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BAYESIAN ESTIMATION OF FRECHET DISTRIBUTION UNDER ASYMMETRIC LOSS FUNCTIONS

Kamran Abbas and Yincai Tang

ABSTRACT

This paper develops the Bayesian estimators for the unknown parameters of Frechet distribution under different asymmetric loss functions. The Bayesian estimators cannot be obtained in closed forms. An approximate Bayesian approach is proposed using Lindley's approximation to obtain the Bayesian estimates. The approximate Bayes estimates obtained under the assumption of non-informative priors are compared with their maximum likelihood estimates via Monte Carlo simulation study. Two real data sets are analyzed for illustrative purposes.

1. INTRODUCTION

Frechet distribution was introduced by French mathematician Maurice Frechet (1878-1973), who identified possible limit distribution for the largest order statistic during 1927. The Frechet distribution have been used as an useful method for modeling and analyzing several extreme events such as accelerated life testing, earthquake, flood, rainfall, sea current and wind speed. Therefore, Frechet distribution is well suited to characterize random variables of large features. The random variable X is said to follow a Frechet distribution with parameters α and β if the cumulative distribution function (CDF) is given by

$$F(x \mid \alpha, \beta) = \exp\left[-(\frac{\beta}{x})^{\alpha} \right], \ x > 0, \ \alpha, \beta > 0, \tag{1}$$

Therefore, the corresponding probability density function (PDF) of the Frechet distribution is;

$$f(x \mid \alpha, \beta) = \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1} \exp\left[-(\frac{\beta}{x})^{\alpha} \right], \ x > 0. \tag{2}$$

where the parameter α determines the shape of the distribution and β is the scale parameter. Frechet distribution is equivalent to taking the reciprocal of values from a standard Weibull distribution. The Frechet distribution has been extensively studied by different authors; see, for example, Harlow (2002) suggested that it is important for modeling the statistical behavior of materials properties for a variety of engineering. Nadarajah and Kotz (2008) discussed the sociological models based on Frechet random variables. Further, Zaharim et al. (2009), abbas and Tang (2014) and Mubarak (2012) studied the application of Frechet distribution. Abbas and Tang (2013) estimated the parameters of Frechet distribution based on type-II censored samples. Moreover, Gumbel (1965) estimated the parameter of Frechet distribution. Abbas and Tang (2012) studied different estimation methods for Frechet distribution with known shape. Mann (1984) discussed the estimation procedures for the Frechet and the three-parameter Weibull distribution. The relationships between Frechet, Weibull and the Gumbel distribution were also discussed. Further, the maximum-likelihood and moment estimators as well as linearly based estimators involving only a few order statistics and properties for large and small samples were also discussed. However, Bayesian estimation under different loss functions is not frequently discussed. Perhaps, the Bayesian estimators under different loss functions involve integral expressions, which are not analytically solvable. In order to reduce the difficult integrals that are in the posterior distribution which cannot explicitly be obtained in close form. Therefore, we employed Lindley's approximation technique for solving such problems.

The main aim of this paper is to develop the Bayesian estimators of the parameters of Frechet distribution under different loss functions using non-informative priors. The rest of the paper unfolds as follows. In Section 2, the maximum likelihood estimators (MLEs) and observed Fisher information matrix for parameters are derived. Bayesian estimation under LINEX (linear exponential) loss function and general entropy loss function is discussed in Section 3. Monte Carlo simulation study is presented in Section 4 to assess the performance of Bayesian and MLEs in terms of mean squared error (MSE). Two real data sets are analyzed in Section 5 and finally conclusion is given in Section 6.

2. MAXIMUM LIKELIHOOD ESTIMATION

Let $X = X_1, X_2, ..., X_n$ be a random sample of size n from the Frechet distribution (2). The likelihood function of (α, β) is

$$L(\alpha,\beta) = \alpha^{n}\beta^{n\alpha}\prod_{i}^{n} X^{-(\alpha+1)} \exp\left[-\sum_{i=1}^{n}(\frac{\beta}{X_i})^{\alpha}\right].$$

Then the log-likelihood function can be written as

$$\log L = n\log\alpha + n\alpha\log\beta - (\alpha+1)\sum_{i=1}^{n}\log X_i - \sum_{i=1}^{n}\left(\frac{\beta}{X_i}\right)^{\alpha}, \qquad (3)$$

and

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + n\log\beta - \sum_{i=1}^{n}\log X_i - \sum_{i=1}^{n}\left(\frac{\beta}{X_i}\right)^{\alpha}\log(\frac{\beta}{X_i}) \qquad (4)$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{\alpha}{\beta}\left[n - \sum_{i=1}^{n}\left(\frac{\beta}{X_i}\right)^{\alpha}\right], \qquad (5)$$

From equation (5), $\hat{\beta}$ is obtained in terms of $\hat{\alpha}$ in the form

$$\hat{\beta}_{ML} = (n\sum_{i=1}^{n} X_i^{\hat{\alpha}})^{\frac{-1}{\hat{\alpha}}} \qquad (6)$$

Obviously it is not easy to obtain the closed form solution for the two non-linear equations (4) and (6). We use BFGS quasi-Newton optimization method to compute MLEs. Further, the observed Fisher information matrix is obtained by taking the second and mixed partial derivatives of $\log L$ with respect to α and β. We have

$$I(\alpha,\beta) = \begin{pmatrix} -\frac{\partial^2 \log L}{\partial \alpha^2} & -\frac{\partial^2 \log L}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \log L}{\partial \alpha \partial \beta} & -\frac{\partial^2 \log L}{\partial \beta^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \sum_{i=1}^{n}\left(\frac{\beta}{X_i}\right)^{\alpha}\left[\log\left(\frac{\beta}{X_i}\right)\right]^2$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = -\frac{n\alpha}{\beta^2} - \frac{\alpha(\alpha-1)}{\beta^2}\sum_{i=1}^{n}\left(\frac{\beta}{X_i}\right)^{\alpha\,2},$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \beta} = \frac{\partial^2 \log L}{\partial \beta \partial \alpha} = \frac{n}{\beta} - \frac{\alpha}{\beta}\sum_{i=1}^{n}\left(\frac{\beta}{X_i}\right)^{\alpha}\log\left(\frac{\beta}{X_i}\right) - \frac{1}{\beta}\sum_{i=1}^{n}\left(\frac{\beta}{X_i}\right)^{\alpha}$$

3. BAYESIAN ESTIMATION

In Bayesian estimation, we consider two types of loss functions. The first one is LINEX loss function, which is asymmetric. The LINEX loss function was introduced by Varian (1975), and several authors, such as Basu and Ebrahimi (1991), Rojo (1987) and Nassar and Eissa (2004), have used this loss function in different estimation problems. This function rises approximately exponentially on one side of zero and approximately linearly on the other side. The LINEX loss function can be expressed as

$$L(\Delta) \propto e^{c\Delta} - c\Delta - 1, \; c \neq 0, \tag{7}$$

where $\Delta = (\hat{\theta} - \theta)$ and $\hat{\theta}$ is an estimate of θ. The sign and magnitude of the shape parameter c represents the direction and degree of symmetry, respectively. Moreover, if $c > 0$ the overestimation is more serious compared to the under estimation and vice-versa. For c close to zero, the LINEX loss is approximately squared error loss and therefore almost symmetric. The posterior expectation of the LINEX loss function (7) is

$$E_\theta\left[L(\hat{\theta} - \theta)\right] \propto e^{c\hat{\theta}} E_\theta\left[e^{-c\theta}\right] - c(\hat{\theta} - E_\theta(\theta)\;) - 1, \tag{8}$$

where $E_\theta(\cdot)$ denotes the posterior expectation with respect to the posterior density of θ. The Bayes estimator of θ, denoted by $\hat{\theta}_{BL}$ under LINEX loss function, is the value $\hat{\theta}$, which minimizes (8). It is

$$\hat{\theta}_{BL} = -\frac{1}{c}\log\left\{E_\theta\left[e^{-c\theta}\right]\right\}, \tag{9}$$

provided that the expectation $E_\theta\left[e^{-c\theta}\right]$ exists and is finite. The problem of choosing the value of the parameter c is discussed in Calabria and Pulcini (1996). The second type of loss function is the generalization of the entropy loss, which is discussed by Dey and Liu (1992) and Dey (1987). The general entropy loss is defined as

$$L_{BE}(\hat{\theta} - \theta) \propto \left(\frac{\hat{\theta}}{\theta}\right)^c - c\log\left(\frac{\hat{\theta}}{\theta}\right) - 1, \tag{10}$$

where $\hat{\theta}$ is an estimate of θ. The Bayes estimator relative to the general entropy loss function is

$$\hat{\theta}_{BE} = [E(\theta^{-c})]^{-\frac{1}{c}}, \tag{11}$$

provided that $E(\theta^{-c})$ exists and is finite. For $c=1$ the Bayes estimator (11) coincides with the Bayes estimator under the weighted squared error loss function, and for $c=-1$ the Bayes estimator (11) coincides with the Bayes estimator under the squared error loss function. Further, the Bayesian estimators under LINEX loss function and general entropy loss function are presented in Appendix.

4. SIMULATION STUDY

Simulation study is conducted in order to compare the performance of the presented Bayesian estimators with the known non-Bayes estimator such as MLE. Since the Bayesian estimators of the model parameters cannot be obtained analytically, approximate Bayesian estimates are computed using Lindley approximation. In computing the estimates samples are generated from the Frechet distribution using the transformation $X_i = \beta(-\ln U_i)^{-\frac{1}{\alpha}}$, where U_i is uniformly distributed random variable and we replicated the process 5000 times for each sample size and the average of estimates is computed. For Bayesian estimators, we consider that α and β each have independent Gamma *(a_1,b_1)* and Gamma(a_2,b_2) priors. Further, the Bayesian estimators of α and β are also obtained using general uniform priors. We use the shape parameter $c=1.5$ and non-informative priors of both α and β, i.e., $a_1=a_2=b_1=b_2=0$. Comparison are made in terms of means and MSEs (with in parenthesis) and results are presented in Tables 1 and 2. Some of the points are quite clear regarding the performance of the estimators, which are summarized below.

1. As expected, it is observed that the performances of both Bayesian and MLEs become better when sample size increased. Sample size is varied to see the effect of small and large samples on the estimators considering fixed values of parameters. Moreover, it is observed that for large sample sizes, the Bayesian estimates and MLEs become closer in terms of MSEs, because for large sample sizes the effect of prior on posterior is minimal.
2. The Bayesian estimators under general entropy loss function and LINEX loss function perform better than MLEs obtained by using Gamma priors and general Uniform priors in terms of their MSEs with all the hyper parameters equal to zero, i.e., $a_1=a_2=b_1=b_2=0$. From Tables 1 and 2, we can see that in each scenario, the Bayesian estimators under assumption

of general entropy loss function and LINEX loss function outperform the MLEs since MSEs are significantly smaller whatever the value of shape parameter c.

3. Considering Tables 1 and 2, we noticed that Bayesian estimators worked remarkable well under general entropy loss function and LINEX loss function with respect to MSEs. As the sample size increases, the MSE values of the general entropy loss function and LINEX loss function decreases to smaller values than any of the others but it must be stated that the others also have their MSE values decreasing with increasing sample size.

Notations:

BLG**:** Bayesian estimators under LINEX loss function using Gamma prior.

BLU**:** Bayesian estimators under LINEX loss function using general Uniform prior.

BEG**:** Bayesian estimators under general entropy loss function using Gamma prior.

BEU**:** Bayesian estimators under general entropy loss function using general Uniform prior.

Table 1: Average estimates and corresponding MSEs (With in parenthesis) for α.

n	**Estimator** ↓ α→	1.0	1.5	2.0
20	ML	1.0768(0.0489)	1.6112(0.1050)	1.1622(0.2002)
	BLG	1.0405(0.0458)	1.5648(0.1014)	2.1029(0.1997)
	BLU	1.0619(0.0487)	1.6034(0.1047)	2.1593(0.2001)
	BEG	1.0248(0.0408)	1.5217(0.0843)	2.0178(0.1535)
	BEU	1.0459(0.0451)	1.5599(0.0932)	2.0737(0.1696)
30	ML	1.0490(0.0267)	1.5691(0.0603)	2.1023(0.1130)
	BLG	1.0244(0.0253)	1.5373(0.0585)	2.0611(0.1110)
	BLU	1.0377(0.0265)	1.5616(0.0602)	2.0970(0.1118)
	BEG	1.0150(0.0235)	1.5111(0.0523)	2.0094(0.0944)
	BEU	1.0282(0.0251)	1.5352(0.0557)	2.0448(0.1006)
50	ML	1.0318(0.0152)	1.5426(0.0316)	2.0554(0.0582)
	BLG	1.0169(0.0146)	1.5233(0.0308)	2.0300(0.0573)
	BLU	1.0246(0.0152)	1.5374(0.0314)	2.0506(0.0580)
	BEG	1.0116(0.0140)	1.5085(0.0288)	2.0018(0.0525)
	BEU	1.0192(0.0146)	1.5225(0.0299)	2.0215(0.0543)
80	ML	1.0162(0.0082)	1.5266(0.0198)	2.0324(0.0333)
	BLG	1.0069(0.0080)	1.5145(0.0193)	2.0164(0.0329)
	BLU	1.0115(0.0081)	1.5230(0.0197)	2.0289(0.0332)
	BEG	1.0037(0.0078)	1.5055(0.0186)	2.0014(0.0312)
	BEU	1.0083(0.0080)	1.5140(0.0190)	2.0114(0.0318)

100	ML	1.0129(0.0067)	1 .5209(0.0144)	2.0327(0.0268)
	BLG	1.0054(0.0065)	1.5112(0.0142)	2.0198(0.0264)
	BLU	1.0091(0.0066)	1.5180(0.0142)	2.0298(0.0266)
	BEG	1.0030(0.0063)	1.5042(0.0138)	2.0010(0.0251)
	BEU	1.0066(0.0065)	1.5109(0.0140)	2.0159(0.0257)

Table 2: Average estimates and corresponding MSEs (With in parenthesis) for β.

n	Estimator ↓ β →	1.0	1.5	2.0
20	ML	1.0452(0.0698)	1.5404(0.0640)	2.0367(0.0611)
	BLG	1.0404(0.0655)	1.5313(0.0615)	2.0333(0.0604)
	BLU	1.0450(0.0696)	1.5401(0.0638)	2.0360(0.0609)
	BEG	1.0449(0.0688)	1.5400(0.0636)	2.0364(0.0608)
	BEU	1.0451(0.0695)	1.5402(0.0637)	2.0366(0.0610)
30	ML	1.0376(0.0449)	1.5303(0.0420)	2.0223(0.0403)
	BLG	1.0349(0.0431)	1.5242(0.0409)	2.0199(0.0385)
	BLU	1.0357(0.0447)	1.5300(0.0416)	2.0219(0.0400)
	BEG	1.0363(0.0446)	1.5301(0.0417)	2.0221(0.0401)
	BEU	1.0366(0.0448)	1.5302(0.0418)	2.0222(0.0402)
50	ML	1.0222(0.0245)	1.5123(0.0232)	2.0143(0.0234)
	BLG	1.0216(0.0241)	1.5088(0.0229)	2.0127(0.0226)
	BLU	1.0219(0.0244)	1.5121(0.0230)	2.0140(0.0228)
	BEG	1.0220(0.0242)	1.5126(0.0231)	2.0141(0.0230)
	BEU	1.0221(0.0243)	1.5122(0.0231)	2.0142(0.0231)
80	ML	1.0129(0.0149)	1.5088(0.0153)	2.0087(0.0147)
	BLG	1.0126(0.0146)	1.5066(0.0145)	2.0077(0.0143)
	BLU	1.0127(0.0147)	1.5083(0.0147)	2.0085(0.0145)
	BEG	1.0128(0.0147)	1.5085(0.0148)	2.0086(0.0146)
	BEU	1.0128(0.0148)	1.5087(0.0148)	2.0087(0.0146)
100	ML	1.0107(0.0116)	1.5045(0.0115)	2.0049(0.0113)
	BLG	1.0101(0.0114)	1.5027(0.0113)	2.0041(0.0112)
	BLU	1.0106(0.0115)	1.5043(0.0114)	2.0047(0.0113)
	BEG	1.0106(0.0115)	1.5043(0.0114)	2.0048(0.0111)
	BEU	1.0107(0.0115)	1.5044(0.0114)	2.0048(0.0113)

5. DATA ANALYSIS

To illustrate the estimation techniques developed in this article, we consider the following two data sets.

Example 1. The data set is about Buoy-46005 which is available online from National Data Buoy Center (NDBC), situated in the North East Pacific: (46 N, 131 W). The data set consist of 21 (moderate sample size) observations and is presented in Table 3.

Table 3: Dataset (Buoy 46005): Yearly maxima of Hs (m).

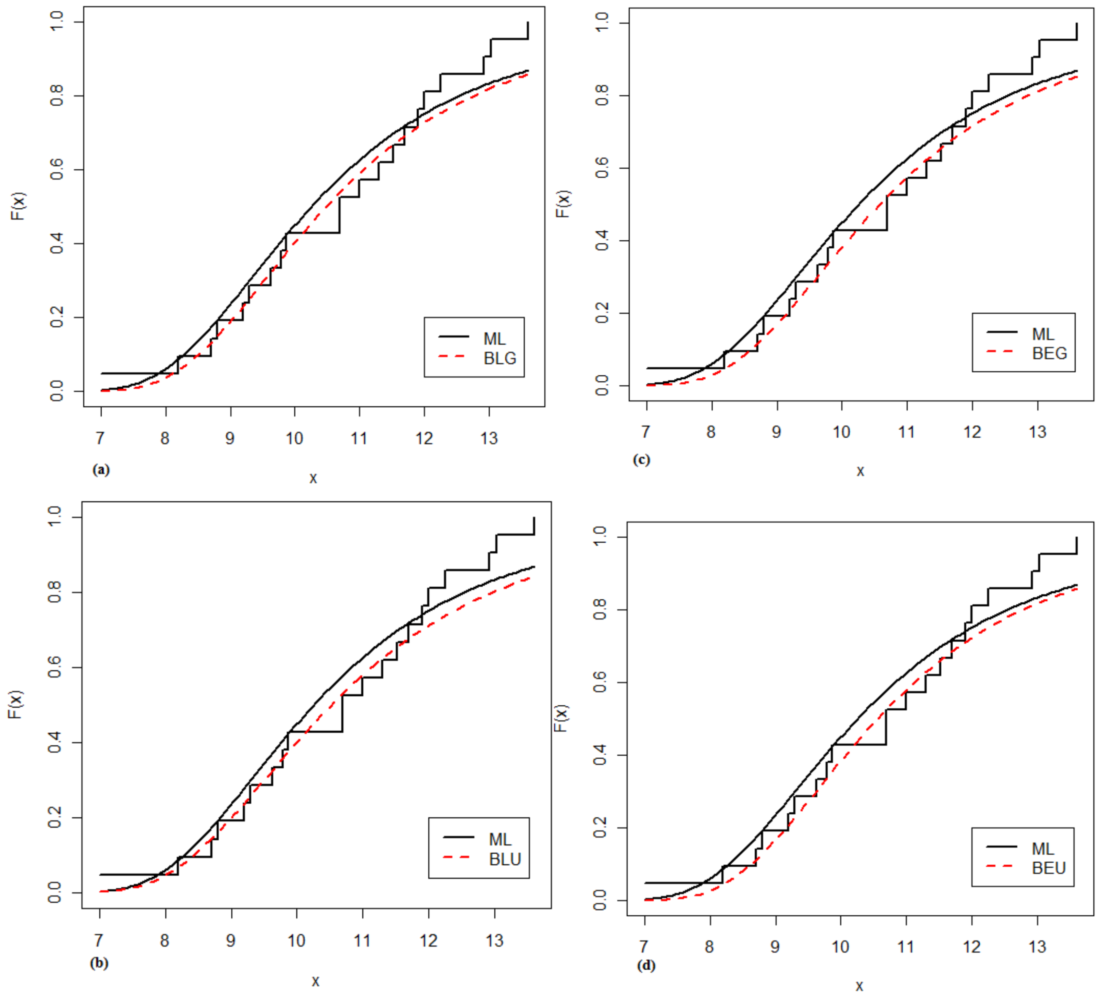
10.70	10.70	7.00	11.30	13.60	11.70	8.20
12.00	9.30	8.80	11.00	11.90	9.20	8.71
9.63	9.87	13.04	9.79	12.26	11.52	12.92

Table 4: Point estimates and standard deviations (SD) of α and β for Example 1.

Estimator	α	SD	β	SD
ML	5.6196	0.8661	9.6109	0.3968
BLG	5.7545	0.8531	9.8389	0.3950
BLU	5.4108	0.8661	9.8340	0.3968
BEG	5.7987	0.8535	9.9330	0.3951
BEU	5.9226	0.8661	9.9284	0.3968

The point estimates of α and β obtained by all the methods are presented in Table 4. In this case, all the estimates are close to the MLE. Since the Bayesian estimators of the model parameters cannot be obtained analytically, approximate Bayesian estimates are obtained numerically using Lindley approximation with $c = 0.5$. We use non-informative priors with all the hyper parameters equal to zero, i.e., $a_1 = a_2 = b_1 = b_2 = 0$, for Bayes estimates. We also plot the empirical and fitted CDFs using these different methods of estimation in Figure1.

Empirical and fitted CDFs using different methods of estimation: (a) BLG, (b)



BLU, (c) BEG and (d) BEU.

Example 2. This example is from Nichols and Padgett (2006), which represents the breaking stress of carbon fibres (in Gba). The data consist of 100 (large sample size) observations. The data are presented in Table 5.

Table 5: Breaking stress of carbon fibres (in Gba).

3.7	2.73	3.6	3.27	1.47	4.42
3.22	3.28	1.87	4.9	2.43	2.97
2.53	2.93	3.39	4.2	2.55	3.31
3.56	2.35	2.59	2.81	2.17	1.92
2.97	0.98	4.91	1.84	3.19	0.81
1.59	1.22	1.71	1.17	2.48	3.51
1.25	1.84	3.68	0.85	2.79	2.03
1.08	1.61	1.89	2.82	3.65	2.85
2.74	2.5	3.11	2.87	3.11	2.56
1.69	3.09	3.15	3.75	2.95	1.41
2.67	3.22	2.81	3.33	3.31	5.56
3.15	2.55	2.38	2.77	2.83	2.17
1.36	2.76	3.68	1.59	1.57	1.8
2.0	1.12	2.17	5.08	1.18	3.68
4.38	0.39	2.48	1.61	4.7	1.73
2.03	2.12	2.88	2.05	1.69	1.57
2.41	3.19	3.39	2.96		

Table 6: Point estimates and standard deviations (SD) of α and β for Example 2.

Estimator	α	SD	β	SD
ML	1.7690	0.1119	1.8916	0.1138
BLG	1.7528	0.1114	1.8886	0.1134
BLU	1.7577	0.1119	1.8929	0.1137
BEG	1.7467	0.1112	1.9007	0.1135
BEU	1.7545	0.1118	1.9052	0.1138

For this example, posterior means and posterior standard deviations (SD) of the two parameters α and β are computed assuming the non-informative priors for each parameter under LINEX loss function and general entropy loss function and results are summarized in Table 6. We choose the shape parameter c=1.0 and it is observed that the Bayesian estimates under general entropy loss function and LINEX loss function are close to the MLEs.

5. CONCLUSION

In this paper, we present Bayesian estimates of the two parameter of Frechet distribution using various loss functions and non-informative priors. It is observed that the Bayesian estimators cannot be obtained in explicit forms. Lindley's approximation is used to obtain the Bayesian estimates numerically, and it is concluded that the approximation works very well even for small sample sizes though the computation of Lindley's approximation based on the MLEs. Comparisons are made between the different estimators based on a simulation study and real data sets considering different values of shape parameter. Simulations showed that the Bayesian estimators under general entropy loss function and LINEX loss function perform better than the MLEs. However, it is observed that for large sample sizes the Bayesian and MLEs become closer in terms of their MSEs and standard deviations.

REFERENCES

Abbas, K. & Tang, Y. (2013) Estimation of parameters for Frechet distribution based on type-II censored samples. *Caspian Journal of Applied Sciences Research*, 2(7), pp 36-43.

Abbas, K. & Tang, Y. (2014) Objective Bayesian analysis of the Frechet stress-strength model. *Statistics and Probability Letters,84,* pp *169-174.*

Abbas, K. & Tang, Y. (2012) Comparison of estimation methods for Frechet distribution with known shape. *Caspian Journal of Applied Sciences Research,* 1(10), pp 58-64.

Al-Baidhani, F. A. & Sinclair, C. D. (1987) Comparison of methods of estimation of parameters of the Weibull distribution. *Commun. Statist. Simula*., 16, pp 373-384.

Basu, A. P. & Ebrahimi, N. (1991) Bayesian approach to life testing and reliability estimation using asymmetric loss function. *Journal of Statistical Planning and Inference*, pp 347-363.

Calabria, R. & Pulcini, G. (1996) Point estimation under asymmetric loss functions for left-truncated exponential samples. *Communications in Statistics - Theory and Methods 25*(3), pp 585-600.

Dey, D. K., Gosh, M., & Srinivasan, C. (1987) Simultaneous estimation of parameters under entropy loss. *Journal of Statistical Planning and Inference*, pp 347-363.

Dey, D. K. & Liao Liu P. S. (1992) On comparison of estimators in a generalized life model. *Microelectronics Reliability 32*, pp 207-221.

Frechet, M. (1927) Sur la loi de probabilite de lecart maximum. *Ann. Soc. Polon. Math*, 6(93).

Gumbel, E. J. (1965) A quick estimation of the parameters in Frechet's distribution. *Review of the International Statistical Institute*. 33(3).

Harlow, D. G. (2002) Applications of the Frechet distribution function. *International Journal of Material and Product Technology,* 5(17), pp 482-495.

Hossain, A. & Howlader, H. A. (1996) Unweighted least squares estimation of Weibull parameters. *Journal of Statistical Computation and Simulation*, 54, pp 265-271.

Lindley, D. V. (1980) Approximate Bayesian method. *Trabajos de Estadistica 31*, pp 223–237.

Mann, N. R. (1984) Statistical estimation of parameters of the Weibull and Frechet distributions.

Statistical Extremes and Application, NATO ASI Series, 131, pp 81-89.

Mubarak, M. (2012) Parameter estimation based on the Frechet progressive type-II censored data with binomial removals. *International Journal of Quality, Statistics and Reliability*, 2012, Article ID 245910

Nadarajah, S. & Kotz, S. (2008) Sociological models based on Frechet random variables. *Quality and Quantity,* 42, pp 89-95.

Nassar, M. M. & Eissa, F. H. (2004) Bayesian estimation for the exponentiated Weibull model. *Communications in Statistics - Theory and Methods 33*(10), pp 2343-2362.

Nichols, M. D. & Padgett, W. J (2006) A bootstrap control chart for Weibull percentiles. *Quality and Reliability Engineering International 22*, pp 141-151.

Rojo, J. (1987) On the admissibility of with respect to the LINEX loss function. *Communications in Statistics - Theory and Methods 16*(12), pp 3745-3748.

Varian, H. R. (1975) A Bayesian Approach to Real Estate Assessment. Amsterdam: North Holland, pp 195-208.

Zaharim, A., Najidi, S. K., Razali, A. M. & Sopian, K. (2009, February, 24-26) Analyzing Malaysian wind speed data using statistical distribution. *Proceedings of the 4th IASME / WSEAS International Conference on Energy and Environment.* University of Cambridge.

6. APPENDIX

For Bayesian estimation, we need prior distribution of α and β. Assuming that α and β each have independent Gamma (a_1, b_1) and Gamma (a_2, b_2) priors respectively for ($a_1,b_1,a_2,b_2>0$), *i.e.*, $\pi_1(\alpha)\propto\alpha^{a_1-1}e^{-b_1\alpha}$ *and* $\pi_2(\beta)\propto\beta^{a_2-1}e^{-b_2\beta}$. Based on the priors, the joint posterior density of α and β can be written as

$$f(\alpha,\beta/x)=\frac{L(data\,|\,\alpha,\beta)\pi_1(\alpha)\pi_2(\beta)}{\int_0^\infty\int_0^\infty L(data|\alpha,\beta)\pi_1(\alpha)\pi_2(\beta)\,d\alpha d\beta} \tag{12}$$

Therefore, the Bayesian estimator of any function of α and β say $g(\alpha,\beta)$ under the LINEX loss function is

$$\hat{g}(\alpha,\beta\,|\,x)=E_{(\alpha,\beta|data)}[g(\alpha,\beta)]=\frac{\int_0^\infty\int_0^\infty g(\alpha,\beta)\,L(data|\,\alpha,\beta)\pi_1(\alpha)\pi_2(\beta)\,d\alpha d\beta}{\int_0^\infty\int_0^\infty L(data|\,\alpha,\beta)\pi_1\ (\alpha)\pi_2\ (\beta)d\alpha d\beta} \tag{13}$$

It is not possible for (13) to have a closed form. Therefore, we adopt Lindley's approximation (1980) procedure to approximate the ratio of the two integrals such as (13), which can be evaluated as

$$\hat{g}=g\left(\hat{\alpha},\hat{\beta}\right)+\frac{1}{2}\left[\sum_{i=1}^{2}\sum_{j=1}^{2}v_{ij}\ s_{ij}+v_{30}A_{12}+v_{03}A_{21}+v_{21}B_{12}+v_{12}B_{21}\right]+P_1C_{12}+P_2C_{21} \tag{14}$$

where $v_{ij}=\dfrac{\partial^{i+j}l(\alpha,\beta)}{\partial\alpha^i\,\partial\beta^j},\ i,j=0,1,2,3,\ \ i+j=3,$

$$p_1=\frac{\partial\log\pi(\alpha,\beta)}{\partial\alpha},P_2=\frac{\partial\log\pi(\alpha,\beta)}{\partial\beta}$$

$$v_{12}=\frac{\partial^2 g(\alpha,\beta)}{\partial\alpha\partial\beta},v_{21}=\frac{\partial^2 g(\alpha,\beta)}{\partial\beta\partial\alpha}$$

$$v_{11}=\frac{\partial^2 g(\alpha,\beta)}{\partial\alpha^2},v_{22}=\frac{\partial^2 g(\alpha,\beta)}{\alpha\beta^2}$$

$$v_1=\frac{\partial g(\alpha,\beta)}{\partial\alpha},v_2=\frac{\partial g(\alpha,\beta)}{\partial\beta}$$

$$A_{ij}=\left(v_i s_{ii}+v_j s_{ij}\right)s_{ii}$$

$$B_{ij}=3v_i s_{ii}\,s_{ij}+v_j\left(s_{ii}\,s_{jj}+2s_{ij}^2\right)$$

$$C_{ij}=v_i S_{ii}+v_j s_{ji},i,j=1,2,$$

where

$l(.)$ is the log likelihood function of the observed data, s_{ij} is the *(i ,j)th* element of the inverse of Fisher's information matrix. Therefore, the approximate Bayesian estimators of α and β under LINEX loss function are

$$\hat{\alpha}_{BLG} = -\frac{1}{c}\log\left[\begin{array}{l} e^{-c\hat{\alpha}} + \frac{1}{2}\left\{\begin{array}{l} c^2 e^{-c\hat{\alpha}} s_{11} - ce^{-c\hat{\alpha}} v_{30} s_{11}^2 - ce^{-c\hat{\alpha}} v_{03} s_{22} s_{11} \\ + 3 s_{12} s_{11} v_{21} ce^{-c\hat{\alpha}} - v_{12} ce^{-c\hat{\alpha}} s_{22} s_{11} + 2 s_{12}^2 \end{array}\right\} \\ - ce^{-c\hat{\alpha}} s_{11}\left(\frac{a_1 - 1}{\hat{\alpha}} - b_1\right) - ce^{-c\hat{\alpha}} s_{12}\left(\frac{a_2 - 1}{\hat{\beta}} - b_2\right) \end{array}\right]$$

$$\hat{\beta}_{BLG} = -\frac{1}{c}\log\left[\begin{array}{l} e^{-c\hat{\beta}} + \frac{1}{2}\left\{\begin{array}{l} c^2 e^{-c\hat{\beta}} s_{22} - ce^{-c\hat{\beta}} v_{30} s_{12} s_{11} - ce^{-c\hat{\beta}} v_{03} s_{22}^2 \\ - (s_{22} s_{11} + 2 s_{12}^2) v_{21} ce^{-c\hat{\beta}} - 3 s_{12} s_{22} v_{12} ce^{-c\hat{\beta}} \end{array}\right\} \\ - ce^{-c\hat{\alpha}} s_{12}\left(\frac{a_1 - 1}{\hat{\alpha}} - b_1\right) - ce^{-c\hat{\beta}} s_{22}\left(\frac{a_2 - 1}{\hat{\beta}} - b_2\right) \end{array}\right]$$

The Bayesian estimators of α and β under general entropy loss function are

$$\hat{\alpha}_{BEG} = \left[\begin{array}{l} \hat{\alpha}^{-c} + \frac{1}{2}\left\{\begin{array}{l} c(c+1)\hat{\alpha}^{-(c+2)} s_{11} - c\hat{\alpha}^{-(c+1)} v_{30} s_{11}^2 - c\hat{\alpha}^{-(c+1)} v_{03} s_{22} s_{11} \\ + 3 s_{12} s_{11} v_{21} c\hat{\alpha}^{-(c+1)} - v_{12} cc\hat{\alpha}^{-(c+1)} s_{22} s_{11} + 2 s_{12}^2 \end{array}\right\} \\ - c\hat{\alpha}^{-(c+1)} s_{11}\left(\frac{a_1 - 1}{\hat{\alpha}} - b_1\right) - c\hat{\alpha}^{-(c+1)} s_{12}\left(\frac{a_2 - 1}{\hat{\beta}} - b_2\right) \end{array}\right]^{-\frac{1}{c}},$$

$$\hat{\beta}_{BEG} = \left[\begin{array}{l} \hat{\beta}^{-c} + \frac{1}{2}\left\{\begin{array}{l} c(c+1)\hat{\beta}^{-(c+2)} s_{22} - c\hat{\beta}^{-(c+1)} v_{30} s_{12} s_{11} - c\hat{\beta}^{-(c+1)} v_{03} s_{22}^2 \\ - (s_{22} s_{11} + 2 s_{12}^2) v_{21} c\hat{\beta}^{-(c+1)} - 3 s_{12} s_{22} v_{12} c\hat{\beta}^{-(c+1)} \end{array}\right\} \\ - c\hat{\beta}^{-(c+1)} s_{21}\left(\frac{a_1 - 1}{\hat{\alpha}} - b_1\right) - c\hat{\beta}^{-(c+1)} s_{22}\left(\frac{a_2 - 1}{\hat{\beta}} - b_2\right) \end{array}\right]^{-\frac{1}{c}}$$

where

$$v_{30} = \frac{2n}{\hat{\alpha}^a} + \sum_{i=1}^{n}\left(\frac{\beta}{X_i}\right)^{\hat{\alpha}}\left[\log\left(\frac{\hat{\beta}}{X_i}\right)\right]^3, \quad v_{03} = \frac{2n\hat{\alpha}}{\hat{\beta}^a} - \frac{\hat{\alpha}(\hat{\alpha}-1)}{\hat{\beta}^2}\sum_{i=1}^{n}\left(\frac{\beta}{X_i}\right)^{\hat{\alpha}},$$

$$v_{21} = -\frac{\hat{\alpha}}{\hat{\beta}}\sum_{i=1}^{n}\left(\frac{\beta}{X_i}\right)^{\hat{\alpha}}\left[\log\left(\frac{\hat{\beta}}{X_i}\right)\right]^2 - \frac{2}{\hat{\beta}}\sum_{i=1}^{n}\left(\frac{\beta}{X_i}\right)^{\hat{\alpha}}\log\left(\frac{\beta}{X_i}\right)$$

$$s_{11} = \frac{v}{uv - w^2}, \; s_{22} = \frac{u}{uv - w^2}, \; s_{12} = s_{21} = \frac{w}{uv - w^2},$$

$$u = \frac{n}{\hat{\alpha}^2} + \sum_{i=1}^{n}\left(\frac{\hat{\beta}}{X_i}\right)^{\hat{\alpha}}\left[\log\left(\frac{\hat{\beta}}{X_i}\right)\right]^2$$

$$w = -\frac{n}{\hat{\beta}} + \frac{\hat{\alpha}}{\hat{\beta}}\sum_{i=1}^{n}\left(\frac{\hat{\beta}}{X_i}\right)^{\hat{\alpha}}\log\left(\frac{\hat{\beta}}{X_i}\right) + \frac{1}{\hat{\beta}}\sum_{i=1}^{n}\left(\frac{\hat{\beta}}{X_i}\right)^{\hat{\alpha}}, \; v = \frac{n\hat{\alpha}}{\hat{\beta}^2} + \frac{\hat{\alpha}(\hat{\alpha}-1)}{\hat{\beta}^2}\sum_{i=1}^{n}\left(\frac{\hat{\beta}}{X_i}\right)^{\hat{\alpha}},$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of α and β. Similarly, the approximate Bayesian estimators of α and β under LINEX loss function and general entropy loss function using general uniform priors *i.e.*, $\pi_3(\alpha) \propto \alpha^{-a_1}$ and $\pi_4(\beta) \propto \beta^{-b_1}$ can be obtained.

Kamran Abbas[*1, 2], **Yincai Tang**[1]

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[1]School of Finance and Statistics,
East China Normal University,
Shanghai 200241, China.
[2]Department of Statistics,
University of Azad Jammu and Kashmir,
Muzaffarabad, Pakistan.

E-mail: *kamiuajk@gmail.com*,
yctang@stat.ecnu.edu.cn