Aligarh Journal of Statistics
Vol. 35(2015), 41-50

CHEBYSHEV GOAL PROGRAMMING APPROACH FOR LINEAR BILEVEL MULTI-FOLLOWER PROGRAMMING PROBLEM

Neha Gupta, Irfan Ali and Abdul Bari

ABSTRACT

Many problems have been formulated as bilevel programming problems in the field of sciences and industries such as traffic assignment, transportation, signal optimization etc. In past most of the research work concentrated on linear bilevel programming in which one leader and only one follower are involved and are linear in nature and many algorithms and approaches are well developed to find the global optimum of the linear bilevel programming problems *viz. K*-th best approach, Kuhn-Tucker approach etc. This paper considers a particular case of linear bilevel programming with one leader and multiple followers' are involved and there is no sharing information among followers. To solve these problems Chebyshev (Fuzzy) Goal programming approach is suggested and the optimal solution is obtained through R *&* LINGO Software. By using a numerical example it is shown that suggested approach obtains the most appropriate optimal solution.

1. INTRODUCTION

Bilevel optimization problems involve two optimization tasks (upper and lower level), in which every feasible upper level solution must correspond to an optimal solution to a lower level optimization problem. The Bilevel Programming Problem (BLP) is a special case of multi-level programming problem with a structure of two levels, viz., upper level and lower level. The upper level decision maker is called the leader's problem and that the lower level is called the followers' problem. The follower executes its policies after and in view of the decisions of the upper level decision maker. Control over the decision variables is partitioned among the levels but a decision variable of one level may affect the objective function of other level. The vast majority of research on bilevel programming has centered on the linear version of the problem, alternatively known as the linear stackelberg game. Several most successful algorithms have been developed by many authors for this case. Such as Bialas and Karwan (1980) proposed a parametric complementary pivot approach for two level linear bilevel programming, Fortuny-Amat, and McCarl (1981) describes the representation and economic interpretation of two level linear bilevel programming problem, Candler and Townsley (1982) & Bialas and Karwan (1984) introduces Two level linear bilevel programming problem, Hansen and Jaumard (1992) gives new branch and bound rules, Shi *et al*. (2005, 2005b) extended the Kuhn-Tucker approach and K th-best approach apart from them some other authors who have contributed in this area are Wen *et al*. (1991), Colson *et al*. (2005), Lucae *et al*. (2008), Dempe and Dutta (2012), and many others.

For $x \in X \subset \Re^n, y \in Y \subset \Re^m, F : X \times Y \to \Re^1$ and $f : X \times Y \to \Re^1$, the general linear bilevel programming problem (LBLPP) can be written as follows:

$$\min_{x \in X} F(x, y) = c_1 x + d_1 y$$

$$\textit{subject to } A_1 x + B_1 y \le b_1$$

$$\min_{y \in Y} f(x, y) = c_2 x + d_2 y$$

$$\textit{subject to } A_2 x + B_2 y \le b_2$$

where

$$c_1, c_2 \in \Re^n,\ d_1, d_2 \in \Re^m, b_1 \in \Re^p, b_2 \in \Re^q,$$
$$A_1 \in \Re^{p \times n}, B_1 \in \Re^{p \times m}, A_2 \in \Re^{q \times n},\ B_2 \in \Re^{q \times m}.$$

Bilevel programming problems occur in diverse applications, such as transportation, economics, ecology, engineering and others.

BPP mainly deals with one leader and one follower decision problems but in real life it is possible that multiple followers' may involved in decision making at the lower level. Some authors who worked in this area are Calvete and Gale (2007), Ansari and Rezai (2011), Taran and Roghanian (2013) and others.

Aim of this paper is to explore the linear bilevel multi-follower programming (LBLMFP) problem with no sharing information among the followers'. To derive the optimal solution of linear BLMFPP Chebyshev goal programming approach is suggested. Chebyshev goal programming (CGP) was introduced by

Flavell (1976). It is known as Chebyshev goal programming because it uses the underlying chebyshev (L_∞) means of measuring distance. That is, the maximal deviation from any goal, as opposed to the sum of all deviations, is minimized. For this reason CGP is sometimes termed as Minmax goal programming. Recently this approach is used by authors in different fields such as Khowaja *et al*. (2012) apply this approach in the field of sampling etc. It has the potential to give the most appropriate solution of the linear BLMFPP by converting it into single objective problem i.e. CGP model. The CGP model is solved by an optimization software LINGO (2013) whereas best and worst solution of each objective function is obtained by R (2011) software.

The paper is organized as follows: Section 2 discusses the linear bilevel multi-follower programming problem. In section 3 we present the Chebyshev Goal Programming approach for solving linear BLMFP problem. In section 4, numerical example is illustrated for better understanding. Section 5 presents the comparative study and finally, section 6 provides the conclusion and Future work.

2. LINEAR BILEVEL MULTI-FOLLOWER PROGRAMMING PROBLEM (LBLMFP)

For

$$x \in X \subset \Re^n,\ y_i \in Y_i \subset \Re^{m_i},\ F : X \times Y_1 \times \cdots \times Y_k \to \Re^1$$
$$\text{and } f_i : X \times Y_i \to \Re^1,\ i = 1, \ldots, k,$$

a linear BLMFP problem in which $k\,(k \geq 2)$ followers are involved and there is no sharing information among them except the leaders is given (Lu, *et al*. (2005)):

$$\begin{aligned} &\underset{x \in X}{Min}\, F(x, y_1, \ldots, y_k) = cx + \sum_{s=1}^{k} d_s y_s \\ &\qquad\qquad\qquad \text{subject to } Ax + \sum_{t=1}^{k} B_t y_t \leq b \\ &\underset{y_i \in Y_i}{Min}\, f_i(x, y_i) = c_i x + e_i y_i \\ &\qquad\qquad\qquad \text{subject to } A_i x + C_i y_i \leq b_i \end{aligned} \tag{1}$$

where

$$c \in \Re^n,\ c_i \in \Re^n, d_i \in \Re^{m_i}, e_i \in \Re^{m_i},\ b \in \Re^p,\ b_i \in \Re^{q_i},$$
$$A \in \Re^{p\times n}, B_i \in \Re^{p\times m_i}, A_i \in \Re^{q_i\times n},\ C_i \in \Re^{q_i\times m_i},\ i = 1,2,\ldots,k.$$

All followers have individual objective function and constraint, since there is not sharing variables among followers.
Basic definition for linear BLMFP solution given by Lu *et al.*, 2005 is given as:
(*a*) Constraint region of the linear BLMFP problem:

$$S = \{(x, y_1,\ldots, y_k) \in X \times Y_1 \times\ldots\times Y_k, Ax + \sum_{t=1}^{k} B_t y_t \le b,\ A_i x + C_i y_i \le b_i,\ \forall\ i\}.$$

The linear BLMFP problem constraint region refers to all possible combinations of choices that the leader and followers may make.
(*b*) Projection of S onto the leader's decision space:

$$S(X) = \{x \in X : \exists\, y_i \in Y_i, Ax + \sum_{t=1}^{k} B_t y_t \le b,\ A_i x + C_i y_i \le b_i,\ \forall\ i\}.$$

Unlike the rules in uncooperative game theory where each player must choose a strategy simultaneously, the definition of BLMFP model requires that the leader moves first by selecting an x in attempting to minimize his objective subjecting to constraints of both upper and each lower level.
(*c*) Feasible set for each follower $\forall x \in S(X)$:

$$S_i(x) = \{y_i \in Y_i : (x, y_1,\ldots, y_k) \in S\},\ i = 1, 2,\ldots, k.$$

The feasible region for the follower is affected by the leader's choice of x, and allowable choices of each follower are the elements of S.
(*d*) Each follower's rational reaction set for $x \in S(X)$:

$$P_i(x) = \{y_i \in Y_i : y_i \in \arg\min[\, f_i(x, \hat{y}_i) : \hat{y}_i \in S_i(x)]\},\quad i = 1, 2,\ldots, k,$$

where $\arg\min[f_i(x,\hat{y}_i) : \hat{y}_i \in S_i(x)] = \{y_i \in S_i(x) : f_i(x, y_i) \le f_i(x,\hat{y}_i), \hat{y}_i \in S_i(x)\}$.T
he followers observe the leader's action and simultaneously react by selecting y_i from their feasible set to minimize their objective functions, respectively.
(*e*) Inducible region:

$$IR = \{(x, y_1,\ldots, y_k) : (x, y_1,\ldots, y_k) \in S, y_i \in P_i(x), i = 1,2,\ldots,k\}.$$

To ensure the optimality of (1), following assumption is given.
(*i*) S is nonempty and compact.
(*ii*) For decisions taken by the leader, each follower has some room to respond; i.e, $P_i(x) \ne \phi$.

(iii) $P_i(x)$ is a point to point map.

Thus the linear BLMFP problem in terms of the above notations can be written as

$$\min\{F(x, y_1, \ldots, y_k) : (x, y_1, \ldots, y_k) \in IR\}. \tag{2}$$

3. CHEBYSHEV (FUZZY) GOAL PROGRAMMING APPROACH

There are numerous forms of Chebyshev goal programming but we restrict our coverage to just one for the solution of linear BLMFP problem. The notion of chebyshev goal programming is that the solution sought is the one that minimizes the maximum deviation from any single soft goal. Returning to our linear BLMFP problem, one possible chebyshev goal programming model is as follows:

$$\begin{aligned} &Min \quad \delta \\ &subject\ to\ Ax + \sum_{t=1}^{k} B_t y_t \le b \\ &\qquad A_i x + C_i y_i \le b_i \\ &\delta \ge (F(x, y_1, \ldots, y_k) - U_1)/(U_1 - L_1) \\ &\delta \ge (f_i(x, y_i) - U_i)/(U_i - L_i) \\ &\delta,\ x,\ y_i \ge 0;\ \ i = 1,2,\ldots,k \end{aligned} \tag{3}$$

where

$U_k =$ the worst possible value for objective *k*.

$L_k =$ the best possible value for objective *k*.

$\delta =$ a dummy variable representing the worst deviation level.

$F(x, y_1, \ldots, y_k) =$ the value of the function representing the leaders' objective.

$f_i(x, y_i) =$ the value of the function representing the followers' problem.

4. A NUMERIC EXAMPLE

Let us give the following example given by Shi *et al*. (2005) to show how the Chebyshev goal programming approach works. Consider the following linear BLMFP problem

$$
\begin{aligned}
&\underset{x\in X}{Min}\, F(x,y,z) = x - 2y - 4z\\
&\text{subject to } -x + 3y \le 4\\
&\qquad\qquad -x + z \le 1\\
&\qquad\qquad \underset{y\in Y}{Min}\ f_1(x,y) = x + y\\
&\qquad\qquad \text{subject to } x - y \le 0\\
&\qquad\qquad\qquad\qquad -x - y \le 0\\
&\qquad\qquad \underset{y\in Y}{Min}\ f_2(x,y) = x + z\\
&\qquad\qquad \text{subject to } x + z \le 4\\
&\qquad\qquad\qquad\qquad 2x - 5z \le 1\\
&\qquad\qquad\qquad\qquad 2x + z \ge 1
\end{aligned} \tag{4}
$$

For formulating the Chebyshev GP model we have to obtain the best (L_k) and worst (U_k) solution of the leaders' objective and followers' objective functions as follows:

$$L_1 = \min F(x^*, y^*, z^*) \quad \text{and} \quad U_1 = \max F(x^*, y^*, z^*)$$

$$L_k = \min f_k(x^*, y^*, z^*) \quad \text{and} \quad U_k = \max f_k(x^*, y^*, z^*); \ k = 1,2$$

Ideal solutions are obtained by R (2011) software by taking one objective at a time subject to the system constraints as

```
> library(lpSolve)
> f.obj <- c(1, -2, -4)
> f.con <- matrix(c(-1, 3, 0, -1, 0, 1, 1, -1, 0, -1, -1, 0, 1,
+ 0, 1, 2, 0, -5, 2, 0, 1), nrow = 7, byrow = TRUE)
> f.dir <- c("<=", "<=", "<=", "<=", "<=", "<=", ">=")
> f.rhs <- c(4, 1, 0, 0, 4, 1, 1)
> lp("min", f.obj, f.con, f.dir, f.rhs, int.vec=1:3)
Success: the objective function is -10
> lp("min", f.obj, f.con, f.dir, f.rhs, int.vec=1:3)$solution
[1] 2 2 2
> f.obj <- c(1, 1, 0)
> lp("min", f.obj, f.con, f.dir, f.rhs, int.vec=1:3)
Success: the objective function is 0
> lp("min", f.obj, f.con, f.dir, f.rhs, int.vec=1:3)$solution
[1] 0 0 1
> f.obj <- c(1, 0, 1)
```

```
> lp("min", f.obj, f.con, f.dir, f.rhs, int.vec=1:3)
Success: the objective function is 1
> lp("min", f.obj, f.con, f.dir, f.rhs, int.vec=1:3)$solution
[1] 0 0 1
```

Similarly, by maximizing the objective functions the worst solutions can be obtained. Hence the best and worst solutions for all the objectives are:

$$L_1 = -10 \quad \text{and} \quad U_1 = -4 \} \text{ Leaders' problem}$$

$$\left.\begin{array}{lll} L_1 = 0 & \text{and} & U_1 = 4 \\ L_2 = 1 & \text{and} & U_2 = 4 \end{array}\right\} \text{Followers' problem}$$

Now the Chebyshev goal programming model will be:

$$\begin{aligned} & Min \quad \delta \\ & subject\ to \quad -x + 3y \le 4 \\ & \qquad -x + z \le 1 \\ & \qquad x - y \le 0 \\ & \qquad -x - y \le 0 \\ & \qquad 2x - 5z \le 1 \\ & \qquad 2x + z \ge 1 \\ & \delta \ge (x - 2y - 4z - (-4))/6 \\ & \qquad \delta \ge (x + y - 4)/4 \\ & \qquad \delta \ge (x + z - 4)/3 \\ & \text{and} \qquad \delta, x, y, z \ge 0 \end{aligned} \tag{5}$$

Above model is solved by LINGO (2013) software and obtains the following optimal solution

$$(x^*, y^*, z^*) = (1,1,1) \ \text{ with } \ F^* = -5, \ \ f_1^* = 2, \ \ f_2^* = 2.$$

5. COMPARATIVE STUDY

The derived solution is compared with the solution of Shi *et al*. (2005), in which *K*-th best approach (algorithm of four steps) is used to solve the linear BLMFPP. The solution is derived in four loops as follows:

$$(x^*, y^*, z^*) = (2, 2, 0.6) \ \text{ with } \ F^* = -4.4, \ \ f_1^* = 4, \ \ f_2^* = 2.6.$$

6. CONCLUSION AND FUTURE WORK

In this paper, theoretical properties of linear BLMFPPs are not discussed because it is already discussed by other authors in past. This paper is designed to suggest a new approach for solving linear BLMFPPs in which there are no sharing variables except the leaders'. Suggested CGP approach provides most appropriate optimal solution simply by converting the linear BLMFPP in single objective problem and this is illustrated through a numerical example and compared with the Shi *et al*. (2005)'s optimal solution which is obtained by *K*-th- best approach. For a clear view and understanding solutions from both the approaches are summarized in the Table below:

Table 1: Optimal solution

Approaches	x^*	y^*	z^*	F^*	f_1^*	f_2^*
Chebyshev Goal Programming	1	1	1	-5	2	2
***K* th-best**	2	2	0.6	-4.4	4	2.6

The further study of the research can be based on exploring the utility of the proposed approach by solving linear BLMFPP with more than two followers' and also for linear bilevel multi-follower programming problems in which there are sharing variables among followers'.

REFERENCES

Ansari, E., and Rezai, Z. H. (2011): Solving multi objective linear Bilevel multi follower programming problem, *International Journal of Industrial Mathematics*, ***3*(4)**, 303-316.

Bialas, W., and Karwan, M. H. (1980): *A parametric complementary pivot approach for two-level linear programming*, State University of New York at Buffalo.

Bialas, W. F. and Karwan, M. H. (1984): Two Level Linear Programming, *Management Science*, **30(8)**, 1004-1020.

Candler, W. and Townsley, R. (1982): A Linear Two Level Programming Problem, *Comput. Oper. Res*., **9**, 59-76.

Calvete, H. I., and Galé, C. (2007): Linear bilevel multi-follower programming with independent followers, *Journal of Global Optimization*, **39(3)**, 409-417.

Colson B, Marcotte P., and Savard G. (2005): A Thrust-Region Method for Nonlinear Bilevel Programming: Algorithm and Computational Experience, *Comp. Optim. Appl*, **30**, 211-227.
Dempe, S., and Dutta, J. (2012): Is bilevel programming a special case of a mathematical program with complementarity constraints, *Math. Program*, **131**, 37–48.
Flavell, R. (1976): A new goal programming formulation, *Omega*, **4(6)**, 731-732.
Fortuny-Amat, J. and McCarl, B. (1981): A Representation and Economic interpretation of a two level programming problem, *J. Opl Res. Soc*., **32**, 783-792.
Hansen, P., Jaumard, B. et al. (1992): New branch and bound rules for linear bilevel programming. *SIAM Journal on Scientific and Statistical Computing*, **13**, 1194–1217.
Jones, D., and Tamiz, M. (2010): *Practical Goal Programming*, Springer.
Khowaja, S., Ghufran, S., and Ahsan, M. J. (2012): On the Problem of Compromise Allocation in Multi-Response Stratified Sample Surveys, *Communications in Statistics – Simulation and Computation*, **42**, 790–799.
LINGO-User's Guide (2001): "*LINGO-User's Guide*", Published by LINDO SYSTEM INC., 1415, North Dayton Street, Chicago, Illinois, 60622, USA.
Lucae Z., Soriac K., and Rosenweig V.V. (2008): Production Planning Problem with Sequence Dependent Setups as a Bilevel Programming Problem, *Eur. Jour. Oper. Res*., **187**, 1504-1512.
Lu, J., Shi, C. and Zhang, G. (2005): On bilevel multi-follower decision-making: general framework and solutions, *Information Sciences* (in press), available online at www.sciencedirect.com
Shi, C., Lu, J. and Zhang, G. (2005): An extended Kuhn-Tucker approach for linear bilevel programming, *Applied Mathematics and Computation*, **162**, 51–63.
Shi, C., Zhang, G. and Lu, J. (2005b): An extended Kth best approach for linear bilevel programming, *Applied Mathematics and Computation*, **164**, 843–855.
Shi, C., Zhang, G., and Lu, J. (2005): The kth-best approach for linear bilevel multi-follower programming, *J. Global Optim*., **33**, 563–578. DOI 10.1007/s10898-004-7739-4.
Taran, M. and Roghanian, E. (2013): A fuzzy multi-objective multi-follower linear Bi-level programming problem to supply chain optimization, *Uncertain Supply Chain Management*, **1**, 193–206.

Venables, W. N., Smith, D. M. and the R Development Core Team (2011): *An Introduction to R- Notes on R*: A Programming Environment for Data Analysis and Graphics, Version 2.14.1.

Wen, Ue-Pyng and Hsu, Shuh-Tzy (1991): Linear Bilevel Programming Problems: A Review, *J. Opl. Res. Soc*. **42(2)**, 125-133.

Neha Gupta, Irfan Ali, and Abdul Bari

Received: 26.09.2013

Department of Statistics and Operations Research
Aligarh Muslim University, Aligarh (INDIA)

Revised: 23.04.2015

E-mail: *ngngupta4@gamil.com*
irfii.st@amu.co.in
bariamu2k3@yahoo.co.in