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RECURRENCE RELATIONS FOR FUNCTION OF DUAL GENERALIZED ORDER STATISTICS FROM MARSHALL-OLKIN EXTENDED GENERAL CLASS OF DISTRIBUTION AND CHARACTERIZATION

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ABSTRACT

In this paper the moment recurrence relations for function of single and two dual generalized order statistics ($dgos$) from Marshall-Olkin extended general class of distribution have been derived. Further, particular cases and examples are also discussed. At the end a characterization theorem is given.

1. INTRODUCTION

The concept of dual (lower) generalized order statistics is first given by Pawlas and Szynal (2001) which is further extensively studied and discussed by Burkschat *et al*. (2003) as below:

The random variables $X'(1,n,m,k), X'(2,n,m,k),..., X'(n,n,m,k)$, $k \geq 1$, $m \in \Re$ are n dual generalized order statistics ($dgos$) from an absolutely continuous distribution function $F()$ with the density function $f()$, if their joint density function is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i)\right)[F(x_n)]^{k-1} f(x_n)\,, \tag{1.1}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq ... \geq x_n > F^{-1}(0)$,

where $\gamma_j = k + (n-j)(m+1) > 0$ for all j, $1 \leq j \leq n$, k is a positive integer and $m \geq -1$.

In view of (1.1), the probability density function of $X'(r,n,m,k)$ is

$$f_{X'(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))\,. \tag{1.2}$$

The joint probability density function of $X'(r,n,m,k)$ and $X'(s,n,m,k)$ is

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x,y)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x)\, g_m^{r-1}(F(x))$$

$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1} f(y)\,,$$

$$\alpha\le y<x\le\beta\,, \qquad (1.3)$$

where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i\,,$$

$$h_m(x)=\begin{cases}-\dfrac{1}{m+1}x^{m+1}, & m\ne-1\\ -\log x & ,\ m=-1\end{cases},$$

and

$$g_m(x)=h_m(x)-h_m(1)\,, \qquad x\in[0,1)\,.$$

If $m=0$, $k=1$, then $X'(r,n,m,k)$ reduces to the $(n-r+1)-th$ order statistic $X_{n-r+1:n}$ from the sample $X_1,X_2,...,X_n$ and when $m=-1$ and $k=1$ then $X'(r,n,m,k)$ reduces to the r^{th} lower $k-$record value.

Recurrence relations for moments of dual generalized order statistics (*dgos*) for some specific distributions as well as for general class of distributions are investigated by several authors in the literature. Pawlas and Szynal (2001) derived the recurrence relations for single and product moments of dual generalized order statistics from the inverse Weibull distribution whereas Khan *et al.* (2008) obtained the recurrence relations for exponentiated Weibull distribution. Further, Khan and Kumar (2010, 2011a, b) established recurrence relations for moments of dual generalized order statistics from Pareto, generalized exponential and exponentiated gamma distributions and they also characterized these distributions using moment properties of *dgos*.

Characterizations of probability distributions have been carried out by several authors using *dgos*. Khan *et al.* (2009) characterized a generalized family of distributions through conditional expectation of *dgos*, when conditioning is non-adjacent. Further, Khan *et al.* (2010a) established characterizing relationships and used it to characterize a general form of distributions whereas Khan *et al.* (2010b) characterized general families of distributions through conditional variance.

Athar and Faizan (2011) obtained explicit expression for the $r-th$ moment of *dgos* for power function distribution and characterized the same through conditional moments of dual generalized order statistics.

In this paper we present a unified approach to obtain certain types of recurrence relations for single and product moments of dual generalized order statistics from Marshall-Olkin extended general class of distribution. These relations are applicable to all specific distributions belong to Marshall-Olkin extended distributions. Further, results for order statistics and lower record values are discussed as special cases.

Adding parameters to a well-established family of distributions is a time honoured device for obtaining more flexible new families of distributions. Marshall and Olkin (1997) introduced a new method of adding a parameter into a family of distributions. Marshall-Olkin extended distributions offer a wide range of behaviour than the basic distributions from which they are derived. Using this concept, here we have introduced Marshall-Olkin general form of distribution, which is as below:

A continuous random variable X is said to have Marshall-Olkin extended general form of distribution, if its distribution function (df) is of the form

$$F(x) = \frac{\lambda[ah(x)+b]^c}{\{1-\bar{\lambda}[ah(x)+b]^c\}}, \ \alpha \le x \le \beta, \ \lambda > 0, \ \bar{\lambda} = 1-\lambda, \tag{1.4}$$

where a, b and c are such that $F(\alpha)=0$, $F(\beta)=1$ and $h(x)$ is a monotonic and differentiable function of x in the interval (α, β).

Also we have,

$$F(x) = \frac{\{[ah(x)+b]-\bar{\lambda}[ah(x)+b]^{c+1}\}}{ach'(x)} f(x). \tag{1.5}$$

The relation (1.5) will be utilized to establish recurrence relations for moments of $dgos$.

2. RECURRENCE RELATIONS FOR SINGLE MOMENTS

We shall assume throughout $\xi(x)$ a measurable function of x and is differentiable.

Theorem 2.1: For the Marshall-Olkin extended general class of distribution as given in (1.4) and $n \in N$, $m \in \Re$, $k > 0$, $1 \le r \le n$, $\lambda > 0$, and $\gamma_r = k + (n-r)(m+1) > 0$,

$$E[\xi\{X'(r,n,m,k)\}] = E[\xi\{X'(r-1,n,m,k)\}] - \frac{1}{ca\gamma_r} E[\psi\{X'(r,n,m,k)\}] + \frac{\bar{\lambda}}{ca\gamma_r} E[\varphi\{X'(r,n,m,k)\}], \tag{2.1}$$

where

$$\psi(x)=[ah(x)+b]\,\omega(x),\ \ \varphi(x)=[ah(x)+b]^{c+1}\,\omega(x),\ \ \omega(x)=\frac{\xi'(x)}{h'(x)}.$$

Proof: From Athar *et al.* (2008), we have

$$E[\xi\{X'(r,n,m,k)\}]-E[\xi\{X'(r-1,n,m,k)\}]$$

$$=-\frac{C_{r-2}}{(r-1)!}\int_{\alpha}^{\beta}\xi'(x)[F(x)]^{\gamma_r}\,g_m^{r-1}(F(x))dx. \tag{2.2}$$

Now in view of (1.5) and (2.2), we get

$$E[\xi\{X'(r,n,m,k)\}]-E[\xi\{X'(r-1,n,m,k)\}]$$

$$=-\frac{C_{r-2}}{ac(r-1)!}\int_{\alpha}^{\beta}\frac{\xi'(x)}{h'(x)}[F(x)]^{\gamma_r-1}\,g_m^{r-1}(F(x))\{[ah(x)+b]-\bar{\lambda}[ah(x)+b]^{c+1}\}f(x)dx,$$

which after simplification yields (2.1).

Remark 2.1: Recurrence relation for moments of function of order statistics (at $m=0,\ k=1$) is

$$E[\xi(X_{n-r+1:n})]\ =E[\xi(X_{n-r+2:n})]-\frac{1}{ca(n-r+1)}\{E[\psi(X_{n-r+1:n})]$$

$$-\bar{\lambda}E[\varphi(X_{n-r+1:n})]\}. \tag{2.3}$$

Remark 2.2: Recurrence relation for moments of function single $k-th$ lower record statistics (at $m=-1$) will be

$$E[\xi\{X(r,n,-1,k)\}]\ =E[\xi\{X(r-1,n,-1,k)\}]-\frac{1}{cak}\{E[\psi\{X(r,n,-1,k)\}]$$

$$-\bar{\lambda}E[\varphi\{X(r,n,-1,k)\}]\}.$$

Remark 2.3: At $\lambda=1$ in (2.1), we get

$$E[\xi\{X'(r,n,m,k)\}]\ =E[\xi\{X'(r-1,n,m,k)\}]-\frac{1}{ca\gamma_r}E[\psi\{X'(r,n,m,k)\}]$$

as obtained by Athar *et al.* (2008).

Remark: 2.4: At $\lambda = 1$ in (2.3), we get

$$E[\xi(X_{n-r+1:n})] = E[\xi(X_{n-r+2:n})] - \frac{1}{ca(n-r+1)} E[\psi(X_{n-r+1:n})]$$

as obtained by Ali and Khan (1997).

Examples

1. Marshall-Olkin Extended Generalized Exponential Distribution

From (1.4) and for $a = -1$, $b = 1$, $c = \beta$ and $h(x) = e^{-\alpha x}$, the distribution function is given by,

$$F(x) = \frac{\lambda(1-e^{-\alpha x})^{\beta}}{[1-\bar{\lambda}(1-e^{-\alpha x})^{\beta}]}, \quad 0 < x < \infty, \ \alpha, \beta > 0.$$

Let $\xi(x) = x^j$, then

$$\psi(x) = \frac{j}{\alpha}(x^{j-1} - \eta(x))$$

and

$$\varphi(x) = -\frac{j}{\alpha}\nu(x),$$

where

$$\eta(x) = x^{j-1}e^{\alpha x} \quad \text{and} \quad \nu(x) = x^{j-1}e^{\alpha x}(1-e^{-\alpha x})^{\beta+1}.$$

Thus from relation (2.1), we have

$$E[X'^{j}(r,n,m,k)] = E[X'^{j}(r-1,n,m,k)] - \frac{j\bar{\lambda}}{\alpha\beta\gamma_r} E\big[\nu\{X'(r,n,m,k)\}\big]$$

$$+ \frac{j}{\alpha\beta\gamma_r}\Big\{ E[X'^{j-1}(r,n,m,k)] - E[\eta\{X'(r,n,m,k)\}]\Big\}. \quad (2.4)$$

Set $\lambda = 1$, in (2.4) to get

$$E[X'^{j}(r,n,m,k)] - E[X'^{j}(r-1,n,m,k)]$$

$$= \frac{j}{\alpha\beta\gamma_r}\Big\{ E[X'^{j-1}(r,n,m,k)] - E[\eta\{X'(r,n,m,k)\}]\Big\}$$

as obtained by Khan and Kumar (2011a).

2. Marshall-Olkin Extended Exponentiated Pareto Distribution

From (1.4) and for $a=-1,\ b=1,\ c=\beta$ and $h(x)=(1+x)^{-\alpha}$, the distribution function is given by,

$$F(x)=\frac{\lambda(1-(1+x)^{-\alpha})^{\beta}}{[1-\bar{\lambda}(1-(1+x)^{-\alpha})^{\beta}]},\ \ 0<x<\infty,\ \ \alpha,\beta>0.$$

Let $\xi(x)=x^j$, then

$$\psi(x)=-\frac{j}{\alpha}\left[\alpha x^j+\sum_{u=2}^{\alpha+1}\binom{\alpha+1}{u}x^{j+u-1}\right]$$

and

$$\varphi(x)=-\frac{j}{\alpha}\sum_{p=0}^{\beta+1}\sum_{q=0}^{\alpha(1-p)+1}(-1)^p\binom{\beta+1}{p}\binom{\alpha(1-p)+1}{q}x^{j+q-1}.$$

Thus from relation (2.1), we have

$$E[X'^j(r,n,m,k)]-E[X'^j(r-1,n,m,k)]$$

$$=-\frac{j}{\beta\gamma_r}\left[E[X'^j(r,n,m,k)]+\frac{1}{\alpha}\sum_{u=2}^{\alpha+1}\binom{\alpha+1}{u}E[X'^{j+u-1}(r,n,m,k)]\right]$$

$$+\frac{j\bar{\lambda}}{\alpha\beta\gamma_r}\sum_{p=0}^{\beta+1}\sum_{q=0}^{\alpha(1-p)+1}(-1)^p\binom{\beta+1}{p}\binom{\alpha(1-p)+1}{q}E[X'^{j+q-1}(r,n,m,k)]. \quad (2.5)$$

Set $\lambda=1$, in (2.5) to get

$$E[X'^j(r,n,m,k)]-E[X'^j(r-1,n,m,k)]$$

$$=-\frac{j}{\beta\gamma_r}\left[E[X'^j(r,n,m,k)]+\frac{1}{\alpha}\sum_{u=2}^{\alpha+1}\binom{\alpha+1}{u}E[X'^{j+u-1}(r,n,m,k)]\right]$$

as obtained by Khan and Kumar (2010).

3. Marshall-Olkin Extended Exponentiated Gamma Distribution

From (1.4) and for $a=-1,\ b=1,\ c=\theta$ and $h(x)=(1+x)e^{-x}$, the distribution function is given by,

$$F(x)=\frac{\lambda[1-e^{-x}(1+x)]^{\theta}}{\{1-\bar{\lambda}[1-e^{-x}(1+x)]^{\theta}\}},\ \ 0<x<\infty,\ \ \theta>0.$$

Let $\xi(x)=x^j$, then

$$\psi(x) = j(x^{j-1} + x^{j-2} - \tau(x))$$

where $\tau(x) = x^{j-2}e^{x}$

and

$$\varphi(x) = -j\sum_{p=0}^{\theta+1}\sum_{q=0}^{p}\sum_{l=0}^{\infty}\frac{(-1)^{p}}{l!}(1-p)^{l}\binom{\theta+1}{p}\binom{p}{q}x^{j+q+l-2}.$$

Thus from relation (2.1), we have

$$E[X'^{j}(r,n,m,k)] - E[X'^{j}(r-1,n,m,k)]$$

$$= \frac{j}{\theta\gamma_r}\left\{E[X'^{j-1}(r,n,m,k)] + E[X'^{j-2}(r,n,m,k)] - E[\tau\{X'(r,n,m,k)\}]\right\}$$

$$+\frac{j\bar{\lambda}}{\theta\gamma_r}\sum_{p=0}^{\theta+1}\sum_{q=0}^{p}\sum_{l=0}^{\infty}\frac{(-1)^{p}}{l!}(1-p)^{l}\binom{\theta+1}{p}\binom{p}{q}E[X'^{j+q+l-2}(r,n,m,k)]. \quad (2.6)$$

At $\lambda = 1$ in (2.6), we get

$$E[X'^{j}(r,n,m,k)] - E[X'^{j}(r-1,n,m,k)]$$

$$= \frac{j}{\theta\gamma_r}\left\{E[X'^{j-1}(r,n,m,k)] + E[X'^{j-2}(r,n,m,k)] - E[\tau\{X'(r,n,m,k)\}]\right\}$$

as obtained by Khan and Kumar (2011b).

4. Marshall-Olkin Extended Inverse Weibull Distribution

From (1.4) and for $a = 1, b = 0,$ $c = 1$ and $h(x) = e^{-(\theta/x)^{p}}$, the distribution function is given by,

$$F(x) = \frac{\lambda e^{-(\theta/x)^{p}}}{[1-\bar{\lambda}e^{-(\theta/x)^{p}}]}, \quad 0 < x < \infty, \quad p, \theta > 0.$$

Let $\xi(x) = x^{j+1}$, then

$$\psi(x) = \frac{(j+1)}{p\,\theta^{p}}x^{j+p+1}$$

and

$$\varphi(x) = \frac{(j+1)}{p\,\theta^{p}}\delta(x)$$

where $\delta(x) = x^{j+p+1}e^{-(\theta/x)^{p}}$.

Thus from relation (2.1), we have

$$E[X'^{j+1}(r,n,m,k)] - E[X'^{j+1}(r-1,n,m,k)]$$

$$= \frac{(j+1)}{p\theta^p \gamma_r}\left\{ \bar{\lambda} E[\delta\{X'(r,n,m,k)\}] - E[X'^{j+p+1}(r,n,m,k)]\right\} \tag{2.7}$$

At $\lambda = 1$ in (2.7), we get

$$E[X'^{j+1}(r,n,m,k)] = E[X'^{j+1}(r-1,n,m,k)] - \frac{(j+1)}{p\theta^p \gamma_r} E[X'^{j+p+1}(r,n,m,k)]$$ as

obtained by Pawlas and Szynal (2001).

5. Marshall-Olkin Extended Logistic Distribution

From (1.4) and for $a=1, b=1, \ c=-1$ and $h(x)=e^{-x}$, the distribution function is given by,

$$F(x) = \frac{\lambda(1+e^{-x})^{-1}}{[1-\bar{\lambda}(1+e^{-x})^{-1}]}, \quad -\infty < x < \infty .$$

Let $\xi(x) = x^j$, then

$$\psi(x) = -j(x^{j-1} + \kappa(x))$$

and

$$\varphi(x) = -j\kappa(x)$$

where $\kappa(x) = x^{j-1}e^x$.

Thus from relation (2.1), we have

$$E[X'^{j}(r,n,m,k)] - E[X'^{j}(r-1,n,m,k)]$$

$$= -\frac{j}{\gamma_r}\left\{ E[X'^{j-1}(r,n,m,k)] + \lambda E[\kappa\{X'(r,n,m,k)\}]\right\}.$$

3. RECURRENCE RELATIONS FOR PRODUCT MOMENTS

Theorem 3.1: For the Marshall-Olkin extended general class of distribution as given in (1.4). Fix a positive integer k and for $n \in N$, $m \in \Re$, $1 \le r < s \le n$, $\lambda > 0$,

$$E[\xi\{X'(r,n,m,k), X'(s,n,m,k)\}] = E[\xi\{X'(r,n,m,k), X'(s-1,n,m,k)\}]$$

$$- \frac{1}{ca\gamma_s} E[\psi\{X'(r,n,m,k), X'(s,n,m,k)\}]$$

$$+ \frac{\bar{\lambda}}{ca\gamma_s} E[\varphi\{X'(r,n,m,k), X'(s,n,m,k)\}], \tag{3.1}$$

where

$$\psi(x,y)=[ah(y)+b]\frac{\frac{\partial}{\partial y}\xi(x,y)}{h'(y)},$$

$$\varphi(x,y)=[ah(y)+b]^{c+1}\frac{\frac{\partial}{\partial y}\xi(x,y)}{h'(y)} \quad .$$

Proof: We have from Athar *et al.* (2008),

$$E[\xi\{X'(r,n,m,k),X'(s,n,m,k)\}]-E[\xi\{X'(r,n,m,k),X'(s-1,n,m,k)\}]$$

$$=-\frac{C_{s-2}}{(r-1)!(s-r-1)!}\int_{\alpha}^{\beta}\int_{\alpha}^{x}\frac{\partial}{\partial y}\xi(x,y)[F(x)]^{m}f(x)g_{m}^{r-1}(F(x))$$

$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s}\,dydx. \tag{3.2}$$

Now in view of (1.5) and (3.2), we have

$$E[\xi\{X'(r,n,m,k),X'(s,n,m,k)\}]-E[\xi\{X'(r,n,m,k),X'(s-1,n,m,k)\}]$$

$$=-\frac{C_{s-2}}{ac(r-1)!(s-r-1)!}\int_{\alpha}^{\beta}\int_{\alpha}^{x}\frac{\frac{\partial}{\partial y}\xi(x,y)}{h'(y)}[F(x)]^{m}f(x)g_{m}^{r-1}(F(x))[F(y)]^{\gamma_s-1}$$

$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}\{[ah(y)+b]-\overline{\lambda}[ah(y)+b]^{c+1}\}f(y)dydx,$$

which leads to (3.1).

Remark 3.1: Recurrence relation for moments of function of two order statistics (at $m=0,\ k=1$) is

$$E[\xi(X_{n-r+1,n-s+1:n})]\ =E[\xi(X_{n-r+1,n-s+2:n})]-\frac{1}{ca(n-s+1)}\{E[\psi(X_{n-r+1,n-s+1:n})]$$

$$-\overline{\lambda}E[\varphi(X_{n-r+1,n-s+1:n})]\}. \tag{3.3}$$

Remark 3.2: Recurrence relation for moments of function of two $k-th$ lower records will be

$$E[\xi\{X(r,n,-1,k),X(s,n,-1,k)\}]=E[\xi\{X(r,n,-1,k),X(s-1,n,-1,k)\}]$$

$$-\frac{1}{cak}\{E[\psi\{X(r,n,-1,k),X(s,n,-1,k)\}]$$

$$-\overline{\lambda}E[\varphi\{X(r,n,-1,k),X(s,n,-1,k)\}]\}.$$

Remark 3.3: At $\lambda = 1$ in (3.1), we get

$$E[\xi\{X'(r,n,m,k), X'(s,n,m,k)\}] = E[\xi\{X'(r,n,m,k), X'(s-1,n,m,k)\}]$$

$$-\frac{1}{ca\gamma_s} E[\psi\{X'(r,n,m,k), X'(s,n,m,k)\}]$$

as obtained by Athar *et al.* (2008).

Remark: 3.4: Set $\lambda = 1$ in (3.3) to get

$$E[\xi(X_{n-r+1,n-s+1:n})] = E[\xi(X_{n-r+1,n-s+2:n})]$$

$$-\frac{1}{ca(n-s+1)} E[\psi(X_{n-r+1,n-s+1:n})]$$

as obtained by Ali and Khan (1998).

Examples

1. Marshall-Olkin Extended Generalized Exponential Distribution

From (1.4) and for $a = -1$, $b = 1$, $c = \beta$ and $h(x) = e^{-\alpha x}$, the distribution function is given by,

$$F(x) = \frac{\lambda(1-e^{-\alpha x})^{\beta}}{[1-\bar{\lambda}(1-e^{-\alpha x})^{\beta}]}, \ \ 0 < x < \infty, \ \ \alpha, \beta > 0.$$

Let $\xi(x, y) = x^i y^j$, then

$$\psi(x, y) = \frac{j}{\alpha}(x^i y^{j-1} - \eta(x, y))$$

and

$$\varphi(x, y) = -\frac{j}{\alpha} \nu(x, y),$$

where

$$\eta(x, y) = x^i y^{j-1} e^{\alpha y}$$

and

$$\nu(x, y) = x^i y^{j-1} e^{\alpha y} (1-e^{-\alpha y})^{\beta+1}.$$

Thus from relation (3.1), we have

$$E[X'^i(r,n,m,k).X'^j(s,n,m,k)] - E[X'^i(r,n,m,k).X'^j(s-1,n,m,k)]$$

$$= \frac{j}{\alpha\beta\gamma_s}\left\{E[X'^i(r,n,m,k).X'^{j-1}(s,n,m,k)] - E[\eta\{X'(r,n,m,k).X'(s,n,m,k)\}]\right\}$$

$$+\frac{j\bar{\lambda}}{\alpha\beta\gamma_s}E[\nu\{X'(r,n,m,k).X'(s,n,m,k)\}]. \tag{3.4}$$

At $\lambda = 1$ in (3.4), we get

$$E[X'^i(r,n,m,k).X'^j(s,n,m,k)] - E[X'^i(r,n,m,k).X'^j(s-1,n,m,k)]$$

$$= \frac{j}{\alpha\beta\gamma_s}\left\{E[X'^i(r,n,m,k).X'^{j-1}(s,n,m,k)] - E[\eta\{X'(r,n,m,k).X'(s,n,m,k)\}]\right\}$$

as obtained by Khan and Kumar (2011a).

2. Marshall-Olkin Extended Exponentiated Pareto Distribution

From (1.4) and for $a = -1,\ b = 1,\ c = \beta$ and $h(x) = (1+x)^{-\alpha}$, the distribution function is given by,

$$F(x) = \frac{\lambda(1-(1+x)^{-\alpha})^{\beta}}{[1-\bar{\lambda}(1-(1+x)^{-\alpha})^{\beta}]},\ \ 0 < x < \infty,\ \ \alpha, \beta > 0.$$

Let $\xi(x, y) = x^i y^j$, then

$$\psi(x, y) = -\frac{j}{\alpha}\left[\alpha x^i y^j + \sum_{u=2}^{\alpha+1}\binom{\alpha+1}{u} x^i y^{u+j-1}\right]$$

and

$$\varphi(x, y) = -\frac{j}{\alpha}\sum_{p=0}^{\beta+1}\sum_{q=0}^{\alpha(1-p)+1}(-1)^p\binom{\beta+1}{p}\binom{\alpha(1-p)+1}{q}x^i y^{j+q-1}.$$

Thus from relation (3.1), we have

$$E[X'^i(r,n,m,k).X'^j(s,n,m,k)] - E[X'^i(r,n,m,k).X'^j(s-1,n,m,k)]$$

$$= \frac{j\bar{\lambda}}{\alpha\beta\gamma_s}\sum_{p=0}^{\beta+1}\sum_{q=0}^{\alpha(1-p)+1}(-1)^p\binom{\beta+1}{p}\binom{\alpha(1-p)+1}{q}$$

$$\times E[X'^i(r,n,m,k).X'^{j+q-1}(s,n,m,k)]$$

$$-\frac{j}{\beta\gamma_s}\Big[E[X'^i(r,n,m,k).X'^j(s,n,m,k)]$$

$$+\frac{1}{\alpha}\sum_{u=2}^{\alpha+1}\binom{\alpha+1}{u}E[X'^i(r,n,m,k).X'^{j+u-1}(s,n,m,k)]\Big]. \qquad (3.5)$$

Set $\lambda=1,$ in (3.5) to get

$$E[X'^i(r,n,m,k).X'^j(s,n,m,k)]-E[X'^i(r,n,m,k).X'^j(s-1,n,m,k)]$$

$$=-\frac{j}{\beta\gamma_s}\Big[E[X'^i(r,n,m,k).X'^j(s,n,m,k)]$$

$$+\frac{1}{\alpha}\sum_{u=2}^{\alpha+1}\binom{\alpha+1}{u}E[X'^i(r,n,m,k).X'^{j+u-1}(s,n,m,k)]\Big].$$

as obtained by Khan and Kumar (2010).

3. Marshall-Olkin Extended Exponentiated Gamma Distribution

From (1.4) and for $a=-1,$ $b=1,$ $c=\theta$ and $h(x)=(1+x)e^{-x},$ the distribution function is given by,

$$F(x)=\frac{\lambda[1-e^{-x}(1+x)]^{\theta}}{\{1-\bar{\lambda}[1-e^{-x}(1+x)]^{\theta}\}}, \quad 0<x<\infty, \quad \theta>0.$$

Let $\xi(x,y)=x^i y^j$, then

$$\psi(x,y)=j[x^i y^{j-2}+x^i y^{j-1}-\tau(x,y)]$$

where

$$\tau(x,y)=x^i y^{j-2}e^{y}$$

and

$$\varphi(x,y)=-j\sum_{p=0}^{\theta+1}\sum_{q=0}^{p}\sum_{l=0}^{\infty}(-1)^p\binom{\theta+1}{p}\binom{p}{q}(1-p)^l x^i y^{j+q+l-2}$$

Thus from relation (3.1), we have

$$E[X'^i(r,n,m,k).X'^j(s,n,m,k)]-E[X'^i(r,n,m,k).X'^j(s-1,n,m,k)]$$

$$=\frac{j}{\theta\gamma_s}\{E[X'^i(r,n,m,k).X'^{j-1}(s,n,m,k)]+E[X'^i(r,n,m,k).X'^{j-2}(s,n,m,k)]$$

$$-E[\tau\{X'(r,n,m,k).X'(s,n,m,k)\}]+\frac{j\overline{\lambda}}{\theta\gamma_s}\sum_{p=0}^{\theta+1}\sum_{q=0}^{p}\sum_{l=0}^{\infty}(-1)^p\binom{\theta+1}{p}\binom{p}{q}(1-p)^l$$

$$\times E[X'^i(r,n,m,k).X'^{j+q+l-2}(s,n,m,k)]. \tag{3.6}$$

At $\lambda = 1$ in (3.6), we get

$$E[X'^i(r,n,m,k).X'^j(s,n,m,k)] - E[X'^i(r,n,m,k).X'^j(s-1,n,m,k)]$$

$$=\frac{j}{\theta\gamma_s}\{E[X'^i(r,n,m,k).X'^{j-1}(s,n,m,k)]+E[X'^i(r,n,m,k).X'^{j-2}(s,n,m,k)]$$

$$-E[\tau\{X'(r,n,m,k).X'(s,n,m,k)\}]\}$$

as obtained by Khan and Kumar (2011b).

4. Marshall-Olkin Extended Inverse Weibull Distribution

From (1.4) and for $a=1, b=0,$ $c=1$ and $h(x)=e^{-(\theta/x)^p}$, the distribution function is given by,

$$F(x)=\frac{\lambda e^{-(\theta/x)^p}}{[1-\overline{\lambda}e^{-(\theta/x)^p}]},\ \ 0<x<\infty,\ \ p,\theta>0.$$

Let $\xi(x,y)=x^i y^{j+1}$, then

$$\psi(x,y)=\frac{(j+1)}{p\,\theta^p}x^i y^{j+p+1}$$

and

$$\varphi(x,y)=\frac{(j+1)}{p\theta^p}\delta(x,y)$$

where

$$\delta(x,y)=x^i y^{j+p+1}e^{-(\theta/y)^p}.$$

Thus from relation (3.1), we have

$$E[X'^i(r,n,m,k).X'^{j+1}(s,n,m,k)] - E[X'^i(r,n,m,k).X'^{j+1}(s-1,n,m,k)]$$

$$=-\frac{(j+1)}{p\,\theta^p\gamma_s}E[X'^i(r,n,m,k).X'^{j+p+1}(s,n,m,k)]$$

$$+\frac{\overline{\lambda}(j+1)}{p\,\theta^p\gamma_s}E[\delta\{X'(r,n,m,k).X'(s,n,m,k)\}]. \tag{3.7}$$

Set $\lambda = 1,$ in (3.7) to get

$$E[X'^{i}(r,n,m,k).X'^{j+1}(s,n,m,k)] - E[X'^{i}(r,n,m,k).X'^{j+1}(s-1,n,m,k)]$$

$$= -\frac{j+1}{p\,\theta^{p}\gamma_{s}} E[X'^{i}(r,n,m,k).X'^{j+p+1}(s,n,m,k)]$$

as obtained by Pawlas and Szynal (2001).

5. Marshall-Olkin Extended Logistic Distribution

From (1.4) and for $a=1, b=1,\ c=-1$ and $h(x)=e^{-x},$ the distribution function is given by,

$$F(x) = \frac{\lambda(1+e^{-x})^{-1}}{[1-\bar{\lambda}(1+e^{-x})^{-1}]}, \quad -\infty < x < \infty\,.$$

Let $\xi(x,y) = x^{i} y^{j}$, then

$$\psi(x,y) = -j[x^{i} y^{j-1} + \kappa(x,y)]$$

and

$$\varphi(x,y) = -j\kappa(x,y)$$

where $\kappa(x,y) = x^{i} y^{j-1} e^{y}.$

Thus from relation (3.1), we have

$$E[X'^{i}(r,n,m,k).X'^{j}(s,n,m,k)] - E[X'^{i}(r,n,m,k).X'^{j}(s-1,n,m,k)]$$

$$= -\frac{j}{\gamma_{s}}\{E[X'^{i}(r,n,m,k).X'^{j-1}(s,n,m,k)] + \lambda E[\kappa\{X'(r,n,m,k).X'(s,n,m,k)\}]\}.$$

4. CHARACTERIZATION THEOREM

Theorem 4.1: Let X be an absolutely continuous random variable (rv) with distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$ over the support $(\alpha,\beta),$ and $\Phi(x)$ be a monotonic and differentiable function of $x,$ then for two consecutive values r and $r+1,\ 2 \le r+1 \le s \le n,$

$$E[\Phi\{X'(s,n,m,k)\} \mid X'(l,n,m,k) = x] = \bar{\lambda} + \left[\Phi(x) - \bar{\lambda}\right] \prod_{j=r+1}^{s} \left(\frac{\gamma_{j}}{\gamma_{j}-1}\right)$$

$$l = r, r+1 \text{ and } \gamma_{j} \neq 1 \qquad (4.1)$$

if and only if

$$F(x)=\frac{\lambda[ah(x)+b]^{c}}{\{1-\bar{\lambda}[ah(x)+b]^{c}\}} \tag{4.2}$$

where, $\Phi(x)=[ah(x)+b]^{-c}$.

Proof: To prove necessary part, for $s\geq r+1$,

$$E[\Phi\{X'(s,n,m,k)\}\mid X'(r,n,m,k)=x]$$

$$=\frac{c_{s-1}}{(s-r-1)!c_{r-1}(m+1)^{s-r-1}}$$

$$\times\int_{\alpha}^{x}[ah(y)+b]^{-c}\left[1-\left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1}\left[\frac{F(y)}{F(x)}\right]^{\gamma_s-1}\frac{f(y)}{F(x)}dy \tag{4.3}$$

Set

$$t=\left[\frac{F(y)}{F(x)}\right]^{m+1}=\left[\frac{\lambda[ah(y)+b]^{c}}{\{1-\bar{\lambda}[ah(y)+b]^{c}\}}\times\frac{\{1-\bar{\lambda}[ah(x)+b]^{c}\}}{\lambda[ah(x)+b]^{c}}\right]^{m+1},$$

which implies

$$[ah(y)+b]^{-c}=\bar{\lambda}+\{[ah(x)+b]^{-c}-\bar{\lambda}\}t^{-1/m+1}.$$

Then the RHS of (4.3) reduces to

$$=\frac{c_{s-1}}{(s-r-1)!c_{r-1}(m+1)^{s-r}}\int_{0}^{1}[\bar{\lambda}+\{[ah(x)+b]^{-c}-\bar{\lambda}\}t^{-\frac{1}{m+1}}](1-t)^{s-r-1}t^{\frac{\gamma_s}{m+1}-1}dt$$

$$=\bar{\lambda}+\frac{\{[ah(x)+b]^{-c}-\bar{\lambda}\}\prod_{j=r+1}^{s}\gamma_j}{(s-r-1)!(m+1)^{s-r}}B\left(\frac{\gamma_s-1}{(m+1)},\ s-r\right).$$

Which after simplification, yields

$$E[\{ah(X'(s,n,m,k)+b\}^{-c}\mid X'(r,n,m,k)=x]$$

$$=\bar{\lambda}+\left\{[ah(x)+b]^{-c}-\bar{\lambda}\right\}\prod_{j=r+1}^{s}\left(\frac{\gamma_j}{\gamma_j-1}\right)$$

hence the ‘*if*’ part.

To prove sufficiency part, let

$$E[\{ah(X'(s,n,m,k)+b\}^{-c} \mid X'(r,n,m,k)=x]=g_{s|r}(x)$$

or

$$g_{s|r}(x)=\bar{\lambda}+\left\{\left[ah(x)+b\right]^{-c}-\bar{\lambda}\right\}\prod_{j=r+1}^{s}\left(\frac{\gamma_j}{\gamma_j-1}\right).$$

therefore,

$$g_{s|r+1}(x)=\bar{\lambda}+\left\{\left[ah(x)+b\right]^{-c}-\bar{\lambda}\right\}\prod_{j=r+2}^{s}\left(\frac{\gamma_j}{\gamma_j-1}\right).$$

Thus in view of Khan *et al.* (2010a)

$$\frac{f(x)}{F(x)}=\frac{1}{\gamma_{r+1}}\frac{\frac{\partial}{\partial x}g_{s|r}(x)}{[g_{s|r+1}(x)-g_{s|r}(x)]}$$

$$=\frac{1}{\gamma_j+1}\frac{ach'(x)\left[ah(x)+b\right]^{-(c+1)}\prod_{j=r+1}^{s}\left(\frac{\gamma_j}{\gamma_j-1}\right)}{\left[\bar{\lambda}+\left\{\left[ah(x)+b\right]^{-c}-\bar{\lambda}\right\}\prod_{j=r+2}^{s}\left(\frac{\gamma_j}{\gamma_j-1}\right)-\bar{\lambda}-\left\{\left[ah(x)+b\right]^{-c}-\bar{\lambda}\right\}\prod_{j=r+2}^{s}\left(\frac{\gamma_j}{\gamma_j-1}\right)\right]}$$

$$=\frac{a\,c\,h'(x)}{\{[ah(x)+b]-\bar{\lambda}[ah(x)+b]^{c+1}\}}$$

implying that

$$F(x)=\frac{\lambda[ah(x)+b]^{c}}{\{1-\bar{\lambda}[ah(x)+b]^{c}\}}.$$

Hence (4.2).

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