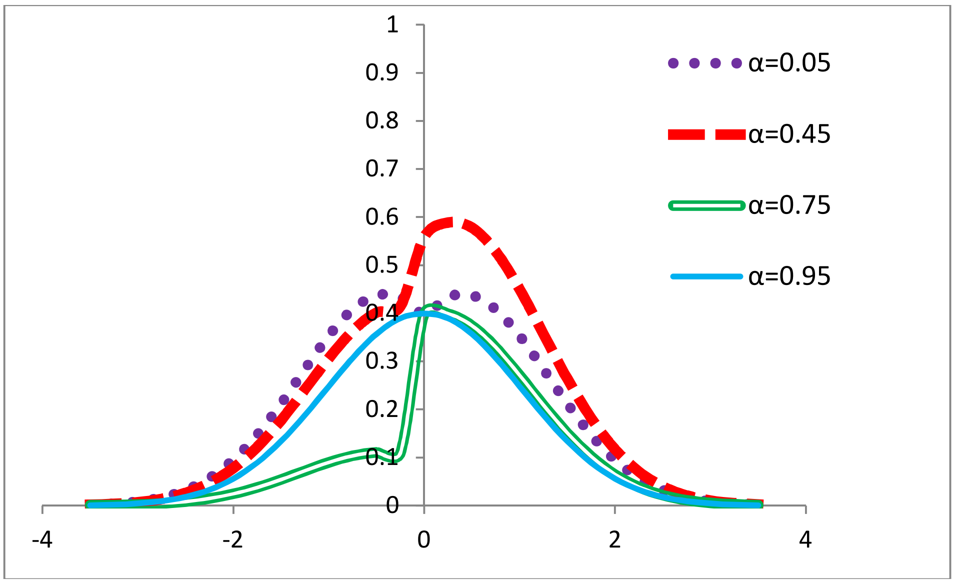

Figure1: Probability plots of $TMSND(0.65,\alpha)$ for some particular choices of α ($\alpha = 0.05, 0.45, 0.75, 0.95$).

Result 2.1 *If* Z *follows the* $TMSND(\lambda,\alpha)$ *with p.d.f.* $h(z;\lambda,\alpha)$, *then*

a.) $Y_1 = -Z$ *follows* $TMSND(\lambda,\alpha)$ *and*

b) $Y_2 = Z^2$ *has the following p.d.f.*

$$h_2(y_2;\lambda,\alpha) = \left(\frac{f(\sqrt{y_2})}{2\sqrt{y_2}}\right)\left[2\alpha + C_{\lambda,\alpha}\right] \tag{2.2}$$

Remark 2.1 *Note that when* $\alpha = 1$ *or* $\lambda = 0$, *(2.2) reduces to the p.d.f. of a Chi-square variate with one degree of freedom.*

Result 2.2 *If Z is a* $TMSND(\lambda,\alpha)$ *variate, then for any reals* d_1, d_2 *such that* $d_1 \le d_2$,

$$P(d_1 \le Z \le d_2) = \begin{cases} \alpha[F(d_2) - F(d_1)] + \dfrac{C_{\lambda,\alpha}}{2}\left[G(d_2,-\lambda) - G(d_1,-\lambda)\right], d_1 \le d_2 < 0 \\ \alpha[F(d_2) - F(d_1)] + \dfrac{C_{\lambda,\alpha}}{2}\left[G(d_2,\lambda) - G(d_1,\lambda)\right], 0 \le d_1 \le d_2, \end{cases} \quad (2.3)$$

where $G(.,\lambda)$ is the c.d.f. of the $SND(\lambda)$.

Proof: For any $d_1 \le d_2 < 0$, by definition,

$$P(d_1 \le Z \le d_2) = \int_{d_1}^{d_2} h(z;\lambda,\alpha)dz$$

$$= \int_{d_1}^{d_2} [\alpha f(z) + \frac{C_{\lambda,\alpha}}{2} 2 f(z)F(-\lambda z)]dz$$

$$= \alpha[F(d_2) - F(d_1)] + \frac{C_{\lambda,\alpha}}{2}[G(d_2,-\lambda) - G(d_1,-\lambda)]. \quad (2.4)$$

Now, for the case $0 \le d_1 \le d_2$,

$$P(d_1 \le Z \le d_2) = \int_{d_1}^{d_2} h(z;\lambda,\alpha)dz$$

$$= \int_{d_1}^{d_2} [\alpha f(z) + \frac{C_{\lambda,\alpha}}{2} 2 f(z)F(\lambda z)]dz$$

$$= \alpha[F(d_2) - F(d_1)] + \frac{C_{\lambda,\alpha}}{2}[G(d_2,\lambda) - G(d_1,\lambda)]. \quad (2.5)$$

Thus, (2.4) and (2.5) implies (2.3).

Result 2.3 *The c.d.f.* $H(z;\lambda,\alpha)$ *of a random variable Z with p.d.f. (2.1) is the following.*

$$H(z;\lambda,\alpha) = \begin{cases} \alpha F(z) + \dfrac{C_{\lambda,\alpha}}{2}\left[F(z) - 2\xi_0(z,-\lambda)\right], z < 0 \\ \alpha F(z) + \dfrac{C_{\lambda,\alpha}}{2}\left[F(z) + \dfrac{2}{\pi} tan^{-1}(\lambda) - 2\xi_0(z,\lambda)\right], z \ge 0, \end{cases} \quad (2.6)$$

where $\xi_0(a,b)$ is as defined in (1.3).

Proof: Let Z be a random variable with p.d.f. (2.1). Then the c.d.f. $H(z;\lambda,\alpha)$ is

$$H(z;\lambda,\alpha)=\begin{cases}L_1, z<0\\ \\ L_2, z\geq 0,\end{cases} \tag{2.7}$$

in which

$$\begin{aligned}L_1 &= \int_{-\infty}^{z}\alpha f(t)dt+\frac{C_{\lambda,\alpha}}{2}\int_{-\infty}^{z} f(t)F(-\lambda t)dt\\ &=\alpha F(z)+\frac{C_{\lambda,\alpha}}{2}G(z,-\lambda)\\ &=\alpha F(z)+\frac{C_{\lambda,\alpha}}{2}\left(F(z)-2\xi_0(z,-\lambda)\right)\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}L_2 = \alpha\int_{-\infty}^{z} f(t)dt+\frac{C_{\lambda,\alpha}}{2}\int_{-\infty}^{z}2f(t)F(\lambda t)dt\\ &=\int_{-\infty}^{0}\alpha f(t)dt+C_{\lambda,\alpha}\int_{-\infty}^{0}f(t)F(-\lambda t)dt+\\ &\int_{0}^{z}\alpha f(t)dt+C_{\lambda,\alpha}\int_{0}^{z}f(t)F(\lambda t)dt\\ &=\alpha F(z)+\frac{C_{\lambda,\alpha}}{2}G(0,-\lambda)+\frac{C_{\lambda,\alpha}}{2}\left[G(z,\lambda)-G(0,\lambda)\right]\\ &=\alpha F(z)+\frac{C_{\lambda,\alpha}}{2}\left(\frac{1}{2}+\frac{\tan^{-1}(\lambda)}{\pi}\right)+\\ &\frac{C_{\lambda,\alpha}}{2}\left(F(z)-\frac{1}{2}+\frac{\tan^{-1}(\lambda)}{\pi}-2\xi_0(z,\lambda)\right).\end{aligned} \tag{2.9}$$

Now on substituting (2.8) and (2.9) in (2.7) we get (2.6).

In order to obtain the characteristic function of $TMSND(\lambda,\alpha)$ we need the following lemma.

Lemma 2.2 *If U is a standard normal variable, then for any real* λ *and* k_0,

$$E[F(\lambda\,|U|+k_0)]=F\left\{\frac{k_0}{\sqrt{1+\lambda^2}}\right\}+2\xi\left(\frac{k_0}{\sqrt{1+\lambda^2}},\lambda\right),$$

where for any $a\in R$ and $b>0$,

$$\xi(a;b) = \int_a^{\infty}\int_0^{bx} f(x)f(y)dydx. \quad (2.10)$$

Result 2.4 *The characteristic function,* $\phi_Z(t)$ *of a random variable* Z *following* $TMSND(\lambda,\alpha)$ *with p.d.f (2.1) is the following, for any* $t \in R$ *and* $i = \sqrt{-1}$.

$$\phi_Z(t) = e^{-\frac{t^2}{2}}[\alpha + C_{\lambda,\alpha}F(-i\delta t)] - C_{\lambda,\alpha}e^{-\frac{t^2}{2}}\left[\xi_s(-it,-\lambda) - \xi_{-s}(-it,\lambda)\right],$$

in which $s = -\lambda it$, $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$ and $\xi_k(a,b)$ is as defined in (1.3).

Proof: Let Z follows $TMSND(\lambda,\alpha)$ with p.d.f. (2.1). By the definition of characteristic function, for any $t \in R$ and $i = \sqrt{-1}$, we have

$$\phi_Z(t) = E\left(e^{itZ}\right)$$

$$= \int_{-\infty}^{\infty} e^{itz} h(z;\lambda,\alpha)dz$$

$$= \int_{-\infty}^{\infty} e^{itz} f(z)[\alpha + C_{\lambda,\alpha}F(-\lambda z)]dz - \int_0^{\infty} e^{itz} f(z)[\alpha + C_{\lambda,\alpha}F(-\lambda z)]dz$$

$$+ \int_0^{\infty} e^{itz} f(z)[\alpha + C_{\lambda,\alpha}F(\lambda z)]dz$$

$$= e^{-\frac{t^2}{2}}\left\{\alpha + C_{\lambda,\alpha}\int_{-\infty}^{\infty}\frac{e^{-\frac{(z-it)^2}{2}}F(-\lambda z)dz}{\sqrt{2\pi}}dz - C_{\lambda,\alpha}\int_0^{\infty}\frac{e^{-\frac{(z-it)^2}{2}}F(-\lambda z)dz}{\sqrt{2\pi}}\right.$$

$$\left. + C_{\lambda,\alpha}\int_0^{\infty}\frac{e^{-\frac{(z-it)^2}{2}}F(\lambda z)dz}{\sqrt{2\pi}}\right\}.$$

On substituting $z - it = x$, $\phi_Z(t)$ reduces to

$$\phi_Z(t) = e^{-\frac{t^2}{2}}[\alpha + C_{\lambda,\alpha}\int_{-\infty}^{\infty}\frac{e^{-\frac{x^2}{2}}F(-\lambda(x+it))dx}{\sqrt{2\pi}}]$$

$$-C_{\lambda,\alpha}\int_{-it}^{\infty}\frac{e^{-\frac{x^2}{2}}F(-\lambda(x+it))dx}{\sqrt{2\pi}}+C_{\lambda,\alpha}\int_{-it}^{\infty}\frac{e^{-\frac{x^2}{2}}F(\lambda(x+it))dx}{\sqrt{2\pi}}$$

$$=e^{-\frac{t^2}{2}}\left\{\alpha+C_{\lambda,\alpha}F(-i\delta t)-C_{\lambda,\alpha}\int_{-it}^{\infty}f(x)F(-\lambda(x+it))dx\right.$$

$$\left.+C_{\lambda,\alpha}\int_{-it}^{\infty}f(x)F(\lambda(x+it))dx\right\},$$

in the light of Lemma 2.2

$$\phi_Z(t)=e^{-\frac{t^2}{2}}[\alpha+C_{\lambda,\alpha}F(-i\delta t)]-C_{\lambda,\alpha}e^{-\frac{t^2}{2}}\int_{-it}^{\infty}f(x)\left[\int_{-\infty}^{0}f(u)du+\int_{0}^{-\lambda x+s}f(u)du\right]dx+$$

$$C_{\lambda,\alpha}e^{-\frac{t^2}{2}}\int_{-it}^{\infty}f(x)\left[\int_{-\infty}^{0}f(u)du+\int_{0}^{\lambda x-s}f(u)du\right]dx,$$

which implies (2.10)

3. MODE

In this section we discuss some important aspects of the mode of the $TMSND(\lambda,\alpha)$, through the following results.

Result 3.1 *The p.d.f. of* $TMSND(\lambda,\alpha)$ *is bimodal with unimodes in the region of* $z\in(-\infty,0]$ *if*

(i) $\lambda z<0$ or (ii) $\lambda z>0$ provided $|k_2(z;\lambda,\alpha)|>k_1(z;\lambda,\alpha)$ and in the region of $z\in[0,\infty)$ if

(i) $\lambda z>0$ or (ii) $\lambda z<0$ provided $|k_4(z;\lambda,\alpha)|>k_3(z;\lambda,\alpha)]|$,

where

$$k_1(z;\lambda,\alpha)=\frac{-\lambda^2 zf(-\lambda z)}{[\alpha+C_{\lambda,\alpha}F(-\lambda z)]},\tag{3.1}$$

$$k_2(z;\lambda,\alpha)=\frac{\lambda C_{\lambda,\alpha}f^2(-\lambda z)}{[\alpha+C_{\lambda,\alpha}F(-\lambda z)]^2},\tag{3.2}$$

$$k_3(z;\lambda,\alpha)=\frac{\lambda^2 zf(\lambda z)}{[\alpha+C_{\lambda,\alpha}F(\lambda z)]}\tag{3.3}$$

and

$$k_4(z;\lambda,\alpha)=\frac{\lambda C_{\lambda,\alpha}f^2(\lambda z)}{[\alpha+C_{\lambda,\alpha}F(\lambda z)]^2}. \qquad (3.4)$$

Proof: In order to show that there exists unimodes in regions of $z\in(-\infty,0]$ and $z\in[0,\infty)$, it is enough to show that the second derivative of $h(z;\lambda,\alpha)$ is negative for all α and λ in the respective regions.

For $z\in(-\infty,0]$, we have

$$\frac{d^2}{dz^2}\{\ln[h(z;\lambda,\alpha)]\}=-1-\lambda C_{\lambda,\alpha}[k_1(z;\lambda,\alpha)+k_2(z;\lambda,\alpha)] \qquad (3.5)$$

and for $z\in[0,\infty)$, we have

$$\frac{d^2}{dz^2}\{\ln[h(z;\lambda,\alpha)]\}=-1-\lambda C_{\lambda,\alpha}[k_3(z;\lambda,\alpha)+k_4(z;\lambda,\alpha)], \qquad (3.6)$$

where $k_j(z;\lambda,\alpha)$ for j=1, 2, 3, and 4 are as given in (3.1) to (3.4). Note that $f(\lambda z)$ and $F(\lambda z)$ are positive for all $z\in R$ and hence $[\alpha+C_{\lambda,\alpha}F(-\lambda Z)]$ is positive for all $\alpha>0$. For $z<0$ and $\lambda>0$, (3.5) is negative and hence the density is unimodal. For $\lambda<0$, (3.5) is negative if $|k_2(z;\lambda,\alpha)|>k_1(z;\lambda,\alpha)$. Similarly for $z>0$ and $\lambda>0$, (3.6) is negative and hence the density is unimodal and for $\lambda<0$, (3.6) is negative if $|k_4(z;\lambda,\alpha)|>k_3(z;\lambda,\alpha)$. Thus the proof of the result follows.

As a consequence of Result 3.1 we obtain the following result.

Result 3.2 The p.d.f. of $TMSND(\lambda,\alpha)$ is plurimodal in the region of $z\in(-\infty,0]$, for $\lambda z>0$ if $|k_2(z;\lambda,\alpha)|<k_1(z;\lambda,\alpha)$ and $|\lambda C_{\lambda,\alpha}[k_1(z;\lambda,\alpha)+k_2(z;\lambda,\alpha)]|>1$, and in the region of $z\in[0,\infty)$, for $\lambda z<0$ if $|k_4(z;\lambda,\alpha)|<k_3(z;\lambda,\alpha)$ and $|\lambda C_{\lambda,\alpha}[k_3(z;\lambda,\alpha)+k_4(z;\lambda,\alpha)]|>1$.

4. RELIABILITY ASPECTS

Here we derive some properties of the $TMSND(\lambda,\alpha)$ with p.d.f.(2.1) useful in reliability studies.

Let Z follows $TMSND(\lambda,\alpha)$ with p.d.f.(2.1). Now from the definition of reliability function $R(t;\lambda,\alpha)$ and failure rate $r(t;\lambda,\alpha)$ of Z we obtain the following results.

Result 4.1 *The reliability function* $R(t;\lambda,\alpha)$ *of the* $TMSND(\lambda,\alpha)$ *is*

$$R(t;\lambda,\alpha)=1-\begin{cases}\alpha F(t)+\dfrac{C_{\lambda,\alpha}}{2}\left[F(t)-2\xi_0(t,-\lambda)\right], t<0\\[2ex] \alpha F(t)+\dfrac{C_{\lambda,\alpha}}{2}\left[F(t)+2\dfrac{\tan^{-1}(\lambda)}{\pi}-2\xi_0(t,\lambda)\right], t\geq 0,\end{cases} \tag{4.1}$$

where $\xi_0(t,\lambda)$ is as defined in (1.3).

Proof follows from the definition of reliability function $R(t;\lambda,\alpha)=1-H(t;\lambda,\alpha)$, where $H(t;\lambda,\alpha)$ is as given in Result 2.3.

Result 4.2 *The failure rate* $r(t;\lambda,\alpha)$ *of* Z *following the* $TMSND(\lambda,\alpha)$ *is*

$$r(t;\lambda,\alpha)=\begin{cases}\dfrac{f(t)[\alpha+C_{\lambda,\alpha}F(-\lambda t)]}{1-\alpha F(t)-\dfrac{C_{\lambda,\alpha}}{2}[F(t)-2\xi_0(t,-\lambda)]}, t<0\\[3ex] \dfrac{f(t)[\alpha+C_{\lambda,\alpha}F(\lambda t)]}{1-\alpha F(t)-\dfrac{C_{\lambda,\alpha}}{2}\left[F(t)+\dfrac{2}{\pi}\tan^{-1}(\lambda)-2\xi_0(t,\lambda)\right]}, t\geq 0\end{cases} \tag{4.2}$$

Proof follows from the definition of failure rate, $r(t;\lambda,\alpha)=\dfrac{h(t;\lambda,\alpha)}{R(t;\lambda,\alpha)}$, where $R(t;\lambda,\alpha)$ is as defined in Result 4.1.

Further we derive the following result regarding the mean residual life function of $TMSND(\lambda,\alpha)$.

Result 4.3 *The mean residual life function* $\mu(t;\lambda,\alpha)$ *of* Z *following the* $TMSND(\lambda,\alpha)$ *is*

$$\mu(t;\lambda,\alpha)=\frac{1}{R(t;\lambda,\alpha)}\begin{cases}\alpha f(t)+C_{\lambda,\alpha}f(t)F(-\lambda t)+\frac{\delta}{\sqrt{2\pi}}C_{\lambda,\alpha}+\\ \frac{\delta}{\sqrt{2\pi}}C_{\lambda,\alpha}F\left(t\sqrt{1+\lambda^2}\right)-t, t\le 0\\ \\ \alpha f(t)+C_{\lambda,\alpha}f(t)F(\lambda t)+\frac{\delta}{2\sqrt{2\pi}}C_{\lambda,\alpha}\\ -\frac{\delta}{\sqrt{2\pi}}C_{\lambda,\alpha}F\left(t\sqrt{1+\lambda^2}\right)-t, t>0,\end{cases} \tag{4.3}$$

where $\delta=\frac{\lambda}{\sqrt{1+\lambda^2}}$

Proof: By definition, the mean residual life function of Z following the $TMSND(\lambda,\alpha,\rho)$ is given by

$$\mu(t;\lambda,\alpha)=E(Z\mid Z>t)-t$$

$$=\frac{1}{R(t;\lambda,\alpha)}\begin{cases}\int_t^0 zf(z)\left[\alpha+C_{\lambda,\alpha}F(-\lambda z)\right]dz+\alpha\int_0^\infty zf(z)dz+\\ C_{\lambda,\alpha}\int_0^\infty zf(z)F(\lambda z)dz-t, t\le 0\\ \int_t^\infty zf(z)\left[\alpha+C_{\lambda,\alpha}F(\lambda z)\right]dz-t, t>0.\end{cases} \tag{4.4}$$

Now for any $t<0$,

$$\int_t^0 zf(z)[\alpha+C_{\lambda,\alpha}F(-\lambda z)]dz=-\int_t^0 f'(z)[\alpha+C_{\lambda,\alpha}F(-\lambda z)]dz$$

$$=\alpha f(t)-\frac{\alpha}{\sqrt{2\pi}}-C_{\lambda,\alpha}\left[\frac{1}{2\sqrt{(2\pi)}}-F(-\lambda t)f(t)\right]$$

$$+C_{\lambda,\alpha}\lambda\int_t^0 f(\lambda z)f(z)dz$$

$$=\alpha f(t)-\frac{\alpha}{\sqrt{2\pi}}-\frac{\delta(\lambda,\alpha)}{2\sqrt{2\pi}}+C_{\lambda,\alpha}F(-\lambda t)f(t)$$

$$+\frac{\lambda C_{\lambda,\alpha}\left[\frac{1}{2}-F(t\sqrt{1+\lambda^2})\right]}{\sqrt{2\pi}\sqrt{1+\lambda^2}} \tag{4.5}$$

and for any $t>0$,

$$\int_t^\infty zf(z)[\alpha+C_{\lambda,\alpha}F(\lambda z)]dz=-\int_t^\infty f'(z)[\alpha+C_{\lambda,\alpha}F(\lambda z)]dz$$

$$=\alpha f(t)+C_{\lambda,\alpha}\left[F(\lambda t)f(t)+\frac{\lambda}{\sqrt{2\pi}\sqrt{1+\lambda^2}}\right]$$

$$-C_{\lambda,\alpha}\frac{\lambda}{\sqrt{2\pi}\sqrt{1+\lambda^2}}F(t\sqrt{1+\lambda^2}). \tag{4.6}$$

In particular, when t=0 in (4.6) we have

$$\int_0^\infty zf(z)[\alpha+C_{\lambda,\alpha}F(\lambda z)]dz=\frac{\alpha}{\sqrt{(2\pi)}}+\frac{C_{\lambda,\alpha}}{2\sqrt{2\pi}}+\frac{\lambda C_{\lambda,\alpha}}{2\sqrt{2\pi}\sqrt{1+\lambda^2}} \tag{4.7}$$

On substituting (4.5), (4.6) and (4.7) in (4.4), we get (4.3).

5. LOCATION-SCALE EXTENSION

In this section we consider the location-scale extension of the $TMSND(\lambda,\alpha)$ and discuss some of its important properties.

Definition 5.1 Let Z follows $TMSND(\lambda,\alpha)$ with p.d.f. (2.1). Then $X=\mu+\sigma Z$ is said to have an extended two-piece mixture skew normal distribution with location parameter μ, scale parameter σ and shape parameter λ and α, denoted as $ETMSND(\mu,\sigma;\lambda,\alpha)$, if its p.d.f. is of the following form, in which $\mu,\lambda\in R$, $\sigma>0$ and $\alpha\in[0,1]$.

$$h(x;\mu,\sigma,\lambda,\alpha)=\frac{1}{\sigma}f\left(\frac{x-\mu}{\sigma}\right)\left[\alpha+C_{\lambda,\alpha}F\left(\lambda\left|\frac{x-\mu}{\sigma}\right|\right)\right] \tag{5.1}$$

Clearly, $ETMSND(\mu,\sigma;\lambda,1)$ or $ETMSND(\mu,\sigma;0,\alpha)$ is the normal distribution $N(\mu,\sigma)$ and $ETMSND(\mu,\sigma;\lambda,0)$ is the extended skew normal distribution [$ETSND(\mu,\sigma,\lambda)$] of Kim (2005). Further, if $\lambda\to-\infty$ and $\alpha=2$ $ETMSND(\mu,\sigma;\lambda,\alpha)$ reduces to the half normal distribution $HN(\mu,\sigma)$.

Corresponding to the results obtained for $TMSND(\lambda,\alpha)$, here we have the following for $ETMSND(\mu,\sigma;\lambda,\alpha)$.

Result 5.1 *The characteristic function* $\phi_X(t)$ *of a random variable* X *following* $ETMSND(\mu,\sigma;\lambda,\alpha)$ *is the following, in which* $s=-\lambda it$ *and* $\delta=\frac{\lambda}{\sqrt{1+\lambda^2}}$. *For* $i=\sqrt{-1}$ *and* $t\in R$,

$$\phi_Y(t)=e^{i\mu t-\frac{t^2\sigma^2}{2}}\left[\alpha+C_{\lambda,\alpha}F(-i\delta\sigma t)\right]-C_{\lambda,\alpha}e^{i\mu t-\frac{t^2\sigma^2}{2}}\left[\xi_{s\sigma}(-it\sigma,-\lambda)-\xi_{-s\sigma}(-it\sigma,\lambda)\right].$$

Result 5.2 *The c.d.f.* $H(t;\mu,\sigma,\lambda,\alpha)$ *of a random variable* X *following* $ETMSND(\mu,\sigma;\lambda,\alpha)$ *is the following, in which* $\xi_0(\frac{t-\mu}{\sigma},\lambda)$ *is as defined in (1.3).*

$$H(t;\mu,\sigma,\lambda,\alpha)=\begin{cases}\alpha F\left(\frac{t-\mu}{\sigma}\right)+\frac{C_{\lambda,\alpha}}{2}\left[F\left(\frac{t-\mu}{\sigma}\right)-2\xi_0\left(\frac{t-\mu}{\sigma},-\lambda\right)\right], t<\mu\\ \alpha F\left(\frac{t-\mu}{\sigma}\right)+\frac{C_{\lambda,\alpha}}{2}\left[F\left(\frac{t-\mu}{\sigma}\right)+2\frac{\tan^{-1}(\lambda)}{\pi}-2\xi_0\left(\frac{t-\mu}{\sigma},\lambda\right)\right], t\geq\mu\end{cases}$$

Result 5.3 *The reliability function* $R(t;\mu,\sigma,\lambda,\alpha)$ *of a random variable* X *following* $ETMSND(\mu,\sigma;\lambda,\alpha)$ *is the following.*

$$R(t;\mu,\sigma,\lambda,\alpha)=1-\begin{cases}\alpha F\left(\frac{t-\mu}{\sigma}\right)+\frac{C_{\lambda,\alpha}}{2}\left[F\left(\frac{t-\mu}{\sigma}\right)-2\xi_0\left(\frac{t-\mu}{\sigma},-\lambda\right)\right], t<\mu\\ \alpha F\left(\frac{t-\mu}{\sigma}\right)+\frac{C_{\lambda,\alpha}}{2}\left[F\left(\frac{t-\mu}{\sigma}\right)+2\frac{\tan^{-1}(\lambda)}{\pi}-2\xi_0\left(\frac{t-\mu}{\sigma},\lambda\right)\right], t\geq\mu\end{cases}$$

Result 5.4 *The failure rate* $r(t;\mu,\sigma,\lambda,\alpha)$ *of a random variable* X *following* $ETMSND(\mu,\sigma;\lambda,\alpha)$ *is the following,*

$$r(t;\mu,\sigma,\lambda,\alpha)=\begin{cases}\dfrac{f\left(\frac{t-\mu}{\sigma}\right)\left[\alpha+C_{\lambda,\alpha}F\left(-\lambda\frac{t-\mu}{\sigma}\right)\right]}{\sigma-\alpha\sigma F\left(\frac{t-\mu}{\sigma}\right)-\sigma\frac{C_{\lambda,\alpha}}{2}[F\left(\frac{t-\mu}{\sigma}\right)-2\xi_0\left(\frac{t-\mu}{\sigma},-\lambda\right)]}, t<\mu\\ \dfrac{f\left(\frac{t-\mu}{\sigma}\right)\left[\alpha+C_{\lambda,\alpha}F\left(\lambda\frac{t-\mu}{\sigma}\right)\right]}{\sigma-\alpha\sigma F\left(\frac{t-\mu}{\sigma}\right)-\sigma\frac{C_{\lambda,\alpha}}{2}\left[F\left(\frac{t-\mu}{\sigma}\right)+\frac{2}{\pi}\tan^{-1}(\lambda)-2\xi_0\left(\frac{t-\mu}{\sigma},\lambda\right)\right]}, t\geq\mu\end{cases}$$

6. ESTIMATION

Let $X_1, X_2, ..., X_n$ be a random sample taken from $ETMSND(\mu,\sigma;\lambda,\alpha)$ with p.d.f. (5.1). Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be the ordered sample. Assume $X_{(r)} < \mu < X_{(r+1)}$, for a particular r=1,2,...,n. Then the log-likelihood function of the sample is the following, in which Σ_{I_j}, denote the summation over the set I_j such that

$I_1 = \{i : X_{(i)} < \mu \text{ , } for\ i = 1,2,...,r\}$ and $I_2 = \{i : X_{(i)} \geq \mu \text{ , } for\ i = r+1,..,n\}$.

$$\ln L = -n \ln \sigma + \sum_{I_1} \ln f\left(\frac{x_i - \mu}{\sigma}\right)\left[\alpha + C_{\lambda,\alpha} F\left(\frac{-\lambda(x_i - \mu)}{\sigma}\right)\right] + \sum_{I_1} \ln f\left(\frac{\lambda(x_i - \mu)}{\sigma}\right)\left[\alpha + C_{\lambda,\alpha} F\left(\frac{\lambda(x_i - \mu)}{\sigma}\right)\right]$$

Differentiate (6.1) with respect to the parameters μ, σ, λ and α, and equate to zero to obtain the following likelihood equations:

$$\sum_{I_1}\left(\frac{x_i - \mu}{\sigma^2}\right) + \frac{\lambda C_{\lambda,\alpha}}{\sigma} \sum_{I_1} \frac{f\left(-\frac{\lambda}{\sigma}(x_i - \mu)\right)}{[\alpha + C_{\lambda,\alpha} F\left(\frac{-\lambda}{\sigma}(x_i - \mu)\right)]}$$

$$+ \sum_{I_2}\left(\frac{x_i - \mu}{\sigma^2}\right) - \frac{\lambda}{\sigma} \sum_{I_2} \frac{f\left(\frac{\lambda}{\sigma}(x_i - \mu)\right)}{[\alpha + C_{\lambda,\alpha} F\left(\frac{\lambda}{\sigma}(x_i - \mu)\right)]} = 0, \qquad (6.2)$$

$$-\frac{n}{2\sigma^2} + \frac{1}{2}\sum_{I_1} \frac{(x_i - \mu)^2}{\sigma^4} - \sum_{I_1} \frac{f\left(-\frac{\lambda}{\sigma}(x_i - \mu)\right)}{[\alpha + C_{\lambda,\alpha} F\left(\frac{-\lambda}{\sigma}(x_i - \mu)\right)]} \frac{1}{2} \frac{C_{\lambda,\alpha}\lambda}{\sigma^3}(x_i - \mu)$$

(6.3)

$$-\delta(\lambda,\alpha)\sum_{I_1}\frac{f\left(\frac{-\lambda(x_i-\mu)}{\sigma}\right)\frac{(x_i-\mu)}{\sigma}}{[\alpha+C_{\lambda,\alpha}F\left(\frac{-\lambda}{\sigma}(x_i-\mu)\right)]}+C_{\lambda,\alpha}\sum_{I_2}\frac{f\left(\frac{\lambda(x_i-\mu)}{\sigma}\right)\frac{(x_i-\mu)}{\sigma}}{[\alpha+C_{\lambda,\alpha}F\left(\lambda\frac{(x_i-\mu)}{\sigma}\right)]}=0$$

and

$$\sum_{I_1}\frac{1-2F\left(\frac{-\lambda(x_i-\mu)}{\sigma}\right)}{\left[\alpha+C_{\lambda,\alpha}F\left(\frac{-\lambda(x_i-\mu)}{\sigma}\right)\right]}+\sum_{I_2}\frac{1-2F\left(\frac{\lambda(x_i-\mu)}{\sigma}\right)}{\left[\alpha+C_{\lambda,\alpha}F\left(\frac{\lambda(x_i-\mu)}{\sigma}\right)\right]}=0. \qquad (6.5)$$

Let

$$w(x_i)=\frac{f\left(-\lambda\frac{(x_i-\mu)}{\sigma}\right)}{\left[[\alpha+C_{\lambda,\alpha}F\left(-\lambda\frac{(x_i-\mu)}{\sigma}\right)]\right]},$$

and

$$\Omega(x_i)=\frac{f\left(\lambda\frac{(x_i-\mu)}{\sigma}\right)}{\left[[\alpha+C_{\lambda,\alpha}F\left(\lambda\frac{(x_i-\mu)}{\sigma}\right)]\right]},$$

$$W(x_i)=\frac{F\left(-\lambda\frac{(x_i-\mu)}{\sigma}\right)}{\left[[\alpha+C_{\lambda,\alpha}F\left(-\lambda\frac{(x_i-\mu)}{\sigma}\right)]\right]}$$

and

$$\Delta(x_i)=\frac{F\left(\lambda\frac{(x_i-\mu)}{\sigma}\right)}{\left[[\alpha+C_{\lambda,\alpha}F\left(\lambda\frac{(x_i-\mu)}{\sigma}\right)]\right]}.$$

Then the equations from (6.2) to (6.5) becomes

$$\sum_{I_1}\frac{(x_i-\mu)}{\sigma}+\sum_{I_2}\frac{(x_i-\mu)}{\sigma}=C_{\lambda,\alpha}\lambda\left[\sum_{I_1}-w(x_i)+\sum_{I_2}\Omega(x_i)\right], \qquad (6.6)$$

$$\frac{n}{2\sigma^2} = \frac{1}{2}\sum_{I_1}\frac{(x_i-\mu)^2}{\sigma^4} + \frac{1}{2}\sum_{I_2}\frac{(x_i-\mu)^2}{\sigma^4}$$

$$+\frac{C_{\lambda,\alpha}\lambda}{2\sigma^3}\left(\sum_{I_1} w(x_i)(x_i-\mu) - \sum_{I_2}\Omega(x_i)(x_i-\mu)\right), \qquad (6.7)$$

$$C_{\lambda,\alpha}\sum_{I_1} w(x_i)\left(\frac{x_i-\mu}{\sigma}\right) + C_{\lambda,\alpha}\sum_{I_2}\Omega(x_i)\left(\frac{x_i-\mu}{\sigma}\right) = 0 \qquad (6.8)$$

and

$$\sum_{I_1} W(x_i) + \sum_{I_2}\Delta(x_i) = \frac{n}{(\alpha+2)} = 0. \qquad (6.9)$$

On solving the non-linear system of equations (6.6) to (6.9) by simultaneous solution method using some mathematical softwares such as *MATHCAD*, *MATLAB*, *MATHEMATICA* etc. we get the maximum likelihood estimates (MLE) of the parameters of $ETMSND(\mu,\sigma;\lambda,\alpha)$.

7. NUMERICAL COMPUTATIONS

For illustrating the above procedure, we have considered two data sets.

Dataset 1: The data on average length of stay for patients who are in hospital for acute care because of problems, hepatobiliary system and pancreas, and die for this cause. The sample, under study, corresponds to 1082 hospitals in 10 states of the United States (For details, see columns 4 in http://lib.stat.cmu.edu/data-expo/1997/ascii/p07.dat).

Dataset 2: The data on the heights (in centimeters) of 100 Australian athletes, given in Cook and Weisberg (1994). The data recorded is as given below.
148.9 149 156 156.9 157.9 158.9 162 162 162.5 163 163.9 165 166.1 166.7 167.3 167.9 168 168.6 169.1 169.8 169.9 170 170 170.3 170.8 171.1 171.4 171.4 171.6 171.7 172 172.2 172.3 172.5 172.6 172.7 173 173.3 173.3 173.5 173.6 173.7 173.8 174 174 174 174.1 174.1 174.4 175 175 175 175.3 175.6 176 176 176 176 176.8 177 177.3 177.3 177.5 177.5 177.8 177.9 178 178.2 178.7 178.9 179.3 179.5 179.6 179.6 179.7 179.7 179.8 179.9 180.2 180.2 180.5 180.5 180.9 181 181.3 182.1 182.7 183 183.3 183.3 184.6 184.7 185 185.2 186.2 186.3 188.7 189.7 193.4 195.9.

We obtained the MLE of the parameters of the Normal, $ESND(\mu,\sigma;\lambda)$, $ETSND(\mu,\sigma;\lambda)$ of Kim (2005) and $ETMSND(\mu,\sigma;\lambda,\alpha)$ with the help of equations (6.6) to (6.9), and $MATHCAD$ software. The values of log-likelihood (l), the Akaike's Information Criterion (AIC), the Bayesian Information Criterion (BIC) and the corrected Akaike's Information Criterion (AICc) are computed for each fitted models corresponding to Data set 1 and Data set 2 and presented in Tables 1 and 2 respectively.

Table 1: Estimated values of the parameters and computed values of l, the AIC the BIC and the $AICc$ for the models - $N(\mu,\sigma)$, $ESND(\mu,\sigma;\lambda)$, $ETSND(\mu,\sigma;\lambda)$ and $ETMSND(\mu,\sigma;\lambda,\alpha)$.

Distribution:	Normal	ESND	ETSND	ETMSND
	(μ,σ)	$(\mu,\sigma;\lambda)$	$(\mu,\sigma;\lambda)$	$(\mu,\sigma;\lambda,\alpha)$
$\hat{\mu}$	5.76	4.12	4.72	4.65
$\hat{\sigma}$	1.61	2.3	1.64	1.82
$\hat{\lambda}$	-	2.06	-0.18	0.28
$\hat{\alpha}$	-	-	-	0.785
l	-2052.82	-2023.65	-2020	-1860
AIC	4109.64	4053.31	4046	3728
BIC	4119.61	4068.26	4060.96	3747.95
AICc	4109.65	4053.32	4046.022	3728.03

Table 2: Estimated values of the parameters and computed values of l, the AIC, the BIC and the $AICc$ for the models - $N(\mu,\sigma)$, $ESND(\mu,\sigma;\lambda)$, $ETSND(\mu,\sigma;\lambda)$ and $ETMSND(\mu,\sigma;\lambda,\alpha)$.

Distribution:	Normal	ESND	ETSND	ETMSND
	(μ,σ)	(μ,σ,λ)	$(\mu,\sigma;\lambda)$	$(\mu,\sigma;\lambda,\alpha)$
$\hat{\mu}$	174.594	174.58	172.25	174.7
$\hat{\sigma}$	8.24	8.20	9.2	8.38
$\hat{\lambda}$	-	0.0016	-0.04	-0.065
$\hat{\alpha}$	-	-	-	0.0085
l	-352.318	-351.9	-352.318	-346.667
AIC	708.64	709.8	710.636	701.33
BIC	713.85	717.6155	718.45	711.75
AICc	708.76	710.05	710.89	701.75

From Table 1 and Table 2, it can be observed that the $ETMSND(\mu,\sigma;\lambda,\alpha)$ gives better fit to both the data sets compared to the existing models - $N(\mu,\sigma)$, the $ESND(\mu,\sigma;\lambda)$ and the $ETSND(\mu,\sigma;\lambda)$.

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REFERENCES

Azzalini, A. (1985). A class of distributions which includes the normal ones, *Scandinavian Journal of Statistics*, **12**, 171–178.

Cook, R. D. and Weisberg, S. (1994). An Introduction to Regression Analysis, Wiley, New York

Genton, M.G. (2004). Skew-elliptical Distributions and their Applications: A Journey Beyond Normality. Chapman and Hall/CRC, London.

Kim, H. J. (2005). On a class of two-piece skew-normal distributions, *Statistics*, **39**(6), 537–553.

Kumar, C. S. and Anusree, M. R. (2011). On a generalized mixture of standard normal and skew normal distributions, *Statistics and Probability Letters*, **81**, 1813–1821.

Kumar, C. S. and Anusree, M. R. (2014). On a modified class of generalized skew normal distribution, *South African Statistical Journal*, **31**, 111–124,

Ma, Y. and Genton, M. G. A flexible class of skew-symmetric distributions, *Scandinavian Journal of Statistics*,**31** , 459–468, (2004).