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QUASI-BAYESIAN ESTIMATION OF LORENZ CURVE AND GINI-INDEX IN THE POWER MODEL

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ABSTRACT

In this article, we estimate the shape and scale parameters, Lorenz curve and Gini-index for the power function distribution using quasi-likelihood and quasi-Bayesian methods. Quasi-Bayes estimators have been developed under squared error loss function as well as under LINEX loss function. We demonstrate the use of the proposed estimation procedure with the U. S. income data for the period 1913-2010. Our proposed quasi-likelihood and quasi-Bayesian estimators are compared with the ML estimators proposed by Belzunce et al. (1998).

1. INTRODUCTION

The Lorenz curve is a graphical representation, usually adopted to depict the distribution of income and wealth in a population. Let *X* be a continuous non-negative random variable representing income of a society or community with distribution function *F(x)*, Gastwirth (1971) defined the Lorenz curve corresponding to *X* as

$$L(p)=\frac{1}{E(X)}\int_0^p Q(u)\,\mathrm{du}, \quad 0\le p\le 1,\ E(X)<\infty, \tag{1}$$

where *Q(u)* is the quantile function. Clearly *L(p)* gives the fraction of total income that the holders of the lowest p^{th} fraction of income possesses. Most of the measures of income inequality are derived from the Lorenz curve. An important measure of inequality is the Gini-index associated with *F* and is defined by

$$G=1-2\int_o^1 L(p)\,\mathrm{dp}. \tag{2}$$

This is a ratio of the area between the Lorenz curve and the 45^0 line to the area under the 45^0 line. In general, these notions are useful for measuring concentration and inequality in the distributions of resources and in size distributions. For the applications of Lorenz curve and Gini-index we refer to Moothathu (1985, 1990) and the references therein. These measures have also been found applications in reliability theory. For more details, see Chandra and Singpurwalla (1981), Sathar et al. (2007) and Sathar and Nair (2009).

Moothathu (1991) has derived uniformly minimum variance unbiased estimators of Lorenz curve and Gini-index for lognormal and Pareto distributions respectively. Sathar et al. (2005), Sathar and Suresh (2006) and Sathar and Jeevanand (2009) have discussed the Bayesian estimation of Lorenz curve and Gini-index of the Pareto and exponential distributions respectively. For recent works on the estimation of Lorenz curve and Gini-index, we refer to Hasegawa and Kozumi (2003), Rohde (2009), Sarabia et al. (2010), Fellman (2012) and the references therein.

In many cases of practical work in economics and social sciences, models are approximates of the true data generation process. Such an approximate model is often misspecified or only partially correct. A useful approach in this case is the quasi maximum likelihood method studied by Huber (1967) and White (1982), among others that generalize the traditional maximum likelihood (ML) method to the case of possible model misspecification. An important extension proposed by Wedderburn (1974) is the quasi-likelihood function, which requires assumptions on the first two moments only, rather than the entire distribution of the data. The quasi-likelihood approach is useful because in many situations the exact distribution of the observations is unknown. Moreover, a quasi-likelihood function has statistical properties similar to those of a log-likelihood function. For recent works on quasi-likelihood estimation, we refer to Annis (2007), Kim (2014), Elshahat and Ismail (2014) and the references therein. The present paper focuses attention on estimating the parameters, Lorenz curve and Gini-index for power function distribution using quasi-likelihood and quasi-Bayesian estimation.

The present article is organized as follows. In Section 2, we consider the model and quasi-likelihood estimates of the shape and scale parameters, Lorenz curve and Gini-index of the power distribution. Section 3 deals with the quasi-Bayesian estimation of the parameters, Lorenz curve and Gini-index of the power distribution when both scale and shape parameters of the distribution are unknown. In Section 4, we demonstrate the use of the proposed estimation procedure with the U. S. income data for the period 1913-2010. Based on a Monte Carlo simulation study, comparisons are made between the proposed estimators and ML estimators and are presented in Section 5. We utilize Section 6 for some concluding remarks and for the description of the summary of the results developed in this work.

2. THE MODEL AND QUASI-LIKELIHOOD ESTIMATES

Among the models which provide a better fit to the whole income "distribution, there". are the Singh-Maddala model and the Dagum Model Type-I (Dagum, 1980). Belzunce et al. (1998) observed that for low values of the parameters of Singh-Maddala distribution, the right residual income follows, asymptotically, the power function distribution. Therefore in the study of poverty, it is important to consider the estimation of the Lorenz curve and the Gini index for this model.

Let $\{X_i\}, i = 1, 2, ..., n$ be a sequence of independent and identically distributed random variables from a power function distribution with pdf

$$f(x,\alpha,\beta) = \frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{-(\alpha-1)}, \ 0 < x < \beta, \ \alpha, \beta > 0, \tag{3}$$

where β and α are scale and shape parameters, respectively. The Lorenz curve and the Gini-index for (3) can be simplified respectively as

$$L = p^{1+\alpha^{-1}}, \ 0 < p < 1, \tag{4}$$

and

$$G = (1+2\alpha)^{-1}. \tag{5}$$

Bagchi and Sarkar (1986) discussed the Bayes interval estimation for the shape parameter of the power distribution. For recent works on estimation of the parameters of the power function distribution, we refer to Sinha et al. (2008), Sultan et al. (2014) and the references therein.

Belzunce et al. (1998) obtained the ML estimates of the parameters α, β, the Lorenz curve and the Gini-index and are given respectively as

$$\hat{\alpha}_{MLE} = \frac{1}{\hat{\lambda}_1}, \ \hat{\beta}_{MLE} = \exp\left(-Y_{(1)}\right), \ \hat{L}_{MLE} = p^{1+\hat{\lambda}_1} \text{ and } \hat{G}_{MLE} = \hat{\lambda}_2, \tag{6}$$

where $\hat{\lambda}_1 = S, \ \hat{\lambda}_2 = \frac{S}{2+S}, \ S = \frac{1}{n}\sum_{j=1}^{n}\left(Y_j - Y_{(1)}\right), \ Y_{(1)} = \min\left(Y_i\right)$ and $Y_i = -\ln X_i$.

2.1 Quasi-Likelihood Estimation

In this section, we derived the maximum quasi-likelihood estimates for the unknown parameters of the power function distribution. The quasi-likelihood function was introduced by Wedderburn (1974) to be used for estimating the unknown parameters in generalized linear models when only the mean-variance relationship is specified. Wedderburn defined the quasi-likelihood function as

$$Q(x,\mu)=\int_{\mu}\frac{x-\mu}{V(\mu)}\mathrm{d}\mu+o(x), \tag{7}$$

where $\mu=E(x)$, $V(\mu)=Var(x)$ and $o(x)$ is some function of x only. The variance assumption is generalized to $Var(x)=\phi\,V(\mu)$, where the variance function $V(.)$ is assumed to be known and the parameter ϕ may be unknown. The quasi-likelihood function has properties similar to those of the log-likelihood function. For a sample $\underline{x}=(x_1,x_2,...,x_n)$ of size n from (3), the quasi-likelihood function simplifies to

$$Q(x,\alpha,\beta)=-\frac{\alpha+1}{\alpha\beta}v-\log\left(\frac{\alpha\beta}{\alpha+1}\right)^{n}, \text{ where } v=\sum_{i=1}^{n}x_i. \tag{8}$$

The natural exponent of $Q(x,\alpha,\beta)$ is the likelihood function and is given as

$$l(\underline{x}\,|\,\alpha,\beta)=\left(\frac{\alpha+1}{\alpha\beta}\right)^{n}\exp\left[-\frac{(\alpha+1)v}{\alpha\beta}\right]. \tag{9}$$

Using (9), the maximum quasi-likelihood estimates of the parameters α and β, denoted by $\hat{\alpha}_{MQL}$ and $\hat{\beta}_{MQL}$, and are simplified respectively as

$$\hat{\alpha}_{MQL}=\frac{v}{n\beta-v}, \tag{10}$$

and

$$\hat{\beta}_{MQL}=\frac{(\alpha+1)}{n\alpha}v. \tag{11}$$

The maximum quasi-likelihood estimators for the Lorenz curve and Gini-index, denoted by $\hat{L}_{MQL}$ and $\hat{G}_{MQL}$ can be obtained from (4) and (5) after replacing α and β by $\hat{\alpha}_{MQL}$ and $\hat{\beta}_{MQL}$, respectively.

3. QUASI-BAYESIAN ESTIMATION

Recently, the Bayesian approach has received large attention for analyzing statistical data and has been often proposed as a valid alternative to traditional statistical perspectives. The Bayesian approach allows prior subjective knowledge on parameters to be incorporated into the inferential procedure.

Hence, Bayesian methods usually require less sample data to achieve the same quality of inferences than methods based on sampling theory, which becomes extremely important in case of expensive testing procedures.

Bayesian statistics provide a conceptually simple process for updating uncertainty in the light of evidence. From a decision-theoretic view point, in order to select the 'best' estimator, a loss function must be specified and is used to represent a penalty associated with each of the possible estimates. Nonetheless, it has been observed that in certain situations when one loss is the true loss function, Bayes estimate under another loss function performs better than the Bayes estimate under the true loss. Therefore, we consider symmetric as well as asymmetric loss functions for getting better understanding in our Bayesian analysis.
Squared error loss function (SELF) is a commonly used loss function and is defined as

$$L\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^2,$$

which is a symmetrical loss function that assigns equal losses to over estimation and underestimation. The Bayes estimator under the above loss function is the posterior mean given by

$$\hat{\theta}_{BS}=E_{\theta|x}\left(\theta\right), \tag{12}$$

and the posterior expected loss of θ, denoted by $R\left(\theta,\hat{\theta}_{BS}\right)$ under the SELF is the posterior variance of θ and is given by

$$R\left(\theta,\hat{\theta}_{BS}\right)=E_{\theta|x}\left(\theta^2\right)-\left[E_{\theta|x}\left(\theta\right)\right]^2. \tag{13}$$

The SELF is a frequently used symmetric loss function, because it does not lead to extensive numerical computation. No doubt, the use of squared error loss function is well justified when the loss is symmetric in nature. Its use is also very popular, perhaps, because of its mathematical simplicity.

In most situations of interest, overestimating is more harmful than underestimating. Due to this, we use the linear exponential (LINEX) loss function (LLF), the most frequently used asymmetric loss function. The LLF is introduced by Varian (1975) in response to the criticisms of SELF function and is defined by

$$L\left(\hat{\theta},\theta\right)=b\left[e^{a\left(\hat{\theta}-\theta\right)}-a\left(\hat{\theta}-\theta\right)-1\right],\ a\neq 0,\ b>0, \tag{14}$$

where a and b are the shape and scale parameters of the loss function (14). Obviously, the nature of LINEX loss function changes according to the choice of a, and assume, in what follows, that $b\equiv 1$ in (14). It is to be noted as a tends to zero, the LINEX loss is approximately squared error loss and therefore almost symmetric. Writing $M_{\theta|x}\left(t\right)=E_{\theta|x}\left[e^{t\theta}\right]$ for the moment generating function of the posterior distribution of θ, it is easy to verify that the value of $\hat{\theta}$ that minimizes $E_{\theta|x}\left[L\left(\hat{\theta},\theta\right)\right]$ in (14) is

$$\hat{\theta}_{BL}=\frac{-1}{a}\ln M_{\theta|x}\left(-a\right)=\frac{-1}{a}\ln\left[E_{\theta|x}\left(e^{-a\theta}\right)\right], \tag{15}$$

provided $M_{\theta|x}\left(.\right)$ exists and is finite. The posterior expected loss of θ, denoted by $R\left(\theta,\hat{\theta}_{BL}\right)$ under the LLF is given by

$$R\left(\theta,\hat{\theta}_{BL}\right)=\ln\left[E_{\theta|x}\left(e^{-a\theta}\right)\right]+aE_{\theta|x}\left(\theta\right). \tag{16}$$

Zellner (1986) discussed the Bayesian estimation and prediction using LLF. For recent works on the Bayes estimation using LLF, we refer to Pandey and Rao (2009), Ahmadi *et al.* (2010) and the references therein.

3.1 Estimation When α, And β, Unknown

The most general and perhaps a more realistic situation is when both the shape and scale parameters of the distribution are unknown. In this section, we attend the problem of estimation of α, β, L and G when α and β are unknown. In Bayesian inference, a prior probability distribution, often called simply the prior, of an uncertain parameter θ or latent variable is a probability distribution that expresses uncertainty about θ before the data are taken into account. The parameters of a prior distribution are called hyper-parameters, to distinguish them from the parameters $\left(\Theta\right)$ of the model. The Bayesian deduction requires appropriate choice of priors for the parameters. Arnold and Press (1983) pointed out that, from a strict Bayesian viewpoint, there is clearly no way in which one can say that one prior is better than any other. Presumably one has one's own subjective prior and must live with all of its lumps and bumps. But if we have enough information about the parameters then it is better to make use of the informative prior which may certainly be preferred over all other choices.

Here we suggest the joint conjugate prior distribution for the parameters and is given by

$$g(\alpha,\beta)=\frac{\tau^r}{\Gamma(r)}\alpha^{r-2}e^{-\alpha\tau}. \tag{17}$$

Combining (2.7) and (3.6), the joint posterior density is obtained as

$$f(\alpha,\beta\,|\,\underline{x})\propto\frac{\tau^r}{\Gamma(r)}\alpha^{r-n-2}\left(\frac{\alpha+1}{\beta}\right)^n\exp\left[-\alpha\tau-\frac{(\alpha+1)v}{\alpha\beta}\right]. \tag{18}$$

From (18) integrating out β, the marginal posterior density of α is obtained as

$$f(\alpha\,|\,\underline{x})=\frac{\alpha^{M-1}}{C_1(0)}(\alpha+1)^n\exp(-\alpha\tau)\int_0^{\sigma}\frac{1}{\beta^n}\exp\left[-\frac{(\alpha+1)v}{\alpha\beta}\right]\mathrm{d}\beta, \tag{19}$$

where $$C_1(d)=\int_0^{\infty}\int_0^{\sigma}\alpha^{M+d-1}\left(\frac{\alpha+1}{\beta}\right)^n\exp\left[-\alpha\tau-\frac{(\alpha+1)v}{\alpha\beta}\right]\mathrm{d}\beta\mathrm{d}\alpha, \tag{20}$$

with $M=r-n-1$, $\sigma=\max\left(\sigma_0,X_{(n)}\right)$ and $X_{(n)}=\max\left(X_1,X_2,\ldots,X_n\right)$. The symbol *C* with various suffixes stands for the normalizing constants. Similarly, the marginal posterior density of β is obtained as

$$f(\beta\,|\,\underline{x})=\frac{1}{C_2(0)}\frac{1}{\beta^n}\int_0^{\infty}\alpha^{M-1}(\alpha+1)^n\exp\left[-\alpha\tau-\frac{(\alpha+1)v}{\alpha\beta}\right]\mathrm{d}\alpha, \tag{21}$$

where $$C_2(d)=\int_0^{\sigma}\int_0^{\infty}\beta^{d-n}\alpha^{M-1}(\alpha+1)^n\exp\left[-\alpha\tau-\frac{(\alpha+1)v}{\alpha\beta}\right]\mathrm{d}\alpha\mathrm{d}\beta. \tag{22}$$

Let the Lorenz curve *L(p)* be a parameter itself and denote it by L for simplicity. Replacing α in (19) in terms of *L* by that (4), we get the posterior density of the Lorenz curve as

$$f(L\,|\,\underline{x})=\frac{(A_L)^{M+1}}{C_3(p,0)}\frac{(1+A_L)^n}{L}\exp(-A_L\tau)\int_0^{\sigma}\frac{1}{\beta^n}\exp\left[-\frac{(A_L+1)v}{\beta A_L}\right]\mathrm{d}\beta, \tag{23}$$

where $$C_3(p,d)=\int_0^{p}\int_0^{\sigma}\frac{(A_L)^{M+1}}{L^{1-d}}\left(\frac{1+A_L}{\beta}\right)^n\exp\left[-A_L\tau-\frac{(A_L+1)v}{\beta A_L}\right]\mathrm{d}\beta\mathrm{d}L, \tag{24}$$

with $$A_L=\frac{\ln L}{\ln p}-1.$$

Now we derive posterior distribution for the Gini-index under this situation. Let the Gini-index G be a parameter itself. Replacing α in (19) in terms of G by that of (5), we get the posterior distribution of Gini-index as

$$f\left(G \mid \underline{x}\right)=\frac{D_G{}^{M-1}}{C_4\left(0\right)}\frac{\left(1+D_G\right)^n}{G^2}\exp\left(-D_G\tau\right)\int_0^{\sigma}\frac{1}{\beta^n}\exp\left[-\frac{\left(D_G+1\right)v}{\beta D_G}\right]d\beta, \quad (25)$$

where $$C_4\left(d\right)=\int_0^1\int_0^{\sigma}\frac{D_G{}^{M-1}}{G^{2-d}}\left(\frac{1+D_G}{\beta}\right)^n\exp\left[-D_G\tau-\frac{\left(D_G+1\right)v}{\beta D_G}\right]d\beta dG, \quad (26)$$

with $$D_G=\frac{1-G}{2G}.$$

In the following theorems, we derived the quasi-Bayes estimators of α, β, L and G using SELF and LLF. The proofs are straight forward using (12), (13), (15) and (16) and also using the posterior densities (19), (21), (23) and (25).

Theorem 1: For the power function distribution (3), the quasi-Bayes estimator and the posterior risk of α under SELF are given by

$$\begin{aligned}\hat{\alpha}_{QBS}&=E\left(\alpha \mid \underline{x}\right)=\frac{C_1\left(1\right)}{C_1\left(0\right)},\\ R\left(\alpha,\hat{\alpha}_{QBS}\right)&=Var\left(\alpha \mid \underline{x}\right)=\frac{C_1\left(2\right)}{C_1\left(0\right)}-\left(\frac{C_1\left(1\right)}{C_1\left(0\right)}\right)^2,\end{aligned} \quad (27)$$

where $C_1\left(d\right)$ is given in (20).

Theorem 2: For the power function distribution (3), the quasi-Bayes estimator and the posterior risk of β under SELF are given by

$$\begin{aligned}\hat{\beta}_{QBS}&=E\left(\beta \mid \underline{x}\right)=\frac{C_2\left(1\right)}{C_2\left(0\right)},\\ R\left(\beta,\hat{\beta}_{QBS}\right)&=Var\left(\beta \mid \underline{x}\right)=\frac{C_2\left(2\right)}{C_2\left(0\right)}-\left(\frac{C_2\left(1\right)}{C_2\left(0\right)}\right)^2,\end{aligned} \quad (28)$$

where $C_2\left(d\right)$ *is given in* (22).

Theorem 3: For the power function distribution (3), the quasi-Bayes estimator and the posterior risk of L under SELF are given by

$$\hat{L}_{QBS} = E\left(L \mid \underline{x}\right) = \frac{C_3\left(p,1\right)}{C_3\left(p,0\right)},$$
$$R\left(L,\hat{L}_{QBS}\right) = Var\left(L \mid \underline{x}\right) = \frac{C_3\left(p,2\right)}{C_3\left(p,0\right)} - \left(\frac{C_3\left(p,1\right)}{C_3\left(p,0\right)}\right)^2, \tag{29}$$

where $C_3\left(p,d\right)$ is given in (24).

Theorem 4: For the power function distribution (4), the quasi-Bayes estimator and the posterior risk of L under SELF are given by

$$\hat{G}_{QBS} = E\left(G \mid \underline{x}\right) = \frac{C_4\left(1\right)}{C_4\left(0\right)},$$
$$R\left(G,\hat{G}_{QBS}\right) = Var\left(G \mid \underline{x}\right) = \frac{C_4\left(2\right)}{C_4\left(0\right)} - \left(\frac{C_4\left(1\right)}{C_4\left(0\right)}\right)^2, \tag{30}$$

where $C_4\left(d\right)$ is given in (26).

Theorem 5: For the power function distribution (3), the quasi-Bayes estimator and the posterior risk of α under LLF are given by

$$\hat{\alpha}_{QBL} = \frac{-1}{a}\ln\left[E\left(e^{-a\alpha} \mid \underline{x}\right)\right] = \frac{-1}{a}\ln B_1,$$
$$R\left(\alpha,\hat{\alpha}_{QBL}\right) = \ln B_1 + a\frac{C_1\left(1\right)}{C_1\left(0\right)}, \tag{31}$$

where $B_1 = \dfrac{\int_0^\infty\int_0^\sigma \alpha^{M-1}\left(\frac{\alpha+1}{\beta}\right)^n \exp\left[-\tau\alpha - a\alpha - \frac{\left(\alpha+1\right)\nu}{\alpha\beta}\right] \mathrm{d}\beta\mathrm{d}\alpha}{C_1\left(0\right)}$ and $C_1\left(d\right)$ is given in (20).

Theorem 6: For the power function distribution (3), the quasi-Bayes estimator and the posterior risk of β under LLF are given by

$$\hat{\beta}_{QBL} = \frac{-1}{a}\ln\left[E\left(e^{-a\beta} \mid \underline{x}\right)\right] = \frac{-1}{a}\ln B_2,$$
$$R\left(\beta,\hat{\beta}_{QBL}\right) = \ln B_2 + a\frac{C_2\left(1\right)}{C_2\left(0\right)}, \tag{32}$$

where $B_2 = \dfrac{\int_0^{\sigma}\int_0^{\infty} \alpha^{M-1}\left(\dfrac{\alpha+1}{\beta}\right)^n \exp\left[-\tau\alpha - a\beta - \dfrac{(\alpha+1)v}{\alpha\beta}\right]\mathrm{d}\alpha\mathrm{d}\beta}{C_2(0)}$ and $C_2(d)$ is given in (22).

Theorem 7: For the power function distribution (3), the quasi-Bayes estimator and the posterior risk of L under LLF are given by

$$\hat{L}_{QBL} = \frac{-1}{a}\ln\left[E\left(e^{-aL} \mid \underline{x}\right)\right] = \frac{-1}{a}\ln B_3,$$
$$R\left(L, \hat{L}_{QBL}\right) = \ln B_3 + a\frac{C_3(p,1)}{C_3(p,0)}, \tag{33}$$

where $B_3 = \dfrac{\int_0^{p}\int_0^{\sigma} \dfrac{(A_L)^{M+1}}{L}\left(\dfrac{1+A_L}{\beta}\right)^n \exp\left[-\tau A_L - aL - \dfrac{(A_L+1)v}{\beta A_L}\right]\mathrm{d}\beta\mathrm{d}L}{C_3(p,0)}$

and $C_3(p,d)$ is given in (24).

Theorem 8: For the power function distribution (3), the quasi-Bayes estimator and the posterior risk of G under LLF are given by

$$\hat{G}_{QBL} = \frac{-1}{a}\ln\left[E\left(e^{-aG} \mid \underline{x}\right)\right] = \frac{-1}{a}\ln B_4,$$
$$R\left(G, \hat{G}_{QBL}\right) = \ln B_4 + a\frac{C_4(1)}{C_4(0)}, \tag{34}$$

where $B_4 = \dfrac{\int_0^{1}\int_0^{\sigma} \dfrac{(D_G)^{M-1}}{G^2}\left(\dfrac{1+D_G}{\beta}\right)^n \exp\left[-\tau D_G - aG - \dfrac{(D_G+1)v}{\beta D_G}\right]\mathrm{d}\beta\mathrm{d}G}{C_4(0)}$

and $C_4(d)$ is given in (26).

It may be noted here that the quasi-Bayes estimators of Lorenz curve and Gini index under both loss functions are not reducible in nice closed form; however, we propose to use 16-point Gaussian quadrature formulas for their evaluation.

4. A NUMERICAL EXAMPLE

To illustrate the usefulness of the proposed estimators obtained in sections 2 and 3 with real situations, we considered here the real data-set reported by Saez (2012) representing the average income of United States for the period 1913-

2010 (Table 1). We fit the power function distribution to the right proportional residual income of this data. For finding the right residual income, we choose the limit as 50,000. The fit seems to be quite well. (Anderson-Darling statistic = 2.4762, *p*-value = 0.0511). For this model, using MLE, the estimated parameters are α = 1.57 and β = 0.9966. We use the value p = 0.5, for evaluating the estimates of Lorenz curve and choose σ = 0.9966.

Table 1: Estimates for U. S. income data

	MLE	MQL	QBS		QBL	
α	1.5700	1.4377	1.5651	(0.2757)	1.5709	(0.1526)
β	1.0034	0.9622	1.1053	(0.0143)	1.0989	(0.0064)
L	0.3215	0.3087	0.3220	(0.0023)	0.3208	(0.0012)
G	0.2415	0.2580	0.2406	(0.0039)	0.2387	(0.0020)

Based on this data, we evaluate and present the proposed estimates and the posterior risks (in parenthesis) of α, β, L and G in table 1. It is clear from table 1 that the performance of the Lorenz curve and Gini-index estimators using SELF and LLF are more or less similar. Obviously, we do not expect much to conclude from this analysis, perhaps or we are capable to show that the proposed estimators can be easily obtained in the practical situations.

5. MONTE CARLO SIMULATION

In order to assess the performance of the estimators obtained in sections 2 and 3, we present here a simulation study. All the programmes were written using the Mathematica 7 package. Simulation study has been done according to the following steps.

Step-1: Generate a sample of size n = 50, 100 and 200 from (3) with α = 0.5, 1.0, 1.5 and β = 1.0, 1.5, 2.0. For the simulation study, we choose the value of a = 0.8, the LINEX shape parameter and set p = 0.5.

Step-2: For the vector (r, τ) of hyper parameters, calculate the estimates of α, β, L and G by using the estimators obtained in sections 2 and 3.

Step-3: Repeat steps 1 and 2, 1000 times and calculate the mean value for the estimates. The Bayes estimates and the posterior risks (in parenthesis) for each estimates using quasi-Bayesian estimation and the mean for ML and MQL estimates are tabulated in tables 2-5.

Table 2: Estimates of α

n	α	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{MQL}$	$\hat{\alpha}_{QBS}$		$\hat{\alpha}_{QBL}$	
50	0.5	0.51682	0.48384	0.50705	(0.01171)	0.50243	(0.00370)
	1.0	1.06180	0.99299	1.03324	(0.03492)	1.01955	(0.01095)
	1.5	1.50047	1.47526	1.53916	(0.05749)	1.51665	(0.01801)
100	0.5	0.53790	0.52059	0.52356	(0.00881)	0.52006	(0.00280)
	1.0	1.04262	1.02429	1.00366	(0.02864)	0.99238	(0.00902)
	1.5	1.49315	1.47747	1.50081	(0.05103)	1.48080	(0.01601)
200	0.5	0.50987	0.52156	0.49696	(0.00554)	0.49474	(0.00178)
	1.0	1.03006	1.04194	0.97853	(0.02218)	0.96973	(0.00704)
	1.5	1.47029	1.45029	1.53730	(0.04427)	1.51990	(0.01392)

Table 3: Estimates of β

n	β	$\hat{\beta}_{MLE}$	$\hat{\beta}_{MQL}$	$\hat{\beta}_{QBS}$		$\hat{\beta}_{QBL}$	
50	1.0	0.98229	1.00532	1.06800	(0.00430)	1.06633	(0.00133)
	1.5	1.46924	1.47739	1.59736	(0.00889)	1.59396	(0.00272)
	2.0	1.98184	1.99767	2.13274	(0.01641)	2.12656	(0.00494)
100	1.0	0.99348	0.99794	1.03568	(0.00121)	1.03520	(0.00038)
	1.5	1.49289	1.50433	1.55421	(0.00278)	1.55313	(0.00087)
	2.0	1.98330	2.08057	2.07764	(0.00559)	2.07549	(0.00173)
200	1.0	0.99557	1.00722	1.02183	(0.00048)	1.02163	(0.00015)
	1.5	1.49193	1.47899	1.53213	(0.00107)	1.53172	(0.00033)
	2.0	1.99494	2.00497	2.04354	(0.00193)	2.04278	(0.00060)

Table 4: Estimates of *L*

n	α	True *L*	$\hat{L}_{MLE}$	$\hat{L}_{MQL}$	$\hat{L}_{QBS}$		$\hat{L}_{QBL}$	
50	0.5	0.125	0.13026	0.11913	0.12506	(0.00128)	0.12455	(0.00041)
	1.0	0.250	0.25704	0.24597	0.25185	(0.00097)	0.25147	(0.00031)
	1.5	0.315	0.31306	0.30954	0.31598	(0.00052)	0.31577	(0.00017)
100	0.5	0.125	0.13724	0.13059	0.13066	(0.00093)	0.13029	(0.00030)
	1.0	0.250	0.25600	0.25252	0.24723	(0.00086)	0.24689	(0.00027)
	1.5	0.315	0.31284	0.31073	0.31245	(0.00049)	0.31226	(0.00016)
200	0.5	0.125	0.12817	0.13192	0.12260	(0.00065)	0.12234	(0.00021)
	1.0	0.250	0.25419	0.25635	0.24345	(0.00072)	0.24316	(0.00023)
	1.5	0.315	0.31153	0.30948	0.31638	(0.00039)	0.31623	(0.00013)

Table 5: Estimates of *G*

n	α	True *G*	$\hat{G}_{MLE}$	$\hat{G}_{MQL}$	$\hat{G}_{QBS}$		$\hat{G}_{QBL}$	
50	0.5	0.500	0.49290	0.50878	0.50228	(0.00282)	0.50116	(0.00090)
	1.0	0.333	0.32436	0.33858	0.33107	(0.00161)	0.33043	(0.00051)
	1.5	0.250	0.25242	0.25692	0.24864	(0.00086)	0.24829	(0.00027)
100	0.5	0.500	0.48303	0.49307	0.49385	(0.00197)	0.49306	(0.00063)
	1.0	0.333	0.32567	0.33015	0.33702	(0.00143)	0.33645	(0.00045)
	1.5	0.250	0.25271	0.25542	0.25318	(0.00082)	0.25286	(0.00026)
200	0.5	0.500	0.49567	0.49046	0.50495	(0.00145)	0.50437	(0.00046)
	1.0	0.333	0.32798	0.32521	0.34187	(0.00120)	0.34139	(0.00038)
	1.5	0.250	0.25443	0.25706	0.24813	(0.00065)	0.24787	(0.00021)

6. CONCLUSION

The present paper proposes quasi-Bayesian approaches to estimate α, β, L and G of power function distribution. The estimators are obtained using both symmetric and asymmetric loss functions. Comparisons are made between the different estimators based on a simulation study and practical example using a set of real data set. The effect of symmetric and asymmetric loss functions was examined and the following were observed:

1. From Tables 2-5, we can conclude that, as we increase the sample size, the posterior risks for the quasi-Bayes estimates decreases.
2. When we consider smaller values for the LLF shape parameter *a*, we get more or less similar means for the quasi-Bayes estimates for both squared error and LINEX loss functions, but the posterior risks are different.
3. Our proposed estimates using quasi-likelihood techniques performs better than the ML estimates.

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