Aligarh Journal of Statistics
Vol. 35(2015), 1-22

CLASSICAL AND BAYESIAN STOCHASTIC ANALYSIS OF *k-out-of-n*:*G* LOAD SHARING TRICHOTOMOUS SYSTEM

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ABSTRACT

The paper deals with the stochastic analysis of a k-out-of-n:G trichotomous system with load sharing. When any of the n components fails due to open mode failure, the entire load is distributed among the remaining (n-1) components and the system operates with increased failure rate of each operating component i.e. the entire load is shared by the remaining components. This process remains continued till we have k good components. The system may also break down when all the components fail due to some common cause or there is a close mode failure in any component during its operation. The various reliability and cost effectiveness measures useful to system designers have been obtained by supplementary variable technique. The Classical and Bayesian estimates have been obtained for reliability and other characteristics. Monte Carlo Simulation technique is used to derive the posterior distribution for steady state availability and MTSF in a 2-out-of-3:G system.

1. INTRODUCTION

Redundancy is one of the methods to enhance the reliability and other measures of system effectiveness and can be achieved by duplicacy of components or units in the system. There may be various forms of redundancy such as active, passive, element/component redundancy. A common form of redundancy may be considered in a *k-out-of-n:G* system in which at least k out of n components must work satisfactorily for the successful operation of the system so that (*n-k*) components work as redundant. However an *n*-component system that fails if and only if at least k out of n components fail is called a *k-out-of-n:F* system. Thus a *k-out-of-n:G* system is equivalent to an (*n-k+1*)*-out-of-n:F* system. Such type of configuration is very popular in fault tolerant systems which include the multi-display system in a cockpit, the multi-engine system in an airplane and the multi-pump system in a hydraulic control system. For example, in a multi-stranded (20-

25 strands) electric wire system the current will pass if at-least few of them (5-7 strands) are good; Similarly to drive a car with *V*8 engine at least four cylinders are necessary to fire and it will not be driven if less than four cylinders fire. Thus, functioning of engine may be represented by a *4-out-of-8:G* system. Thus in real world we find numerous applications of *k-out-of-n: G* system model. Several examples of *k- out-of-n:G* system are available in Kuo and Zuo (2003), Gurler and Bainomov (2009).

Besides these, another important systems existing in real life are Trichotomous systems (Balaguruswamy (1984)) consisting of a number of components/units that can fail in two mutually exclusive modes-open and close. Most of the electronic goods such as diode circuits, thryristor convertor, and capacitor banks are examples of trichotomous systems. For instance, in an electrical system having components connected in series, if a short circuit occurs in one of the components, then short circuited component will not operate but permits the flow of current through the remaining components so that they continue to operate. However an open circuit failure of any of the components will cause an open mode failure of the system. On the other hand, if the components are connected in parallel, a short circuit will cause failure of all the components and the system breaks down whereas an open circuit failure of any of the components does not cause others to fail. Gupta et. al. (1992, 1996) first time introduced the concept of trichotomous systems in analyzing *k-out-of-n* system and parallel system models by using supplementary variable technique.

In most cases, while analyzing parallel or *k-out-of-n* system, it has been assumed that failure of one of the components doesn't affect the failure of remaining components i.e. lifetimes of components working in parallel or k-out-of-n system configuration are assumed to be independent. However in real existing systems the situation arises where failure of any of the components of the system affects the lifetimes of the remaining components. This aspect may be interpreted in terms of load sharing concept. In load sharing systems if a component fails the entire workload has to be shared by surviving components resulting in the increased load shared by the surviving components. For example, in a power plant, we have electric generators arranged in parallel which can share the electric load if any or many of these generators fail. Mostly increased load induces a higher component failure rate. Many empirical studies by Kapur and Lamberson (1997) and Lui (1998) of mechanical systems and computer systems have proved that workload strongly affects the component failure rate.

As the life testing experiments are time consuming , therefore the parameters involved in lifetime distribution can't be a fixed constant up to a long time and behave like a random variable represented by a prior distribution. In past, many authors have considered the Bayesian study (Martz and Waller (1982), Berger (1985) and Box and Tiao (1992)) that incorporates prior knowledge of the system parameters based on past experience with similar reliability data and this prior knowledge can be put mathematically in the form of suitable prior density. Yadavalli et al. (2005) presented Bayesian analysis in a two component system with common cause shock failure by considering prior distributions on the parameters of exponential failure and repair patterns. Their Bayesian study focuses on steady state availability of two different configurations (series and parallel). Lee, Ke and Hsu (2008, 2009) treated the Bayesian analysis for the repairable standby systems with imperfect coverage and imperfect switching with reboot delay. But so far no study has been done considering a trichotomous *k-out-of-n: G* system model with the concept of load sharing and Bayesian estimation of parameters.

In view of the above considerations, the present study introduces the concept of load sharing in a k-out-of-n:G trichotomous system in which each component may fail due to operation or due to impact of some common cause. Further, due to operation a unit may fail in any of the two mutually exclusive modes –open mode and close mode. The repair is carried out only when the system breaks down i.e. does not work at all and each repair makes the system as good as new. Due to open mode failure, the failed unit does not operate but remaining (n-1) units operate with increased failure rate due to load sharing. The failure rates of the components at each time are taken to be constant whereas all repair rates are general. The analysis of system model under study has been carried out by supplementary variable technique to evaluate following characteristics: point-wise and steady state availabilities of the system, expected up-time of the system in (0,t) and in steady state; reliability and Mean Time To System Failure (MTSF), expected busy period of the repairman in (0,t) and in steady state and net expected profit earned by the system in (0,t) and in steady state. The results are also obtained in a particular case of *2-out-of-3:G* system when the repair time distributions are exponential with different parameters.

Conceptualizing the above model, simulation study is presented for analyzing the *2-out-of-3:G* system model in classical and Bayesian setup. *Monte Carlo Simulation Technique* is used for numerical study. In classical setup maximum likelihood estimators of the parameters involved in the model and reliability characteristics along with their standard error and confidence interval have been obtained. In Bayesian approach, Bayes estimates of parameters and reliability characteristics along with their posterior standard error (PSE) and Highest Posterior Density (HPD) intervals have been computed.

Thus the purpose of the present study is twofold: one is to evaluate the various measures of reliability and cost effectiveness by using supplementary variable technique and other is to evaluate classical and Bayesian estimates of parameters involved in the model and reliability characteristics in a *2-out-of-3 :G* system. Monte Carlo Simulation technique is used to prepare the tables regarding the MTSF, posterior mean and HPD intervals for steady state availability and MTSF as well as estimates of MTSF and A (∞).

2. MODEL DESCRIPTION AND ASSUMPTIONS

Initially the system comprises of n good components that form a parallel network. Each component may fail due to operation or due to impact of random shock. Also a component may fail in any of the two mutually exclusive modes (open and close). The close mode failure in a component is defined as failure due to short circuit in the component. Due to short circuit failure in any of the component, the system breaks down whereas due to open mode failure in any of the component, the failed component does not operate but the system still operates with remaining (*n-1*) components with increased failure rate of each of the components owing to the concept of load sharing. This process goes on until we have k good component in the system. The repair is carried out only when system breaks down and each repair makes the system as good as new. All failure time distributions are taken to be exponential while repair time distributions as general.

3. NOTATIONS AND STATES OF THE SYSTEM

λ_j : Constant failure rate of each component when j components

$\left(\lambda_j < \lambda_{j-1}\right)$ are operative in the system. $\left(j = n, n-1, ..., k\right)$

β : Constant close mode failure rate of the component

λ_{cc} : Constant failure rate of the system due to common cause

$\eta(x), g(x)$: Repair rate and corresponding pdf of repair time when the system breaks down due to close mode failure in a component, so that

$$g(x) = \eta(x)\exp\left[-\int_0^x \eta(u)\,du\right]$$

$\mu(x), q(x)$: Repair rate and corresponding pdf of repair time when the system breaks down due to open mode failure in the components, so that

$$q(x) = \mu(x)\exp\left[-\int_0^x \mu(u)\,du\right]$$

$\theta(x), h(x)$: Repair rate and corresponding pdf of repair time when the system breaks down due to common cause failure in a component, so that

$$h(x) = \theta(x)\exp\left[-\int_0^x \theta(u)\,du\right]$$

$P_w(t)$: P[system is in state S_w at time t] ; $w = 0,1,2,....,(n-k+3)$.

$Q_m(x,t)dx$: P[system is in state S_m at time t and has sojourned in this state for duration $(x,\ x+dx)$; $m = (n-k+1),(n-k+2),(n-k+3)$.

$*, s$: †Symbols for Laplace Transform i.e. $P_k^*(s) = L.T\left[P_k(t)\right] = \int \exp(-st)P_k(t)dt$

The possible states of the system are

S_i : Operative state of the system with the operation of $(n-i)$ components; $i = 0,1,2,....,(n-k)$.

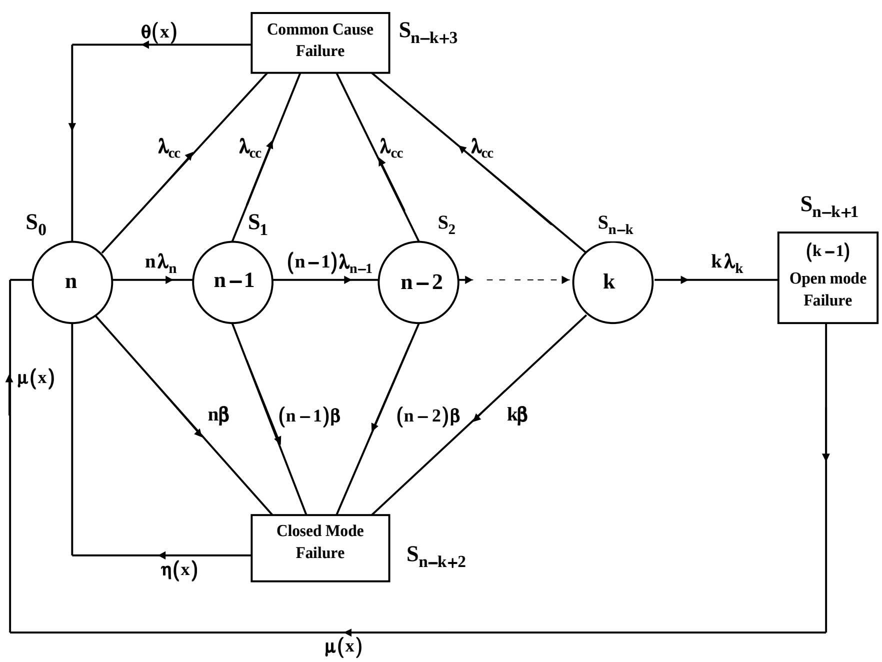
S_{n-k+1} : Failed state of the system when $(n-k+1)$ components have failed one by one due to open mode failure

S_{n-k+2} : Failed state of the system due to closed mode failure in the components.

S_{n-k+3} : Failed state of the system when the system breaks down due to common cause failure.

The transition diagram of the system model is shown in fig 1 where S_0 to S_{n-k} are up states and remaining S_{n-k+1}, S_{n-k+2}, S_{n-k+3} are the failed states.

Fig1. TRANSITION DIAGRAM



4. BASIC EQUATIONS AND THEIR LAPLACE TRANSFORM

Probabilistic considerations and limiting procedure yield the following integro-differential equations

$$\left[\frac{d}{dt}+n\left(\lambda_n+\beta\right)+\lambda_{cc}\right]P_0\left(t\right)=\int Q_{n-k+1}\left(x,t\right)\mu\left(x\right)dx+\int Q_{n-k+2}\left(x,t\right)\eta\left(x\right)dx + \int Q_{n-k+3}\left(x,t\right)\theta\left(x\right)dx \tag{1}$$

$$\left[\frac{d}{dt}+(n-i)\left(\lambda_{n-i}+\beta\right)+\lambda_{cc}\right]P_i\left(t\right)=(n-i+1)\lambda_{n-i+1}P_{i-1}\left(t\right);\quad i=1,2,.....(n-k) \tag{2}$$

$$\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\mu(x)\right]Q_{n-k+1}\left(x,t\right)=0 \tag{3}$$

$$\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\eta(x)\right]Q_{n-k+2}\left(x,t\right)=0 \tag{4}$$

$$\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\theta(x)\right]Q_{n-k+3}\left(x,t\right)=0 \tag{5}$$

Boundary conditions are

$$Q_{n-k+1}\left(0,t\right)=k\lambda_k P_{n-k}\left(t\right) \tag{6}$$

The limits of integration are not mentioned whenever they are 0 to ∞

$$Q_{n-k+2}(0,t)=\sum_{j=0}^{n-k}(n-j)\beta P_j(t) \tag{7}$$

$$Q_{n-k+3}(0,t)=\lambda_{cc}\sum_{j=0}^{n-k}P_j(t) \tag{8}$$

It is assumed that the system is initially in normal state S_0 i.e.

$$P_0(0)=1, P_w(0)=0=P_{n-k+1}(x,0)=P_{n-k+2}(x,0)=P_{n-k+3}(x,0)$$

Taking Laplace Transform of above equations (1-8) we get

$$\begin{aligned}&\left[s+n(\lambda_n+\beta)+\lambda_{cc}\right]P_0^*(s)-\int Q_{n-k+1}^*(x,s)\mu(x)dx-\int Q_{n-k+2}^*(x,s)\eta(x)dx\\&-\int Q_{n-k+3}^*(x,s)\theta(x)dx=1\end{aligned} \tag{9}$$

$$\left[s+(n-i)(\lambda_{n-i}+\beta)+\lambda_{cc}\right]P_i^*(s)-(n-i+1)\lambda_{n-i+1}P_{i-1}^*(s)=0 \tag{10}$$

$$\frac{\partial}{\partial x}Q_{n-k+1}^*(x,s)+\left[s+\mu(x)\right]Q_{n-k+1}^*(x,s)=0 \tag{11}$$

$$\frac{\partial}{\partial x}Q_{n-k+2}^*(x,s)+\left[s+\eta(x)\right]Q_{n-k+2}^*(x,s)=0 \tag{12}$$

$$\frac{\partial}{\partial x}Q_{n-k+3}^*(x,s)+\left[s+\theta(x)\right]Q_{n-k+3}^*(x,s)=0 \tag{13}$$

$$Q_{n-k+1}^*(0,s)=k\lambda_k P_{n-k}^*(s) \tag{14}$$

$$Q_{n-k+2}^*(0,s)=\sum_{j=0}^{n-k}(n-j)\beta P_j^*(s) \tag{15}$$

$$Q_{n-k+3}^*(0,s)=\lambda_{cc}\sum_{j=0}^{n-k}P_j^*(s) \tag{16}$$

5. CALCULATIONS OF $\mathbf{P_w^*(s)=L.T.\left[P_w(t)\right]; w=0,1,....(n-k+3)}$

Integrating (11) and using (14),

$$Q_{n-k+1}^*(x,s)=k\lambda_k P_{n-k}^*(s)\exp\left[-sx-\int_0^x\mu(u)du\right] \tag{17}$$

So that

$$\begin{aligned}\int Q_{n-k+1}^*(x,s)\mu(x)dx&=\int k\lambda_k P_{n-k}^*(s)\exp\left[-sx-\int_0^x\mu(u)du\right]\mu(x)dx\\&=k\lambda_k P_{n-k}^*(s)q^*(s)\end{aligned} \tag{18}$$

Also, from (17)

$$P^*_{n-k+1}(s) = \int Q^*_{n-k+1}(x,s)\,dx = k\lambda_k P^*_{n-k}(s)\left[\frac{1-q^*(s)}{s}\right] \tag{19}$$

Similarly integrating (12) and using (15),

$$Q^*_{n-k+2}(x,s) = \sum_{j=0}^{n-k}(n-j)\beta P^*_j(s)\exp\left[-sx-\int_0^x \eta(u)\,du\right] \tag{20}$$

So that,

$$\int Q^*_{n-k+2}(x,s)\eta(x)\,dx = \sum_{j=0}^{n-k}(n-j)\beta P^*_j(s)g^*(s) \tag{21}$$

Also, from (20)

$$P^*_{n-k+2}(s) = \int Q^*_{n-k+2}(x,s)\,dx = \sum_{j=0}^{n-k}(n-j)\beta P^*_j(s)\left[\frac{1-g^*(s)}{s}\right] \tag{.22}$$

Similarly, from (13) and (16) we have,

$$Q^*_{n-k+3}(x,s) = \lambda_{cc}\sum_{j=0}^{n-k}P^*_j(s)\exp\left[-sx-\int_0^x \theta(u)\,du\right] \tag{23}$$

So that

$$\int Q^*_{n-k+3}(x,s)\theta(x)\,dx = \lambda_{cc}\sum_{j=0}^{n-k}P^*_j(s)h^*(s) \tag{24}$$

Also

$$P^*_{n-k+3}(s) = \int Q^*_{n-k+3}(x,s)\,dx = \lambda_{cc}\sum_{j=0}^{n-k}P^*_j(s)\left[\frac{1-h^*(s)}{s}\right] \tag{25}$$

From (10)

$$P^*_i(s) = \left(\frac{(n-i+1)\lambda_{n-i+1}}{s+(n-i)(\lambda_{n-i}+\beta)+\lambda_{cc}}\right)P^*_{i-1}(s) = \frac{P^*_0(s)}{A_i(s)} \quad ;i=1,2,....,(n-k) \tag{26}$$

Where

$$A_i(s) = \prod_{r=1}^{i}\frac{(s+(n-r)(\lambda_{n-r}+\beta)+\lambda_{cc})}{(n-r+1)\lambda_{n-r+1}} \quad ;\ i=1,2,....................,(n-k)$$

Finally from (19), (22) and (25) with the use of (26) we have

$$P^*_{n-k+1}(s) = k\lambda_k\left[\frac{1-q^*(s)}{s}\right]\frac{P^*_0(s)}{A_{n-k}(s)} \tag{27}$$

$$P^*_{n-k+2}(s) = \left[\frac{1-g^*(s)}{s}\right]\left[n\beta+\sum_{i=1}^{n-k}(n-i)\beta\Big/A_i(s)\right]P^*_0(s) \tag{28}$$

$$P^*_{n-k+3}(s)=\lambda_{cc}\left[\frac{1-h^*(s)}{s}\right]\left[1+\sum_{i=1}^{n-k}\frac{1}{A_i(s)}\right]P^*_0(s) \tag{29}$$

Substituting the values of (18), (21) and (24) in (9) we get

$$P^*_0(s)=\left[\left(s+n(\lambda_n+\beta)+\lambda_{cc}\right)-\frac{k\lambda_k q^*(s)}{A_{n-k}(s)}-n\beta g^*(s)-\lambda_{cc}h^*(s)\right.$$
$$\left.-\sum_{j=1}^{n-k}\frac{(n-j)\beta g^*(s)}{A_j(s)}-\lambda_{cc}\sum_{j=1}^{n-k}\frac{h^*(s)}{A_j(s)}\right]^{-1} \tag{30}$$

6. ANALYSIS OF CHARACTERISTICS

6.1 Long Run State Probabilities

The probabilities that system will be in state S_0 in long run is given by:

$$p_0=\lim_{t\to\infty}P_0(t)=s\lim_{s\to 0}P^*_0(s)=\lim_{s\to 0}\frac{1}{\frac{d}{ds}\left(P^*_0(s)\right)^{-1}}$$

Now let

$$\phi=\int x\,q(x)\,dx\,,\qquad \psi=\int x\,g(x)\,dx\,,\qquad \xi=\int x\,h(x)\,dx$$

and $D_i=\left[\prod_{r=1}^{i}(n-r)(\lambda_{n-r}+\beta+\lambda_{cc})\right]^{-1}$

Then we have

$$p_0=\left[1+k\lambda_k(\phi+D_{n-k})A^{-1}_{n-k}+n\beta\psi+\lambda_{cc}\xi+\sum_{i=1}^{n-k}(n-i)\beta_i(\psi+D_i)A_i^{-1}\right.$$
$$\left.+\lambda_{cc}\sum_{i=1}^{n-k}(\xi+D_i)A_i^{-1}\right]^{-1} \tag{31}$$

and,

$$p_i=A_i^{-1}p_0\ \ ;\ i=1,2,\ldots\ldots\ldots,(n-k). \tag{32}$$

$$p_{n-k+1}=k\,\phi\,\lambda_k\,A^{-1}_{n-k}\,p_0 \tag{33}$$

$$p_{n-k+2}=\psi\beta\left[n+\sum_{i=1}^{n-k}(n-i)A_i^{-1}\right]p_0 \tag{34}$$

$$p_{n-k+3}=\xi\lambda_{cc}\left[1+\sum_{i=1}^{n-k}A_i^{-1}\right]p_0 \tag{35}$$

6.2 Point-wise availability

The Point-wise availability of the system in terms of its Laplace Transform is given by:

$$A^*(s) = L.T\left[A(t)\right]$$
$$= L.T\left[P_0(t) + P_1(t) + ... + P_{n-k}(t)\right]$$
$$= \left[1 + \sum_{i=1}^{n-k} \frac{1}{A_i(s)}\right] P_0^*(s) \quad (36)$$

6.3 Steady-state availability

The probability that in long run system will be operative is given by:

$$A(\infty) = \lim_{s \to 0} sA^*(s)$$
$$= \lim_{s \to 0} sP_0^*(s) + \lim_{s \to 0} s \sum_{i=1}^{n-k} \frac{P_0^*(s)}{A_i(s)}$$
$$= \left[1 + \sum_{i=1}^{n-k} \frac{1}{A_i}\right] p_0 \quad (37)$$

6.4 Expected up-time of the system

The expected up-time of the system during (0, t) is given by:

$$\mu_{up}(t) = \int_0^t A(u) du$$

So that,

$$\mu_{up}^*(s) = \frac{A^*(s)}{s} \quad (38)$$

6.5 Expected busy period of the repairman

(a) Expected busy period of the repairman during time interval (0, t) when the system has failed due to short circuit, is given by

$$\mu_b^{(1)}(t) = \int_0^t P_{n-k+1}(u) du$$

So that

$$\mu_b^{(1)^*}(s) = \frac{P_{n-k+1}^*(s)}{s} \quad (39)$$

(b) Expected busy period of the repairman during time interval (0, t) when the system has failed due to open mode failure, is given by

$$\mu_b^{(2)}(t) = \int_0^t P_{n-k+2}(u) du$$

So that

$$\mu_b^{(2)^*}(s)=\frac{P_{n-k+2}^*(s)}{s} \tag{40}$$

(c) Expected busy period of the repairman during time interval (0, t) when the system has failed due to common cause failure, is given by

$$\mu_b^{(3)}(t)=\int_0^t P_{n-k+3}(u)du$$

So that

$$\mu_b^{(3)^*}(s)=\frac{P_{n-k+3}^*(s)}{s} \tag{41}$$

6.6 Reliability and MTSF

The reliability of the system R(t) in terms of its Laplace Transform is

$$R^*(s)=L.T\left[R(t)\right]$$

This can be obtained by assuming the failed states S_{n-k+1}, S_{n-k+2} and S_{n-k+3} of the system as absorbing. Thus

$$R^*(s)=\left[P_0^*(s)+P_1^*(s)+....+P_{n-k}^*(s)\right]_{g^*(s)=q^*(s)=h^*(s)=0}$$

$$=\left[1+\sum_{i=1}^{n-k}\frac{1}{A_i(s)}\right]\left[s+n(\lambda_n+\beta)+\lambda_{cc}\right]^{-1} \tag{42}$$

and MTSF of the system is given by

$$E(T)=\int R(t)dt=\lim_{s\to 0}R^*(s)\quad=\left[1+\sum_{i=1}^{n-k}\frac{1}{A_i(s)}\right]\left[n(\lambda_n+\beta)+\lambda_{cc}\right]^{-1} \tag{43}$$

7. PROFIT FUNCTION ANALYSIS

The net expected profit incurred in (0,t) is given by

$P(t)=$ Total revenue in $(0,t)$ – Expected cost in $(0,t)$

$$P(t)=K_0\mu_{up}(t)-K_1\mu_b^{(1)}(t)-K_2\mu_b^{(2)}(t)-K_3\mu_b^{(3)}(t) \tag{44}$$

Where,

K_0 = revenue per unit of time when the system is in any of the up states.

K_1 = repair cost per unit of time when system has failed due to close mode.

K_2 = repair cost per unit of time when system has failed due to open mode.

K_3 = repair cost per unit of time when system has failed due to some common cause.

The expected profit per unit of time in steady state is given by

$$P=\lim_{t\to\infty}\frac{P(t)}{t}=K_0\lim_{s\to 0}s^2\mu_{up}^{*}(s)-K_1\lim_{s\to 0}s^2\mu_b^{1*}(s)-K_2\lim_{s\to 0}s^2\mu_b^{2*}(s)-K_3\lim_{s\to 0}s^2\mu_b^{3*}(s)$$

$$=K_0A(\infty)-K_1p_{n-k+1}-K_2p_{n-k+2}-K_3p_{n-k+3}$$

$$=\left[(K_0-K_3\xi\lambda_{cc})\left(1+\sum_{i=1}^{n-k}\frac{1}{A_i}\right)-K_1k\lambda_k\frac{\phi}{A_{n-k}}-K_2\psi\left(n\beta+\frac{\sum_{i=1}^{n-k}(n-i)\beta}{A_i}\right)\right]p_0 \qquad (45)$$

8. PARTICULAR CASE: 2-out-of-3:G System

When repair time distributions are also negative exponential with parameters η,μ,θ i.e

$g(x)=\eta\exp(-\eta x) \qquad \psi=\int xg(x)dx$

$q(x)=\mu\exp(-\mu x) \qquad \phi=\int xq(x)dx$

$h(x)=\theta\exp(-\theta x) \qquad \xi=\int xh(x)dx$

Then for n=3 and k=2 we have

a) $$p_0=\left[1+\frac{3\lambda_3}{2(\lambda_2+\beta)+\lambda_{cc}}\left[\frac{2\lambda_2}{\mu}+\frac{2\beta}{\eta}+\frac{\lambda_{cc}}{\theta}+1\right]+\frac{3\beta}{\eta}+\frac{\lambda_{cc}}{\theta}\right]^{-1} \qquad (46)$$

$$p_1=\frac{3\lambda_3}{2(\lambda_2+\beta)+\lambda_{cc}}p_0 \qquad (47)$$

$$p_2=\frac{6\lambda_2\lambda_3}{\mu\left[2(\lambda_2+\beta)+\lambda_{cc}\right]}p_0 \qquad (48)$$

$$p_3=\frac{3\beta}{\eta}\left[1+\frac{2\lambda_3}{2(\lambda_2+\beta)+\lambda_{cc}}\right]p_0 \qquad (49)$$

$$p_4=\frac{\lambda_{cc}}{\theta}\left[1+\frac{3\lambda_3}{2(\lambda_2+\beta)+\lambda_{cc}}\right]p_0 \qquad (50)$$

b) $$A(\infty)=\left[1+\frac{3\lambda_3}{2(\lambda_2+\beta)+\lambda_{cc}}\right]p_0 \qquad (51)$$

c) $R^{*}(s)=\left[1+\frac{3\lambda_3}{s+2(\lambda_2+\beta)+\lambda_{cc}}\right]\left[s+3(\lambda_3+\beta)+\lambda_{cc}\right]^{-1}$

So that

$$R(t)=e^{(-3\lambda_3-3\beta-\lambda_{cc})t}+\frac{3\lambda_3}{2\lambda_2-\beta-3\lambda_3}\left[e^{(-3\lambda_3-3\beta-\lambda_{cc})t}-e^{(-2\lambda_2-2\beta-\lambda_{cc})t}\right] \quad (52)$$

d) $$\text{MTSF}=\left[1+\frac{3\lambda_3}{s+2(\lambda_2+\beta)+\lambda_{cc}}\right]\left[3(\lambda_3+\beta)+\lambda_{cc}\right]^{-1} \quad (53)$$

9. ESTIMATION STUDIES

9.1 Classical Estimation

In view of the assumptions of the model, the likelihood function of load sharing trichotomous k-out-of-n: G system is given below

$$L\left(\Lambda \mid \tilde{U}_1,\tilde{U}_2,\tilde{U}_3,\tilde{U}_4,\tilde{U}_5,\tilde{U}_6,\tilde{U}_7\right)=\lambda_{cc}{}^{n_1}\lambda_2{}^{n_2}\lambda_3{}^{n_3}\beta^{n_4}\mu^{n_5}\theta^{n_6}\eta^{n_7}e^{-\left(\lambda_{cc}T_1+\lambda_2T_2+\lambda_3T_3+\beta T_4+\mu T_5+\theta T_6+\eta T_7\right)}$$

Where $\Lambda=(\lambda_{cc},\lambda_2,\lambda_3,\beta,\mu,\theta,\eta)$ and

$\tilde{U}_1=\left(u_{11},u_{12},....u_{1n_1}\right)$, $\tilde{U}_2=\left(u_{21},u_{22},....u_{2n_2}\right)$, $\tilde{U}_3=\left(u_{31},u_{32},....u_{3n_3}\right)$,

$\tilde{U}_4=\left(u_{41},u_{42},....u_{4n_4}\right)$ are random samples of sizes n_1,n_2,n_3 and n_4 respectively for failure times of operating components and $\tilde{U}_5,\tilde{U}_6$ and $\tilde{U}_7$ are random samples of sizes n_5,n_6 and n_7 respectively for the repair times and $t_i=\sum_{j=1}^{n_i}u_{ij}$; $i=1,2,....,7$.

By using maximum likelihood approach, the maximum likelihood estimates of Λ are

$$\hat{\lambda}_{cc}=\frac{n_1}{t_1},\ \hat{\lambda}_2=\frac{n_2}{t_2},\ \hat{\lambda}_3=\frac{n_3}{t_3},\ \hat{\beta}=\frac{n_4}{t_4},\ \hat{\mu}=\frac{n_5}{t_5},\ \hat{\theta}=\frac{n_6}{t_6},\ \hat{\mu}=\frac{n_7}{t_7}$$

Using large sample theory of M.L.E, the asymptotic sampling distribution of Λ is $N_7\left(0,\Delta^{-1}\right)$ where Δ is observed Fisher Information diagonal matrix of order 7×7. The elements of Δ are given by:

$$\Delta_{11} = E\left(-\frac{\partial^2 \log L}{\partial \lambda_{cc}^2}\right) = \frac{n_1}{\lambda_{cc}^2}, \quad \Delta_{22} = E\left(-\frac{\partial^2 \log L}{\partial \lambda_2^2}\right) = \frac{n_2}{\lambda_2^2},$$

$$\Delta_{33} = E\left(-\frac{\partial^2 \log L}{\partial \lambda_3^2}\right) = \frac{n_3}{\lambda_3^2}, \quad \Delta_{44} = E\left(-\frac{\partial^2 \log L}{\partial \beta^2}\right) = \frac{n_4}{\beta^2},$$

$$\Delta_{55} = E\left(-\frac{\partial^2 \log L}{\partial \mu^2}\right) = \frac{n_5}{\mu^2}, \quad \Delta_{66} = E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right) = \frac{n_6}{\theta^2},$$

$$\Delta_{77} = E\left(-\frac{\partial^2 \log L}{\partial \eta^2}\right) = \frac{n_7}{\eta^2}$$

The asymptotic $(1-\gamma)\times 100\%$ confidence interval for Λ is $\hat{\Lambda} + z_{\gamma/2}\sqrt{V(\hat{\Lambda})}$. Here $V(\hat{\Lambda})$ is variance of $\hat{\Lambda}$ obtained from Δ and $z_{\gamma/2}$ is upper $100\times(\gamma/2)^{th}$ percentile of standard normal distribution. The respective asymptotic distribution of MTSF (M) is $N_7\left(0, M'\Delta^{-1}M\right)$

where $M' = \left(\frac{\partial M}{\partial \lambda_{cc}}, \frac{\partial M}{\partial \lambda_2}, \frac{\partial M}{\partial \lambda_3}, \frac{\partial M}{\partial \beta}, \frac{\partial M}{\partial \mu}, \frac{\partial M}{\partial \theta}, \frac{\partial M}{\partial \eta}\right)$

and that of Availability (A) is $N_7\left(0, A'\Delta^{-1}A\right)$

where $A' = \left(\frac{\partial A}{\partial \lambda_{cc}}, \frac{\partial A}{\partial \lambda_2}, \frac{\partial A}{\partial \lambda_3}, \frac{\partial A}{\partial \beta}, \frac{\partial A}{\partial \mu}, \frac{\partial A}{\partial \theta}, \frac{\partial A}{\partial \eta}\right)$.

9.2 Bayesian Estimation

In this we conduct a Bayesian study by assuming the model parameters as random variables. The prior distribution of parameters $\Lambda = (\lambda_{cc}, \lambda_2, \lambda_3, \beta, \mu, \theta, \eta)$ are assumed to be conjugate i.e. gamma family as follows

$$\lambda_{cc} \sim G(\varphi_1, \nu_1), \lambda_2 \sim G(\varphi_2, \nu_2), \lambda_3 \sim G(\varphi_3, \nu_3), \beta \sim G(\varphi_4, \nu_4),$$
$$\mu \sim G(\varphi_5, \nu_5), \theta \sim G(\varphi_6, \nu_6), \eta \sim G(\varphi_7, \nu_7)$$

Since the prior distribution of λ_{cc} is $G(\varphi_1, \nu_1)$ with density

$$p(\lambda_{cc}) = \frac{\nu_1^{\varphi_1}}{\Gamma(\varphi_1)} \lambda_{cc}^{\varphi-1} e^{-\nu_1 \lambda_{cc}} \quad ; \ \lambda_{cc} > 0$$

And

$$E(\lambda_{cc}) = \frac{\varphi_1}{\nu_1} \qquad ; \ V(\lambda_{cc}) = \frac{\varphi_1}{\nu_1^2}$$

Then according to Bayesian theory, the posterior distribution of λ_{cc} given T_1 is

$$h\left(\lambda_{cc} \mid \tilde{U}_1\right) = \frac{\left(t_1+\nu_1\right)^{n_1+\varphi_1}}{\Gamma\left(n_1+\varphi_1\right)} \lambda_{cc}{}^{n_1+\varphi-1} e^{-\left(\nu_1+t_1\right)\lambda_{cc}} \quad ; \; \lambda_{cc} > 0$$

This is density of Gamma distribution with parameters $\left(n_1+\varphi_1, T_1+\nu_1\right)$

Preceding analogously the posterior distribution of remaining parameters are

$$\pi_2(\lambda_2 | \tilde{U}_2) \sim G(n_2+\varphi_2, T_2+\nu_2) \qquad \pi_3(\lambda_3 | \tilde{U}_3) \sim G(n_3+\varphi_3, T_3+\nu_3)$$

$$\pi_4(\beta | \tilde{U}_4) \sim G(n_4+\varphi_4, T_4+\nu_4) \qquad \pi_5(\mu | \tilde{U}_5) \sim G(n_5+\varphi_5, T_5+\nu_5)$$

$$\pi_6(\theta | \tilde{U}_6) \sim G(n_6+\varphi_6, T_6+\nu_6) \qquad \pi_7(\eta | \tilde{U}_7) \sim G(n_7+\varphi_7, T_7+\nu_7)$$

One can generate the observations from the above posterior distribution for finding the Bayesian estimation and HPD intervals of the parameters.

10. SIMULATION STUDY AND COMPARISONS

Now we shall use the simulation results to discuss posterior performance of A (∞) and MTSF for the redundant repairable system. We have fixed the sample size $n_i = n; i = 1, 2, ..., 7$. We run 100 simulations for each prior distribution. For each simulation run we first generate the values from assumed prior distribution. These simulated values are then used as parameter values. A sample of size n is then generated for all variables and ML and Bayesian estimates including their SE and PSE and confidence/HPD intervals are computed. The samples are generated using R-software and for HPD intervals boa package of R-software has been used.

Tables 1, 2 and 3 provide ML and Bayesian estimates of MTSF and also their SE/PSE and confidence/HPD interval for varying values of $\lambda_{cc}, \lambda_2, \lambda_3$. A common observation in all three cases is that as failure rate increases MTSF decreases. Moreover ML estimates are closer to true values than Bayes estimates. We also observed that both the type of estimates coincide to true value when failure rate increases.

Table 4 and Table 5 provide PM and HPD intervals of $A(\infty)$ and MTSF for the fixed values of parameters $\lambda_2 = 0.05,\ \lambda_3 = 0.005,\ \lambda_{cc} = 0.0001,\ \beta = 0.005,\ \ \mu = 1.5,\ \theta = 1.8,\ \eta = 2.0$.

The tables reveal that as sample size increases HPD intervals become narrower and PM are closer to true values 0.9849 and 37.7488 of $A(\infty)$ and MTSF respectively.

Table 6 and Table 7 give PM and HPD intervals of MTSF and $A(\infty)$ for various sample sizes when the other parameters are kept fixed as $\lambda_2 = 0.05,\ \lambda_3 = 0.005,\ \lambda_{cc} = 0.0001,\ \beta = 0.005,\ \mu = 1.5,\ \theta = 1.8,\ \eta = 2.0$. Here the two parameter gamma prior with various values of its parameters (φ_1, ν_1) are assumed. The results are compared with true values 0.9849 and 37.7488 of $A(\infty)$ and MTSF. It is evident that PM is more stable and closer to true value and HPD intervals are much smaller when sample size is large.

11. CONCLUSION

To study the behaviors of Reliability, MTSF and profit function in case of 2-out-of-3: G system w.r.t various parameters, we plot the curves for these characteristics in figures 2, 3 and 4 respectively. In fig.2 the reliability curves are drawn to study the impact of change of λ_2 and λ_3 on $R(t)$ when other parameters are kept fixed as $\lambda_{cc} = \beta = 0.002,\ \mu = 0.750,\ \theta = 0.250,\ \eta = 0.500$. From the figure we observed that initially at t=0, the reliability of the system is one as it should be and decreases uniformly as mission time t increases. Also, the reliability of the system decreases with the increase of λ_2 and λ_3. In fig.3 the curves are drawn for the MTSF in respect of the common cause failure rate λ_{cc} for two different values of $\lambda_3 (= 0.05,\ 0.08)$ and three different values of $\lambda_2 (= 0.20,\ 0.40,\ 0.80)$ whereas β is kept fixed as 0.002. Similar trends in case of MTSF are observed for change of λ_2 and λ_3 as in case of $R(t)$. Fig.4 depicts the behavior of profit function in respect of λ_{cc} for varying values of λ_2 and λ_3 while the other parameters are kept fixed as $\beta = 0.002,\ \mu = 0.750,\ \theta = 0.250,\ \eta = 0.500,\ K_0 = 50,\ K_1 = 150,\ K_2 = 275,\ K_3 = 350.$ Her e we observed linear decreasing trend as λ_{cc} increases. The curve clearly reveals that profit decreases with the increase in failure rates λ_2 and λ_3. Another important observation is that for $\lambda_3 = 0.08$ system incurs loss for λ_{cc} > 0.050, 0.060 and 0.070 respectively when $\lambda_2 = 0.80,\ 0.40$ and 0.20. Similarly for $\lambda_3 = 0.05$ system is profitable only for λ_{cc} < 0.075, 0.080 and 0.085 respectively when $\lambda_2 = 0.80,\ 0.40$ and 0.20.

The Bayesian approach adopted in this paper using apt prior provides an alternative way of dealing with 2-out-of-3:G load sharing system and also gives reliable estimates of MTSF and Availability. The conclusions drawn from the Tables 1 to 5 representing the Bayesian study in respect of various parameters have already been mentioned in previous section. The computations involved are relatively easy. So we can simply conclude that Bayesian approach is easy to implement for analyzing.

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Table 1: The values of MTSF for varying λ_{cc} and fixed

$\lambda_2 = 0.05,\ \lambda_3 = 0.08,\ \beta = 0.05$

λ_{cc}	True MTSF	ML MTSF	SE	Confidence Interval	Bayes MTSF	PSE	HPD interval
0.005	5.4955	5.5973	0.3838	4.8450, 6.3496	5.6030	0.3019	5.1590, 6.3529
0.007	5.4393	5.5371	0.3765	4.7991, 6.2751	5.5430	0.2962	5.1175, 6.2773
0.009	5.3843	5.4781	0.3765	4.7538, 6.2024	5.4844	0.2908	5.0608, 6.2036
0.010	5.3571	5.4490	0.3661	4.7314, 6.1667	5.4545	0.2882	5.0301, 6.1659
0.030	4.8654	4.9250	0.3106	4.3162, 5.5337	4.9318	0.2473	4.5411, 5.5148
0.050	4.4545	4.4904	0.2726	3.9561, 5.0246	4.4993	0.2212	4.0382, 4.9260
0.090	3.8075	3.8122	0.2265	3.3680, 4.2561	3.8257	0.1907	3.4474, 4.1772

Table2: The values of MTSF for varying λ_2 and fixed

$\lambda_{cc} = 0.005,\ \lambda_3 = 0.080,\ \beta = 0.050$

λ_2	True MTSF	ML MTSF	SE	Confidence Interval	Bayes MTSF	PSE	HPD interval
0.005	7.8150	8.4851	0.7601	6.9951, 9.9751	8.5064	0.7085	7.2183, 9.8527
0.008	7.5530	8.1368	0.7069	6.7512, 9.5223	8.1555	0.6441	7.0059, 9.4220
0.010	7.3924	7.9261	0.6757	6.6017, 9.2505	7.9357	0.6059	6.8617, 9.1408
0.030	6.2140	6.4465	0.4787	5.5082, 7.3849	6.4486	0.3922	5.6374, 7.1676
0.050	5.4955	5.5973	0.3838	4.8450, 6.3495	5.6029	0.3018	5.1589, 6.3528
0.070	5.0116	5.0463	0.3292	4.4010, 5.6916	5.0564	0.2541	4.7001, 5.7126
0.100	4.5237	4.5070	0.2812	3.9558, 5.0582	4.5111	0.2164	4.1878, 5.0492

Table 3: The values of MTSF for varying λ_3 and fixed

$\boldsymbol{\lambda_2 = 0.050,\ \lambda_{cc} = 0.005,\ \beta = 0.050}$

λ_3	True MTSF	ML MTSF	SE	Confidence Interval	Bayes MTSF	PSE	HPD Interval
0.005	6.3127	6.7899	0.5788	5.6554, 7.9245	6.8127	0.5373	5.7540, 7.8221
0.008	6.2406	6.6751	0.5564	5.5846, 7.7657	6.6957	0.5069	5.7031, 7.6370
0.010	6.1964	6.6058	0.5431	5.5412, 7.6703	6.6227	0.4887	5.6704, 7.5228
0.050	5.6777	5.8439	0.4153	5.0299, 6.6580	5.8505	0.3345	5.1360, 6.5008
0.080	5.4955	5.5973	0.3838	4.8450, 6.3495	5.6029	0.3018	5.1589, 6.3528
0.500	5.0254	5.0049	0.3392	4.3400, 5.6697	5.0045	0.2631	4.5735, 5.5964
1.000	4.9553	4.9216	0.3373	4.2604, 5.5828	4.9201	0.2624	4.4815, 5.5054

Table 4: PM and HPD intervals for $\mathbf{A}(\infty)$

$\boldsymbol{\lambda_{cc} = 0.0001,\ \lambda_2 = 0.050,\ \lambda_3 = 0.005,\ \beta = 0.005,\ \mu = 1.50,\ \theta = 1.80,\ \eta = 2.00}$

n	PM	SD	99% HPD	95%HPD
10	0.9741	0.0076	0.9562,0.9841	0.9562,0.9841
20	0.9837	0.0024	0.9788,0.9874	0.9788,0.9874
50	0.9839	0.0019	0.9799,0.9873	0.9786,0.9873
100	0.9852	0.0012	0.9832,0.9873	0.9812,0.9873
500	0.9851	0.0005	0.9841,0.9865	0.9835,0.9867
1000	0.9852	0.0004	0.9843,0.9860	0.9841,0.9863

Table 5: PM and HPD intervals MTSF

$\boldsymbol{\lambda_{cc} = 0.0001,\ \lambda_2 = 0.050,\ \lambda_3 = 0.005,\ \beta = 0.005,\ \mu = 1.50,\ \theta = 1.80,\ \eta = 2.00}$

n	PM	SD	99% HPD	95%HPD
10	29.3523	5.5477	23.4663,39.3976	23.4663,39.3976
20	36.6146	3.3514	28.0552,42.0560	28.0552,42.0560
50	36.2925	3.2211	30.1177,43.2833	31.6792,43.2833
100	37.6523	2.0157	32.8652,43.1579	34.0656,41.9932
500	37.7919	1.0574	35.5195,40.5782	35.6074,39.5424
1000	38.3447	0.7736	36.4766, 40.2702	36.7965,39.7861

Table 6: Estimate of MTSF

$\lambda_{cc} = 0.0001, \lambda_2 = 0.050, \lambda_3 = 0.005, \beta = 0.005, \mu = 1.50, \theta = 1.80, \eta = 2.00$

	(φ_1, ν_1) =(5,100)		(φ_1, ν_1) =(25,500)		(φ_1, ν_1) =(50,1000)	
n	**PM**	**95%HPD**	**PM**	**95%HPD**	**PM**	**95%HPD**
10	29.3523	23.4663, 39.3976	30.41098	24.2095, 40.6840	30.70753	24.25479, 41.08183
20	36.6146	28.0552, 42.0560	37.59969	28.9425 ,43.0019	37.95242	29.23676, 43.38808
50	36.9252	31.6792, 43.2833	36.61148	31.9352, 43.6114	36.82872	31.86025, 43.49909
100	37.6523	34.0656, 41.9932	37.71527	34.1622, 42.0146	37.77059	34.25127, 42.02907
500	37.7919	35.6074, 39.5424	37.79789	35.6143, 39.5487	37.80476	35.62234, 39.55602
1000	38.3448	36.7965, 39.7861	38.34639	36.7995, 39.7867	38.34836	36.80311, 39.78748

Table 7: Estimate of $A(\infty)$

$\lambda_{cc} = 0.0001, \lambda_2 = 0.050, \lambda_3 = 0.005, \beta = 0.005, \mu = 1.50, \theta = 1.80, \eta = 2.00$

	(φ_1, ν_1) =(5,100)		(φ_1, ν_1) =(25,500)		(φ_1, ν_1) =(50,1000)	
n	**PM**	**95%HPD**	**PM**	**95%HPD**	**PM**	**95%HPD**
10	0.9741945	0.9562, 0.9841	0.9749949	0.9574, 0.9846	0.9752049	0.9577, 0.9848
20	0.9837078	0.9788, 0.9874	0.984156	0.9794, 0.9877	0.9843115	0.9797, 0.9878
50	0.9839146	0.9799, 0.9873	0.9840634	0.9801, 0.9874	0.9841635	0.9803, 0.9875
100	0.9852899	0.9832, 0.9873	0.9853162	0.9832, 0.9873	0.9853392	0.9832, 0.9874
500	0.9851777	0.9841, 0.9865	0.9851803	0.9841, 0.9865	0.9851832	0.9841, 0.9865
1000	0.9852444	0.9843, 0.9860	0.9852451	0.9843, 0.9860	0.9852459	0.9843, 0.9860

CURVE FOR RELIABILITY FUNCTION W.R.T MISSION TIME.

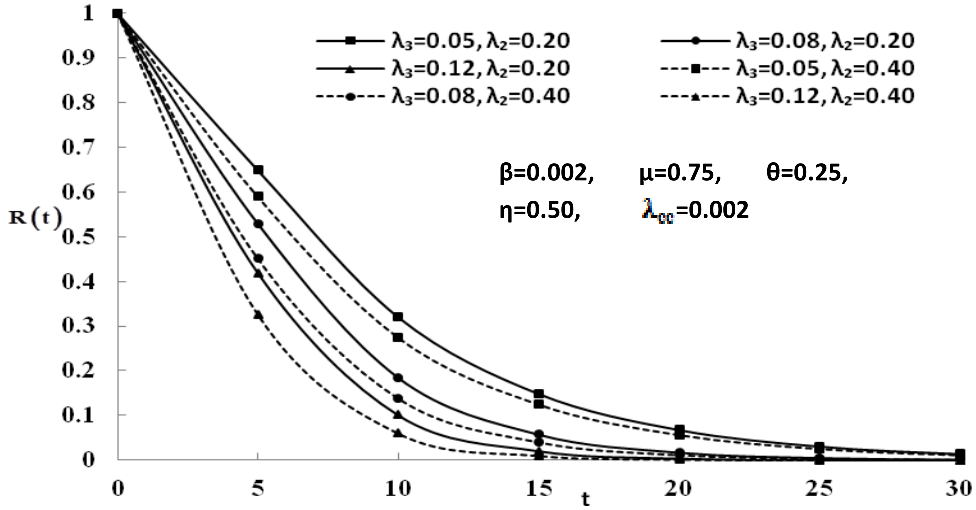

Fig.2

BEHAVIOUR OF MTSF W.R.T. λ_{cc} FOR VARYING VALUES OF λ_2 AND λ_3

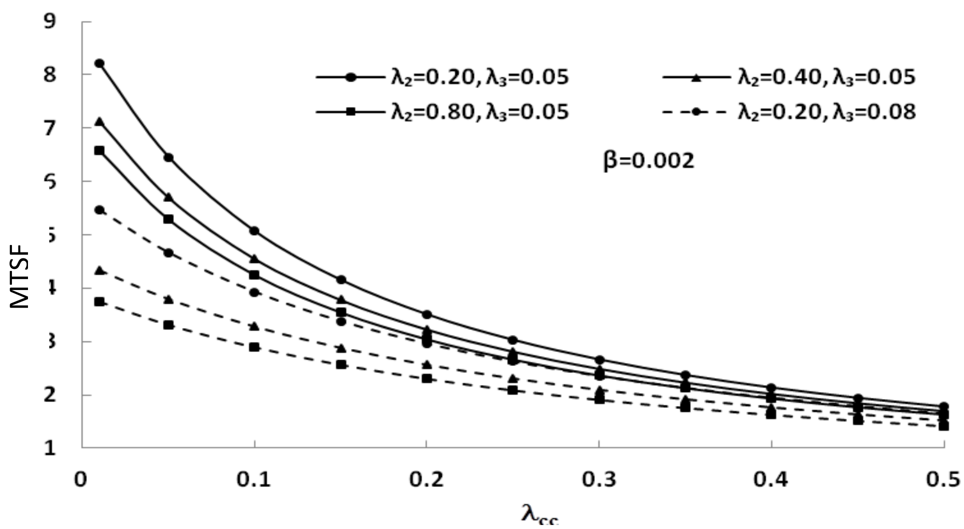

Fig.3

BEHAVIOUR OF PROFIT W.R.T. VARYING VALUES OF λ_{cc}

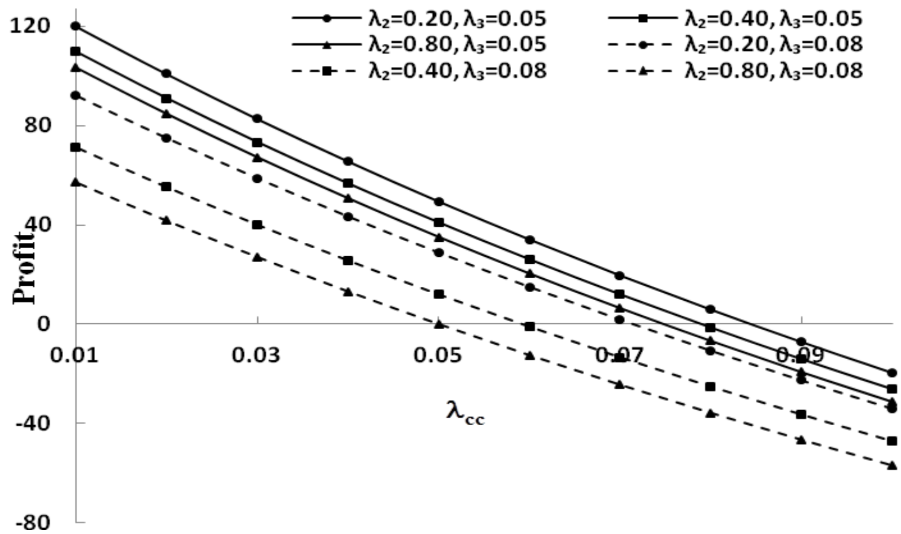

Fig.4

Received: 02.04.2014

Revised: 02.02.2015

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