ISSN 0971-0388

Aligarh Journal of Statistics
Vol. 36 (2016), 1-22

Inferences In Step-Stress Partially Accelerated Life Tests For Generalized Rayleigh Distribution Based On Type-II Censored Samples

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[Received on May, 2015. Revised on November, 2016]

ABSTRACT

In this article, we consider the step-stress partially accelerated life tests for highly reliable units/components to obtain the failure time observations from Generalized Rayleigh distribution under type-II censoring scheme. The maximum likelihood estimates of model parameters and acceleration factor are carried out by using R software. A Monte Carlo simulation study is performed to investigate the precision of the maximum likelihood estimates and also to obtain the coverage probabilities of the bootstrap-t and percentile bootstrap confidence intervals for the parameters. Finally, an example is presented to illustrate both the methods of bootstrap confidence intervals.

1. INTRODUCTION

In the Accelerated Life Test (ALT), the test units are subjected to stress conditions that are more severe than those encountered in normal use so that more failure data can be obtained in a shorter period of time. The failure times observed under accelerated conditions are analyzed by selecting an appropriate statistical model and then extrapolated to estimate the parameters of the life distribution corresponding to the normal use stress condition. Also, instead of holding the stress at a constant level throughout the life of a test unit, a most commonly used technique in reliability or life testing, called step-stress ALT, is the changing the stress setting in different steps at pre-specified times on

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surviving units. Nelson (1980) firstly introduced the concept of step-stress ALT with applications and recently Chandra *et al.* (2014) gave some more applications and attempted the work in this direction.

In the case of ALT, the acceleration factor is assumed to be known value or there is a known mathematical model which specifies the relationship between lifetime and stress conditions. But in some situations such life-stress relationship are not known and cannot be assumed. Therefore, in such cases, Partially Accelerated Life Tests (PALT) is better criterion to perform life test to estimate the acceleration factor and parameters of the life distribution. The concept of PALT was introduced by Goel (1971) in which a test unit is first run at use condition and if it does not fail for a pre-specified time τ, the test is switched to the higher level of stress for testing until all the unit fails or censoring reached. The effect of this switch is to multiply the remaining lifetime of the unit by an unknown factor which is called acceleration factor β. Thus, the total lifetime T of test unit is given by

$$T = \begin{cases} Y, & Y \le \tau \\ \tau + \beta^{-1}\left(Y - \tau\right), & Y > \tau \end{cases} \tag{1}$$

where Y denotes the lifetime of unit at normal use condition.

Goel (1971) was attempted the parameter estimation problem in step-stress PALT by using maximum likelihood (ML) method and Bayesian method when the lifetime data follow both exponential and uniform distribution for complete data. DeGroot and Goel (1979) studied the problem of estimation for acceleration factor and exponential parameters by using Bayesian approach with different loss functions for complete data. Bhattcharyya and Soejoeti (1989) also estimated the parameters of the Weibull distribution and acceleration factor using ML method in step-stress PALT. Bai and Chung (1992) reported ML method for estimating the acceleration factor and scale parameter of exponential distribution under Type-1 censoring. Bai *et al.* (1993) present the parameter estimation of the lognormal parameters and acceleration factor using ML method under Type-1 censoring.

Abdel-Ghally *et al.* (2002) investigated the maximum likelihood estimates for acceleration factor and parameters of the Weibull distribution under Types-I and Type-II censored data. Abd-Elfattah *et al.* (2008) attempted the problem of estimation of the acceleration factor and the parameters of the Burr type XII distribution in step-stress PALT using ML method under Type-I censored data.

Abd-Elfattah and AL-Harbey (2010) studied the estimation problem of the parameters of the Burr-III distribution and acceleration factor under Type-II censoring. Ismail (2010) obtained the Bayesian estimates as well as ML estimates of the Gompertz distribution parameters under SSPALT with type-I censored data. Wang *et al.* (2012) obtained the maximum likelihood estimates of the Weibull distribution parameters and acceleration factor under multiply censored data. A detail discussion on the most commonly used censoring schemes viz., Type-I and Type-II are given by Lawless (2003).

The rest of the article is organized in the following sequence. In section 2, the model description and assumptions of step-stress PALT model are discussed. The ML estimation procedure of the model parameters are derived in section 3. Section 4 presents the bootstrap confidence intervals. The simulation algorithms for constructing bootstrap confidence intervals are given in section 5. In section 6, the simulation study is presented. Finally, conclusion is presented in section 7.

2. MODEL DESCRIPTION AND ASSUMPTIONS

2.1 Model Description

Let T be a non-negative continuous random variable that follow two parameters (α: shape, λ: scale) Generalized Rayleigh (GR) distribution. It is also known as Burr-X distribution. Basically, Burr (1942) introduced twelve different forms of distributions, named as Burr's family of life time distributions. Among this family Burr-X and Burr-XII are well known due to its flexibility and commonly usage in reliability, survival and other important areas. For an excellent review of these distributions, the readers are referred to Johnson *et al.* (1995). A number of author's viz., Sartawi and Abu-Salih (1991), Ahmad *et al.* (1997), and Surles and Padgett (1998) have studied numerous characteristics of the one parameter $(\lambda = 1)$ Burr-X distribution. Surles and Padgett (2001) recommended that the two-parameter GR distribution (Burr-X distribution) is more suitable in the modeling of the stress-strength reliability data. The two-parameter generalized Rayleigh distribution is a particular member of the generalized Weibull or exponentiated Weibull (EW) that was originally proposed by Mudholkar and Srivastava (1993), [see also Mudholkar *et al.* 1995].

The probability density function (p.d.f) of generalized Rayleigh distribution is given by

$$f(t) = 2\alpha\lambda^2 t e^{-(\lambda t)^2}\left(1 - e^{-(\lambda t)^2}\right)^{(\alpha-1)} ;t > 0, \alpha > 0, \lambda > 0 \tag{2}$$

and the cumulative distribution function (c.d.f) is given by

$$F(t) = \left(1 - e^{-(\lambda t)^2}\right)^{\alpha}, t > 0, \alpha, \lambda > 0 \tag{3}$$

Its reliability function is given by

$$R(t) = 1 - \left(1 - e^{-(\lambda t)^2}\right)^{\alpha} \tag{4}$$

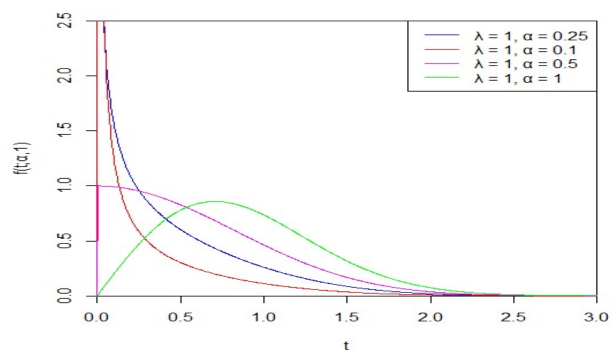

Fig. 1: The probability density functions of GR distribution for different values of shape.

The hazard rate function of the GR distribution is given by

$$h(t) = \frac{2\alpha\lambda^2 t e^{-(\lambda t)^2}\left(1 - e^{-(\lambda t)^2}\right)^{(\alpha-1)}}{1 - \left(1 - e^{-(\lambda t)^2}\right)^{\alpha}} \tag{5}$$

when $\alpha = 1$, then equation (5) becomes, the hazard function of one parameter Rayleigh distribution. Mudholkar *et al.* (1995) observed the hazard rate function of GR(α, λ) is bathtub type for $\alpha \le 1/2$ and for $\alpha > 1/2$, it has an increasing hazard function. Raqab and Kundu (2006) also analyzed the hazard characteristics for GR distribution at several possible values of α. They reported that, if the data are coming from an environment where the failure rate is gradually increasing without any bound, the GR distribution can also be used instead of a Weibull distribution.

2.2 Assumptions in Step-Stress PALT Model

(i) The failure times $t_i, i = 1,2,...,n$ are independent and identically distributed random variables.

(ii) The total lifetime of test units denoted by T, given in (1) pass through two stages that are normal and accelerated conditions.

Therefore, from (1) and (3) the c.d.f. of total lifetime T of an item is given by

$$F(t) = \begin{cases} 1 - \left(1 - e^{-(\lambda t)^2}\right)^{\alpha}, & 0 < t \le \tau \\ 1 - \left(A_2(\mathrm{t})\right)^{\alpha}, & t > \tau \end{cases} \tag{6}$$

Let $A_1(\mathrm{t}) = \left[\tau - \beta\left(t - \tau\right)\right]$ and $A_2(\mathrm{t}) = 1 - \mathrm{e}^{-\left(\lambda\, \mathrm{A}_1(\mathrm{t})\right)^2}$ and the corresponding p.d.f. is given by

$$f(t) = \begin{cases} f_1(t), & 0 < t \le \tau \\ f_2(t), & t > \tau \end{cases} \tag{7}$$

where, $f_1(t) = 2\alpha\lambda^2 t e^{-(\lambda t)^2}\left(1 - e^{-(\lambda t)^2}\right)^{(\alpha-1)}$, which is equivalent form to equation (2), and $f_2(t) = 2\alpha\lambda^2 \beta A_1(\mathrm{t}) e^{-\left(\lambda\, \mathrm{A}_1(\mathrm{t})\right)^2}\left(A_2(\mathrm{t})\right)^{(\alpha-1)}$, it is obtained by the transformation-variable technique using (1) and (2), with $\beta > 1; \alpha, \lambda > 0$.

3. MAXIMUM LIKELIHOOD ESTIMATION AND FISHER INFORMATION MATRIX

This section presents the maximum likelihood estimates of the parameters involved in the proposed model and Fisher information matrix.

In the case of type-II censored data, the test applied to n identical units will terminate when censoring number of failure r is reached. Let n_u and n_a denote the number of failures at normal and accelerated conditions, respectively. Hence, the observed values of the total lifetime T are given by

$t_{(1)} \le \dots \le t_{(n_u)} \le \tau \le t_{(n_u+1)} \le \dots \le t_{(r)}; r = n_u + n_a$.

Therefore, the likelihood function under type-II censoring can be written as

$$L(t) = \prod_{i=1}^{n_u} f_1(t_i) \times \prod_{i=n_u+1}^{r} f_2(t_i) \times \left[R(t_r)\right]^{n-r} \tag{8}$$

$$\begin{aligned} L(t) = &\prod_{i=1}^{n_u} 2\alpha\lambda^2 t_i e^{-(\lambda t_i)^2} \left(1 - e^{-(\lambda t_i)^2}\right)^{(\alpha-1)} \\ &\times \prod_{i=n_u+1}^{r} 2\alpha\lambda^2 \beta A_1(\mathrm{t}_i) e^{-(\lambda a)^2} \left\{A_2(\mathrm{t}_i)\right\}^{(\alpha-1)} \times \left[1 - \left\{A_2(\mathrm{t}_r)\right\}^{\alpha}\right]^{n-r} \end{aligned} \tag{9}$$

Obviously, it is easier to maximize the natural logarithm of the likelihood function than the likelihood function. Therefore, the log-likelihood function is given by

$$\begin{aligned} \log L = &\, r \log\left(2\alpha\lambda^2\right) + \sum_{i=1}^{n_u} \log(\mathrm{t}_i) + (\alpha-1)\sum_{i=1}^{n_u} \log\left[1 - e^{-(\lambda t_i)^2}\right] \\ &- \lambda^2 \sum_{i=1}^{n_u} t_i^2 + \sum_{i=n_u+1}^{r} \log A_1(\mathrm{t}_i) - \lambda^2 \sum_{i=n_u+1}^{r} A_1^2(\mathrm{t}_i) + n_a \log(\beta) \\ &+ (\alpha-1) \sum_{i=n_u+1}^{r} \log\left(A_2(\mathrm{t}_i)\right) + (n-r)\log\left\{1 - \left(A_2(\mathrm{t}_r)\right)^{\alpha}\right\} \end{aligned} \tag{10}$$

The maximum likelihood estimates (MLEs) of α, λ and β are the solution of the following system of equations

$$\left.\begin{aligned}\frac{\partial \log L}{\partial \alpha} = 0\\ \frac{\partial \log L}{\partial \lambda} = 0\\ \frac{\partial \log L}{\partial \beta} = 0\end{aligned}\right\} \tag{11}$$

Since this is system of non-linear equations and do not admit explicit solutions, so *optim()* function is used to solve these equations numerically via R software to obtain $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\beta}$. The asymptotic variance-covariance matrix of the MLEs can be obtained by numerically inverting the asymptotic Fisher Information matrix. It is composed of the negative second partially derivatives of log-likelihood function evaluated at MLEs, is given as

$$F = \begin{bmatrix} -\frac{\partial^2 \log L}{\partial \alpha^2} & -\frac{\partial^2 \log L}{\partial \alpha \partial \lambda} & -\frac{\partial^2 \log L}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \log L}{\partial \lambda \partial \alpha} & -\frac{\partial^2 \log L}{\partial \lambda^2} & -\frac{\partial^2 \log L}{\partial \lambda \partial \beta} \\ -\frac{\partial^2 \log L}{\partial \beta \partial \alpha} & -\frac{\partial^2 \log L}{\partial \beta \partial \lambda} & -\frac{\partial^2 \log L}{\partial \beta^2} \end{bmatrix}$$

The elements of the above Fisher Information matrix are derived in the Appendix.

4. CONFIDENCE INTERVALS

In this section, we present two bootstrap methods to construct confidence intervals (CI) for the unknown parameters α, λ and β , viz., bootstrap-t CI (BTCI) suggested by Hall (1988) and percentile bootstrap CI (PBCI) suggested by Efron (1982).

4.1 Bootstrap-t Confidence Interval

First, find the order statistic $T_l^{*[1]} < \ldots < T_l^{*[m^*]}$ by using the data given in step 8 of subsection 5.1, where

$$T_l^{*[k]} = \frac{\hat{\psi}_l^{*[k]} - \hat{\psi}_l}{\sqrt{Var\left(\hat{\psi}_l^*\right)}}, k = 1, 2, ..., m^*, l = 1, 2, 3.$$

where, $\hat{\psi}_1 = \hat{\alpha}, \hat{\psi}_2 = \hat{\lambda}, \hat{\psi}_3 = \hat{\beta}$.

Next, consider all possible 100(1-γ)% CIs of the form

$$\left(T_l^{*[h]}, T_l^{*[(1-\gamma)m^* + h]}\right), h = 1,2,...\gamma m^*, l = 1,2,3$$

and choose the interval for which the width is minimum, $\left(T_{lL}^*, T_{lU}^*\right)$.

A two-sided 100(1-γ)% BTCI for ψ_l is either

$$\left(\hat{\psi}_1 - T_{lL}^* \sqrt{Var(\hat{\psi}_1)}, \hat{\psi}_1 - T_{lU}^* \sqrt{Var(\hat{\psi}_1)}\right) \text{ or}$$

$$\begin{pmatrix} \hat{\psi}_1 - T_{lL}^{*[(1-\gamma/2)m^*]} \sqrt{Var\left(\hat{\psi}_1\right)}, \\ \hat{\psi}_1 - T_{lU}^{*[\gamma m^*/2]} \sqrt{Var\left(\hat{\psi}_1\right)} \end{pmatrix}$$

where, $Var(\hat{\psi}_1)$ is estimated as the asymptotic variance, obtained from the type-II censored sample.

4.2 Percentile Bootstrap Confidence Interval

First, consider all 100(1-γ)% CIs of the form

$$\left(\hat{\psi}_l^{*[h]}, \hat{\psi}_l^{*[(1-\gamma)m^* + h]}\right), h = 1,2,...\gamma m^*, l = 1,2,3$$

by using the bootstrap sample given in step 8 of subsection 5.1.

And then choose the interval with minimum width, say $\left(\hat{\psi}_{lL}^*, \hat{\psi}_{lU}^*\right)$.

A two-sided 100(1-γ)% PBCI for ψ_l is either

$$\left(\hat{\psi}_{lL}^*, \hat{\psi}_{lU}^*\right) \text{or} \left(\hat{\psi}_{lL}^{*[\gamma m^*/2]}, \hat{\psi}_{lU}^{*[(1-\gamma)m^* + h]}\right).$$

5. SIMULATION PROCEDURES

In this section the following algorithms are used for simulating the failure lifetime data and for obtaining the MLEs of parameters $(\alpha, \lambda \text{ and } \beta)$ and study the performance of their estimates through the mean squared errors (MSEs) and relative absolute biases (RABs) as:

1. First we assumed the initial values of the parameters α, λ and β.
2. Simulate n order statistics from the Uniform (0, 1) distribution, $U_{1:n}, U_{2:n}, \ldots, U_{n:n}$.
3. For a given value of pre-specified switching time τ, find n_1 such that $U_{n_1:n} \le \left[1-\exp\left(-(\lambda\tau)^2\right)\right]^{\alpha} \le U_{n_1+1:n}$.
4. For a given value of censoring number $(r = 0.85n)$, find n_2 such that $n_2 = r - n_1$.
5. From step 3 and 4, the order failure times, $t_{1:n} \le \ldots \le t_{n_1:n} \le t_{n_1+1:n} \le \ldots \le t_r$ are calculated as follows
$$t_{i:n} = \begin{cases} \dfrac{1}{\lambda}\left[-\log\left(1-U_{i:n}^{1/\alpha}\right)\right]^{1/2}, & \text{for } 1 \le i \le n_1 \\ \tau + \dfrac{1}{\beta}\left[\dfrac{1}{\lambda}\left\{-\log\left(1-U_{i:n}^{1/\alpha}\right)\right\} - \tau\right]^{1/2}, & \text{for } n_1+1 \le i \le r \end{cases}$$
6. From n_1, n_2, n, τ, r and ordered failure times $t_{i:n}$ given in step 5, we can obtain the MLEs $\left(\hat{\alpha}, \hat{\lambda}, \hat{\beta}\right)$ by directly maximizing the log-likelihood given in (3.3) through R software.
7. Repeat steps 2-6 for m(1000) times representing m different samples.
8. If $\hat{\psi}_{lk}$ is a MLE of $\psi_l, l = 1, 2, 3$ (where ψ is a general notation that can be replaced by α, λ and β i.e. $(\psi_1 \equiv \alpha, \psi_2 \equiv \lambda, \psi_3 \equiv \beta)$, based on sample $k, k = 1, 2, \ldots, m$, then the average estimate, MSE and RAB of $\hat{\psi}_l$ over the m samples are given respectively, by
$$\overline{\hat{\psi}}_l = \frac{1}{m}\sum_{k=1}^{m}\hat{\psi}_{lk},$$
$$MSE(\hat{\psi}_l) = \frac{1}{m}\sum_{k=1}^{m}(\hat{\psi}_{lk} - \psi_l)^2 \text{ and } RAB(\hat{\psi}_l) = \frac{\left|\overline{\hat{\psi}} - \psi\right|}{\psi}$$

9. From step 8 compute $\left(\bar{\hat{\alpha}}, \bar{\hat{\lambda}}, \bar{\hat{\beta}}\right)$ and also compute $MSE(\hat{\alpha})$, $MSE(\hat{\lambda})$, $MSE(\hat{\beta})$, $RAB(\hat{\alpha})$, $RAB(\hat{\lambda})$, $RAB(\hat{\beta})$ by using above formulas.

5.1 Algorithm for Bootstrap Confidence Intervals

In this subsection, we describe the algorithm to obtain the bootstrap sample. Then, we use this bootstrap sample for constructing the bootstrap confidence intervals. The following steps are followed to obtain a bootstrap sample

1. From the original type-II censored sample, $t_{1:n} \le \ldots \le t_{n_1:n} \le t_{n_1+1:n} \le \ldots \le t_r$, obtain $\left(\hat{\alpha}, \hat{\lambda} \text{ and } \hat{\beta}\right)$.
2. Simulate n^* order statistics from the Uniform (0, 1) distribution, $U^*_{1:n^*}, \ldots, U^*_{n^*:n^*}$.
3. For a given value of pre-specified switching time τ, find n_1^* such that $U^*_{n_1^*:n^*} \le \left[1 - \exp\left(-(\hat{\lambda}\tau)^2\right)\right]^{\hat{\alpha}} \le U^*_{n_1^*+1:n^*}$
4. For a given value of censoring number $\left(r^* = 0.85n^*\right)$, find n_2^* such that $n_2^* = r^* - n_1^*$.
5. From step 3 and 4, the order failure times, $t^*_{(1:n^*)} \le \ldots \le t^*_{(n_1^*:n^*)} \le t^*_{(n_1^*+1:n^*)} \le \ldots \le t^*_{(r^*)}$ are calculated as follows
$$t^*_{i:n^*} = \begin{cases} \left[\dfrac{1}{\hat{\lambda}}\left[-\log\left(1 - U^{*1/\hat{\alpha}}_{i:n^*}\right)\right]\right]^{1/2}, & \text{for } 1 \le i \le n_1^* \\ \left[\tau + \dfrac{1}{\hat{\beta}}\left[\dfrac{1}{\hat{\lambda}}\left\{-\log\left(1 - U^{*1/\hat{\alpha}}_{i:n^*}\right)\right\} - \tau\right]\right]^{1/2}, & \text{for } n_1^* + 1 \le i \le r^* \end{cases}$$
6. From $n_1^*, n_2^*, n^*, \tau^*, r^*$ and ordered failure times given in step 5, we can obtain the MLEs $\left(\hat{\alpha}^*, \hat{\lambda}^* \text{ and } \hat{\beta}^*\right)$ by directly maximizing the log-likelihood given in (10) through R software. The value of m^* has been taken to be equal m.
7. Repeat the above steps m^* times representing m^* different samples. The value of m^* has been taken to be 500.

8. Arrange all the values of $\hat{\alpha}^*$, $\hat{\lambda}^*$ and $\hat{\beta}^*$ in an ascending order to obtain the bootstrap sample $\left(\hat{\psi}_l^{*[1]}, \hat{\psi}_l^{*[2]}, ..., \hat{\psi}_l^{*[m^*]}\right)$, $l = 1, 2, 3$, (where $\psi_1^* \equiv \alpha^*, \psi_2^* \equiv \lambda^*, \psi_3^* \equiv \beta^*$).

6. SIMULATION STUDY

In this section, we present the results of Monte Carlo simulation study carried out in order to compare the performance of two sets of initial values of the parameters. The values of the parameters in two sets are arbitrary chosen to be α=0.50, λ=1.65 and β=1.25 for first set and α=0.50, λ=1.35 and β=1.30 for second set. The value of stress changing time τ is 0.50 and the pre-specified censoring number r=0.85n (n=70, 100, 150, 200). Numerical results are tabulated in Table 1-3 shown in Appendix, based on m=1000 repetitions. Table-1 presents the average value of MLEs, MSEs and RABs of α, λ and β for different sample sizes.

The Table-2 shows the 95% coverage probabilities for BTCIs and PBCIs of α, λ and β based on 1000 Monte Carlo simulations (m=1000) and 500 bootstrap replications (m^*=500).

6.1 An example

Here, we present an example to illustrate the estimation procedure and the two bootstrap CI methods for two sets of the parameters α, λ and β. In this example, we simulate a sample of size n=50, using the same algorithm given in section 5, based on two set of initial value of the parameters; α=0.50, λ=1.65 and β=1.25 for first set and α=0.50, λ=1.35 and β=1.30 for second set. The stress changing time τ and pre-specified censoring number r are taken to be equal 0.50 and 0.85n, respectively. The simulated failure time data are given in Table-3, while the MLEs, MSEs and RABs of the parameters are given in Table-4, under type-II censoring. Using two bootstrap CIs derived in section 4.1 and 4.2. Table-5 shows 95% bootstrap CIs for the parameters α, λ and β.

Table 1: MLEs of $(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$ with their MSEs and RABs for different sample sizes with r=0.85n censoring based on 1000 simulations.

n	Parameters	α=0.50, λ=1.65, β=1.25, τ=0.50			α=0.50, λ=1.35, β=1.30, τ=0.50		
		MLEs	MSEs	RABs	MLEs	MSEs	RABs
70	α	0.5342	0.0095	0.0684	0.5375	0.0107	0.0750
	λ	1.7460	0.0730	0.0582	1.4363	0.0617	0.0639
	β	1.4855	2.7665	0.1884	1.3600	0.1921	0.0462
100	α	0.5079	0.0048	0.0159	0.5166	0.0058	0.0333
	λ	1.6588	0.0432	0.0053	1.3721	0.0381	0.0163
	β	1.3663	0.1941	0.0930	1.3588	0.1297	0.0452
150	α	0.5163	0.0037	0.0325	0.5157	0.0041	0.0313
	λ	1.6991	0.0302	0.0297	1.3863	0.0265	0.0269
	β	1.3123	0.1130	0.0498	1.3272	0.0787	0.0209
200	α	0.5036	0.0025	0.0072	0.5054	0.0028	0.0108
	λ	1.6549	0.0211	0.0030	1.3575	0.0175	0.0056
	β	1.2976	0.0788	0.0381	1.3286	0.0579	0.0220

Table 2: Estimated coverage probability (in %) of bootstrap-t and percentile bootstrap CIs for (α, λ, β) based on 1000 simulations and 500 replications with 95% CI.

n	Parameters	α=0.50, λ=1.65, β=1.25, τ=0.50		α=0.50, λ=1.35, β=1.30, τ=0.50	
		BTCI	PBCI	BTCI	PBCI
70	α	97.80	97.20	99.60	96.80
	λ	99.00	98.70	98.90	98.70
	β	91.10	100.00	99.90	99.30
100	α	98.90	92.80	97.60	77.10
	λ	96.70	100.00	97.50	95.70
	β	97.60	99.60	100.00	100.00
150	α	98.00	91.50	98.40	77.10
	λ	98.40	96.30	97.00	99.10
	β	97.50	98.00	100.00	100.00
200	α	99.20	85.50	100.00	86.60
	λ	98.70	98.70	99.20	100.00
	β	99.80	99.80	99.30	100.00

Table 3: The ordered failure time data form step-stress partially accelerated life test model with n=50, τ=0.50 and censoring r=0.85n.

True values	Use conditions	Accelerated conditions
α=0.50 λ=1.65	0.003, 0.048, 0.069, 0.080, 0.082, 0.103, 0.109, 0.116, 0.119, 0.144, 0.146, 0.153,	0.512, 0.521, 0.548, 0.578,

β=1.25	0.162, 0.167, 0.168, 0.169, 0.170, 0.203, 0.218, 0.242, 0.251, 0.304, 0.314, 0.321, 0.359, 0.391, 0.404, 0.405, 0.406, 0.415, 0.416, 0.430, 0.452, 0.469, 0.498	0.582, 0.586, 0.588
α=0.50 λ=1.35 β=1.30	0.009, 0.038, 0.104, 0.105, 0.109, 0.112, 0.115, 0.125, 0.136, 0.160, 0.169, 0.173, 0.183, 0.184, 0.219, 0.238, 0.254, 0.255, 0.269, 0.272, 0.311, 0.315, 0.322, 0.340, 0.359, 0.381, 0.407, 0.411, 0.417, 0.419, 0.427, 0.439, 0.448, 0.461	0.503, 0.522, 0.538, 0.540, 0.547, 0.555, 0.565, 0.692

Table 4: MLEs of $(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$ with their MSEs and RABs for n=50 with r=0.85n censoring.

Parameters	α=0.50, λ=1.65, β=1.25, τ=0.50			α=0.50, λ=1.35, β=1.30, τ=0.50		
	MLEs	MSEs	RABs	MLEs	MSEs	RABs
α	0.5941	0.0192	0.1882	0.5814	0.0147	0.1628
λ	1.9532	0.1616	0.1837	1.5481	0.0931	0.1468
β	1.2999	0.4978	0.0400	1.3636	0.3563	0.0490

Table 5: Bootstrap-t CIs and percentile bootstrap CIs of based on 500 replications for n=50 with r=0.85n and 95% CI.

Parameters	α=0.50, λ=1.65, β=1.25, τ=0.50		α=0.50, λ=1.35, β=1.30, τ=0.50	
	BTCI	BTCI	BTCI	PBCI
α	(0.397, 1.012)	(0.475,0.623)	(0.384, 1.107)	(0.501, 0.712)
λ	(1.473, 3.022)	(0.783,2.173)	(0.983, 2.359)	(0.989, 2.174)
β	(0.054, 11.62)	(0.594,1.721)	(0.084, 4.929)	(-0.215, 4.21)

From the results of the Table-4, it is observed that the second set have good statistical properties. It can be also observed from the Table-5 that the BTCIs are narrower than the PBCIs and always include the population parameter values.

7. CONCLUSION

In life testing experiment, it is very difficult to continue the test of highly reliable units. So, ALT is recommended to aid estimating the reliability of the unit in a short period of time. But, the main assumption in ALT is that the relationship between the mean lifetime and the stress is known. On the other hand, in the case of the modern products with very intricate technology, it is impossible to know or to expect this relationship relating the lifetime of a unit to the stress. Consequently, in such cases, PALT is the apt procedure of accelerating life tests to be applied where PALT does not assume that this relationship is known.

In this article, the maximum likelihood method is used for estimating the acceleration factor and the parameters of the Generalized Rayleigh distribution under type-II censored data. The mean square error, relative absolute bias and bootstrap (bootstrap-t and percentile bootstrap) confidence intervals are considered. Simulation study with an illustrative example is given.

The concluded remarks on the proposed study are given below:

From results of Table-1 we observe the following:

1. The second set of the population parameters have good statistical properties than first set of the population parameters for all sample sizes.
2. As the sample size increases the MSEs and RABs of the estimated parameters decreases and MLEs also approaches nearer to true values of the population parameters.
3. This indicates that the maximum likelihood estimates provide asymptotically normally distributed and consistent estimators for the parameters and acceleration factor.

From Table-2 we observe the following:

It is observed that for both the sets of the population parameter value the coverage probabilities of the two considered bootstrap methods almost better as expected except for some few cases. It is also seen that CPBTs are better than CPPBs as Hall (1988) suggested.

As future work, this study can be extended to explore the situation under different censoring schemes like as: type-I, progressive type-I, progressive type-II.

ACKNOWLEDGEMENT

We are thankful to the Editor and reviewers who provided valuable comments which have shaped the presentation of this paper.

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APPENDIX

Second partial derivatives of log-likelihood function

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{r}{\alpha^2} - (n-r)\left(A_2(\mathrm{t}_r)\right)^{\alpha}\left[\log\left(A_2(\mathrm{t}_r)\right)\right]^2\left[1-\left(A_2(\mathrm{t}_r)\right)^{\alpha}\right]^{-2}$$

$$\frac{\partial^2 \log L}{\partial\alpha\partial\lambda}=\sum_{i=1}^{n_u}\frac{2\lambda t_i^2 e^{-(\lambda t_i)^2}}{\left(1-e^{-(\lambda t_i)^2}\right)}+\sum_{i=n_u+1}^{n_a}\frac{2\lambda A_1^2(\mathrm{t}_i)e^{-(\lambda A_1(\mathrm{t}_i))^2}}{A_2(\mathrm{t}_i)}$$

$$-(n-r)2\lambda A_1^2(\mathrm{t}_r)e^{-(\lambda A_1(\mathrm{t}_r))^2}\left(A_2(\mathrm{t}_r)\right)^{(\alpha-1)}\left[\alpha\log\left(A_2(\mathrm{t}_r)\right)+1\right]\left[1-\left(A_2(\mathrm{t}_r)\right)^{\alpha}\right]^{-1}$$

$$+(n-r)2\lambda A_1^2(\mathrm{t}_r)e^{-(\lambda A_1(\mathrm{t}_r))^2}\left(A_2(\mathrm{t}_r)\right)^{(2\alpha-1)}\alpha\log\left(A_2(\mathrm{t}_r)\right)\left[1-\left(A_2(\mathrm{t}_r)\right)^{\alpha}\right]^{-2}$$

$$\frac{\partial^2 \log L}{\partial\lambda^2}=-\frac{2r}{\lambda^2}-2\sum_{i=1}^{n_u}t_i^2+(\alpha-1)\sum_{i=1}^{n_u}\frac{2t_i^2 e^{-(\lambda t_i)^2}}{\left(1-e^{-(\lambda t_i)^2}\right)}-(\alpha-1)\sum_{i=1}^{n_u}\frac{4\lambda^2 t_i^4 e^{-(\lambda t_i)^2}}{\left(1-e^{-(\lambda t_i)^2}\right)^2}$$

$$-2\sum_{i=n_u+1}^{n_a}A_1^2(\mathrm{t}_i)-(\alpha-1)\sum_{i=n_u+1}^{n_a}\frac{4\lambda^2 A_1^4(\mathrm{t}_i)e^{-(\lambda A_1(\mathrm{t}_i))^2}}{\left(A_2(\mathrm{t}_i)\right)^2}$$

$$-\alpha(n-r)\frac{(\alpha-1)\left(2\lambda A_1^2(\mathrm{t}_r)e^{-(\lambda A_1(\mathrm{t}_r))^2}\right)^2\left(A_2(\mathrm{t}_r)\right)^{\alpha-1}}{\left[1-\left(A_2(\mathrm{t}_r)\right)^{\alpha}\right]}$$

$$+(\alpha-1)\sum_{i=n_u+1}^{n_a}\frac{2A_1^2(\mathrm{t}_i)e^{-(\lambda A_1(\mathrm{t}_i))^2}}{A_2(\mathrm{t}_i)}$$

$$-\alpha(n-r)\alpha\left(2\lambda A_1^2(\mathrm{t}_r)e^{-(\lambda A_1(\mathrm{t}_r))^2}\right)^2\left(A_2(\mathrm{t}_r)\right)^{\alpha-1}\times\left[1-\left(A_2(\mathrm{t}_r)\right)^{\alpha}\right]^{-2}$$

$$-\alpha(n-r)2A_1^2(\mathrm{t}_r)\left[1-2\left(\lambda A_1(\mathrm{t}_r)\right)^2\right]e^{-(\lambda A_1(\mathrm{t}_r))^2}\left[A_2(\mathrm{t}_r)\right]^{\alpha-1}\left[1-\left(A_2(\mathrm{t}_r)\right)\alpha\right]^{-1}$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = -\frac{n_a}{\beta^2} - (1-2\lambda^2)\sum_{i=n_u+1}^{n_a} \frac{(t_i-\tau)^2}{A_1^2(\mathrm{t}_i)}$$

$$+(\alpha-1)\sum_{i=n_u+1}^{n_a} 2\lambda^2 (t_i-\tau)^2 e^{-(\lambda A_1(\mathrm{t}_i))^2}\left(A_2(\mathrm{t}_i)-2\lambda^2 A_1^2(\mathrm{t}_i)\right)\left(A_2(\mathrm{t}_i)\right)^{-1}$$

$$-\alpha^2(n-r)\left[2\lambda^2 A_1(\mathrm{t}_r)(t_r-\tau)e^{-(\lambda A_1(\mathrm{t}_r))^2}\left(A_2(\mathrm{t}_r)\right)^{\alpha-1}\right]^2\left[1-\left(A_2(\mathrm{t}_r)\right)^{\alpha}\right]^{-2}$$

$$-\alpha(n-r)2\lambda^2(t_r-\tau)^2 e^{-(\lambda A_1)^2}\left(1-2\lambda^2 A_1^2\right)\left(1-e^{-(\lambda A_1)^2}\right)^{\alpha-1}\left[1-\left(A_2(\mathrm{t}_r)\right)^{\alpha}\right]^{-1}$$

$$-\alpha(n-r)(\alpha-1)\left(1-e^{-(\lambda A_1)^2}\right)^{\alpha-2}\left(2\lambda^2 A_1(t_r-\tau)e^{-(\lambda A_1)^2}\right)^2\left[1-\left(A_2(\mathrm{t}_r)\right)^{\alpha}\right]^{-1}$$

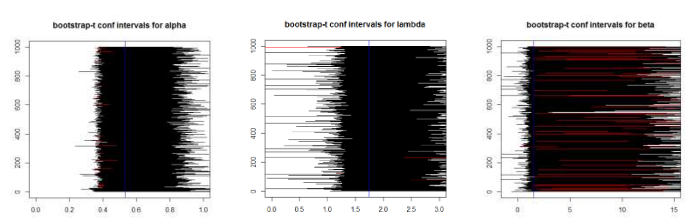
$$\frac{\partial^2 \log L}{\partial \beta \partial \lambda} = -4\lambda \sum_{i=n_u+1}^{n_a} \frac{(t_i-\tau)}{A_1(\mathrm{t}_r)}$$

$$+(\alpha-1)\sum_{i=n_u+1}^{n_a} 4\lambda A_1(\mathrm{t}_i)e^{-(\lambda A_1(\mathrm{t}_i))^2}(t_i-\tau)\left[A_2(\mathrm{t}_i)-\lambda^2 A^2\right]\left[A_2(\mathrm{t}_i)\right]^{-1}$$

$$-\alpha(n-r)(\alpha-1)4(\lambda A_1(\mathrm{t}_r))^3(t_r-\tau)e^{-2(\lambda A_1(\mathrm{t}_r))^2}\left[A_2(\mathrm{t}_r)\right]^{\alpha-2}\left[1-\left(A_2(\mathrm{t}_r)\right)^{\alpha}\right]^{-1}$$

$$-4\alpha^2(n-r)(\lambda A_1(\mathrm{t}_r))^3(t_r-\tau)e^{-2(\lambda A_1(\mathrm{t}_r))^2}\left[A_2(\mathrm{t}_r)\right]^{2(\alpha-1)}\left[1-\left(A_2(\mathrm{t}_r)\right)^{\alpha}\right]^{-2}$$

$$-\alpha(n-r)2\lambda A_1(t_r-\tau)(1-2\lambda^2 A_1^2)e^{-(\lambda A_1)^2}\left[1-e^{-(\lambda A_1)^2}\right]^{\alpha-1}\left[1-\left(A_2(\mathrm{t}_r)\right)^{\alpha}\right]^{-1}$$



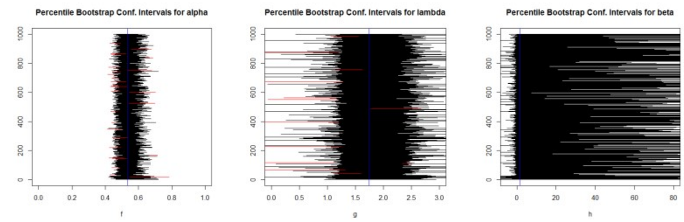


Fig. 2: The coverage probabilities of bootstrap-t and percentile bootstrap are shown in 1(a), 1(b), 1(c) and 1(f), 1(g), 1(h), respectively for n=70 under first set of initial values.

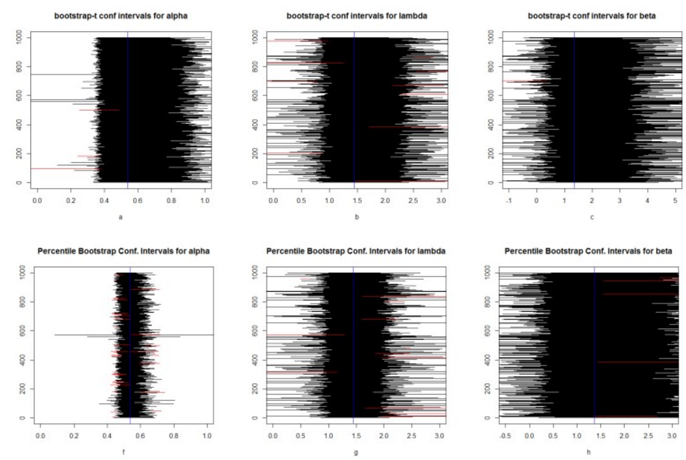


Fig. 3: The coverage probabilities of bootstrap-t and percentile bootstrap are shown in 2(a), 2(b), 2(c) and 2(f), 2(g), 2(h), respectively for n=100 under first set of initial values.

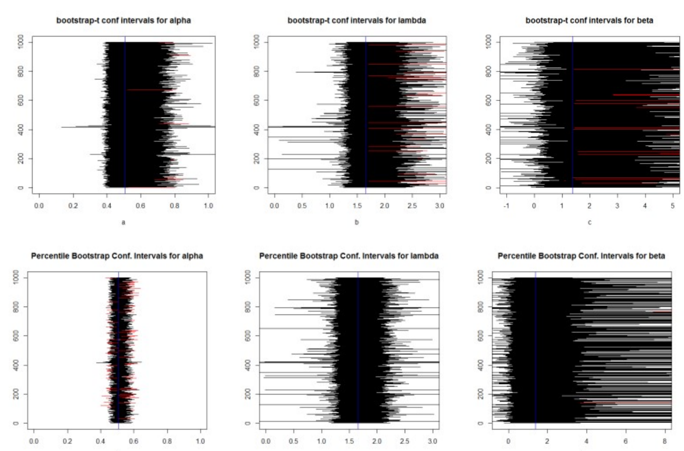


Fig. 4: The coverage probabilities of bootstrap-t and percentile bootstrap are shown in 3(a), 3(b), 3(c) and 3(f), 3(g), 3(h), respectively for n=150 under first set of initial values.

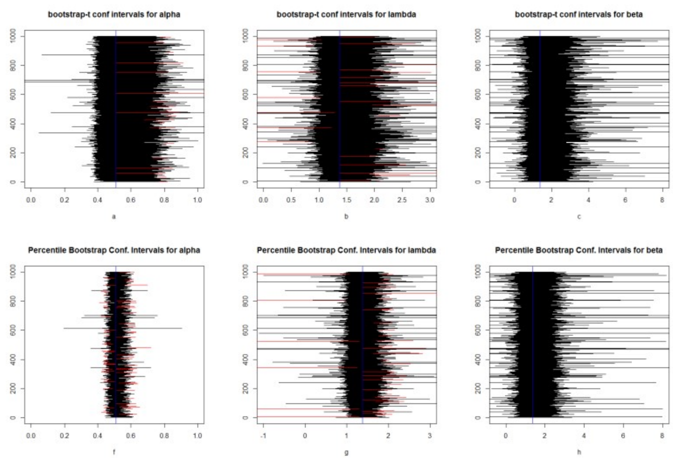


Fig. 5: The coverage probabilities of bootstrap-t and percentile bootstrap are shown in 4(a), 4(b), 4(c) and 4(f), 4(g), 4(h), respectively for n=200 under first set of initial values.

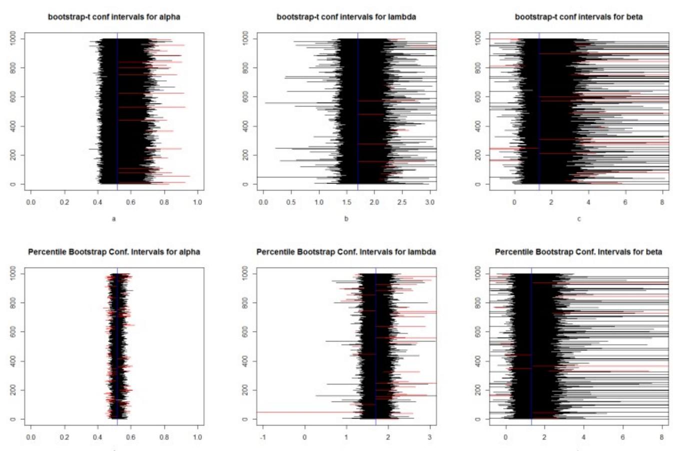


Fig. 6: The coverage probabilities of bootstrap-t and percentile bootstrap are shown in 5(a), 5(b), 5(c) and 5(f), 5(g), 5(h), respectively for n=70 under second set of initial values.

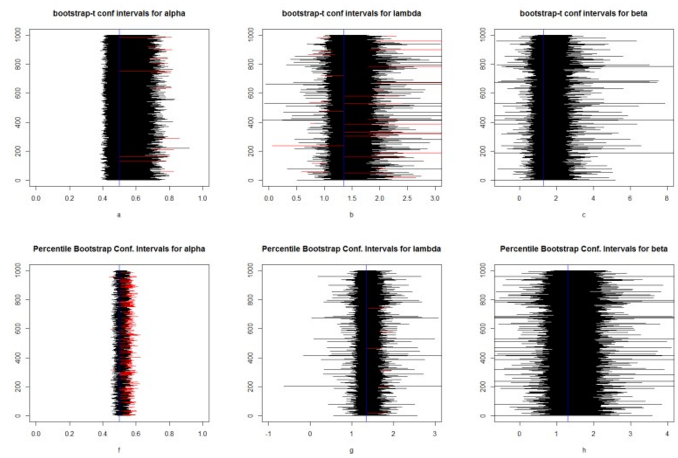


Fig. 7: The coverage probabilities of bootstrap-t and percentile bootstrap are shown in 6(a), 6(b), 6(c) and 6(f), 6(g), 6(h), respectively for n=100 under second set of initial values.

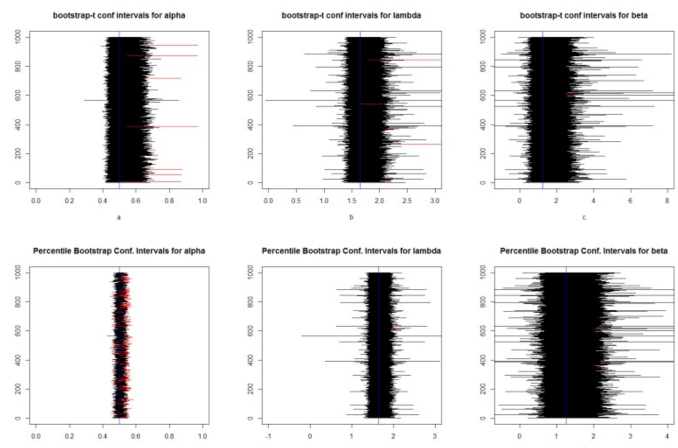
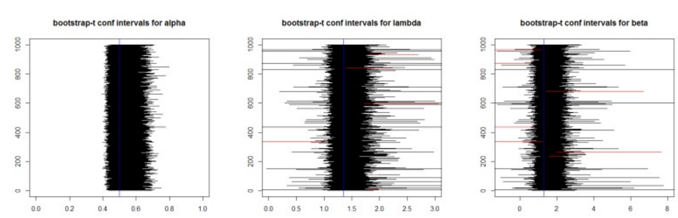


Fig. 8: The coverage probabilities of bootstrap-t and percentile bootstrap are shown in 7(a), 7(b), 7(c) and 7(f), 7(g), 7(h), respectively for n=150 under second set of initial values.



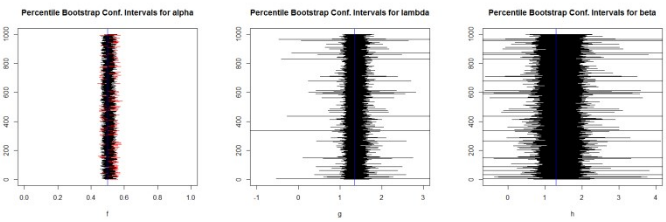


Fig. 9: The coverage probabilities of bootstrap-t and percentile bootstrap are shown in 8(a), 8(b), 8(c) and 8(f), 8(g), 8(h), respectively for n=200 under second set of initial values.