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APPLICATION OF CONCOMITANTS OF ORDER STATISTICS OF INDEPENDENT NON-IDENTICALLY DISTRIBUTED BIVARIATE NORMAL RANDOM VARIABLES IN ESTIMATION

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ABSTRACT

In this paper, we obtain the means, variances and covariances of order statistics arising from independent non-identically distributed bivariate normal random variables. A method of estimation of common parameters involved in several bivariate normal distributions using concomitants of order statistics is also discussed.

1. INTRODUCTION

It is well known that order statistics are very useful in the estimation of location and scale parameters of a distribution. For a survey of literature on the applications of order statistics of *iid* random variables in estimating the location and scale parameters of distributions, see David and Nagaraja (2003) and Balakrishnan and Cohen (1991). Vaughan and Venables (1972) have first discussed about the distribution theory of order statistics of *inid* random variables. For some further results on the order statistics of *inid* random variables, see Beg (1991) and Samuel and Thomas (1998). Sajeevkumar and Thomas (2005) and Thomas and Sajeevkumar (2005) have illustrated some applications of order statistics of independent non-identically distributed random variables in the estimation of common location and scale parameters of several distributions.

In a bivariate setup, study of concomitants of order statistics of *iid* bivariate random variables has gained momentum in a theoretical perspective as well as in terms of its applications. For details, see Beg and Ahsanullah (2007), Chacko (2007), David and Nagaraja (1998) and Nagaraja and David (1994). However as in the case of order statistics of *inid* random variables, not much works have been initiated on the theory and applications of concomitants of order statistics of *inid* random variables. Eryilmaz (2005) introduced the general expression for the *cdf* cdf of concomitants of order statistics of *inid* bivariate random variables. Veena and Thomas (2011) have obtained the general expression for the *pdf* of concomitants of order statistics of *inid* random variables and the means, variances and covariances of order statistics arising from independent non-identically distributed bivariate Pareto distributions. They have also described a method of estimation of common parameters involved in several bivariate Pareto distributions using concomitants of order statistics. Suppose $(X_1,Y_1),(X_2,Y_2),...,(X_n,Y_n)$ are n independent bivariate random variables with (X_i,Y_i) having an absolutely continuous bivariate distribution with *pdf* $f_i(X_i,Y_i), i=1,2,...,n.$. If we order $X_1, X_2,..., X_n$ involved in the above bivariate collection of random variables as $X_{1:n}, X_{2:n},..., X_{n:n}$, , then the accompanying Y value of $X_{r:n}$ in the ordered pair from which $X_{r:n}$ is taken is called the concomitant of the r – th order statistic and is denoted by $Y_{[r:n]}$. If we write $F_{X_i}(x)$ to denote the marginal distribution function of X_i of the bivariate distribution function of the random variable $(X_i,Y_i), i=1,2,...,n$ then from Veena and Thomas (2011), we can write the *pdf* $f_{Y_{[r:n]}}(y)$ of $Y_{[r:n]}$ as

$$f_{Y_{[r:n]}}(y)=\frac{1}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} Per\begin{bmatrix} F_{X_1}(x) & 1-F_{X_1}(x) & f_1(x,y) \\ \vdots & \vdots & \vdots \\ \underbrace{F_{X_n}(x)}_{r-1} & \underbrace{1-F_{X_n}(x)}_{n-r} & \underbrace{f_n(x,y)}_{1} \end{bmatrix} dx \tag{1}$$

where per A is meant to denote the permanent of a square matrix A which is just like the determinant of A except that in per A all terms in its expansion are taken with positive sign and if a symbol k is marked below a column vector a in per A then it means that A includes k copies of a.

For $1 \le r < s \le n$, the joint *pdf* of $Y_{[r:n]}$ and $Y_{[s:n]}$ has been developed by Veena and Thomas (2011) and is given by,

$$f_{Y_{[r,s:n]}}(y,z)=\frac{1}{(r-1)!(s-r-1)!(n-s)!}I$$

where

$$I=\iint_{u<v} Per\begin{bmatrix} F_{X_1}(u) & F_{X_1}(v)-F_{X_1}(u) & 1-F_{X_1}(v) & f_1(u,y) & f_1(v,z) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{X_n}(u) & F_{X_n}(v)-F_{X_n}(u) & 1-F_{X_n}(v) & f_n(u,y) & f_n(v,z) \end{bmatrix} dudv. \tag{2}$$

In this work our main interest lies in establishing an application of the above theory of distribution of concomitants of order statistics of *inid* random variables in the estimation of common parameters involved in several bivariate normal distributions.

In section 2, we have considered the problem of estimation of the common correlation coefficient $°$ (ie., when $°_i = °, i = 1,2,...,n$) involved in several bivariate normal distributions with different $±_i$'s using concomitants of *inid* normal random variables. In section 3, we consider concomitants of order statistics arising from several bivariate normal distributions with $«_{1i} = «_1, ±_{1i} = ±_1, «_{2i} = «_2, ±_{2i} = ±_2$, $i = 1,2,...,n$, but with different known values of the correlation coefficient. Further we illustrate an application of concomitants of order statistics of *inid* normal random variables in estimating the parameters $«_2$ and $±_2$.

2. ESTIMATION OF $°$

Let $(X_i, Y_i), i = 1,2,...,n$ be independent bivariate random variables with (X_i, Y_i) having *pdf* $h_i(x, y)$ of the form

$$\frac{(2®±_1±_{2i})^{-1}}{\sqrt{1-°^2}} \exp\left\{\frac{-2^{-1}}{1-°^2}\left[\frac{(x-«_1)^2}{±_1^2} - 2°\frac{(x-«_1)(y-«_{2i})}{±_1±_{2i}} + \frac{(y-«_{2i})^2}{±_{2i}^2}\right]\right\} \quad (3)$$

for $i = 1,2,...,n$.

Clearly the marginal distributions of X_i and Y_i are $N(«_1, ±_1)$ and $N(«_{2i}, ±_{2i})$ respectively. Let $f(x)$ and $F(x)$ denote the marginal *pdf* and distribution function respectively of each of the X_i 's and $h_i(y|x)$ denote the conditional *pdf* of Y_i given $X_i = x$. Then we have,

$$E\left[Y_{[r:n]}\right] = \int_{-\infty}^{\infty} y f_{Y_{[r:n]}}(y)dy$$

$$= \frac{1}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} Per\begin{bmatrix} F(x) & 1-F(x) & f(x)\int_{-\infty}^{\infty} yh_1(y|x)dy \\ \vdots & \vdots & \vdots \\ \underbrace{F(x)}_{r-1} & \underbrace{1-F(x)}_{n-r} & \underbrace{f(x)\int_{-\infty}^{\infty} yh_n(y|x)dy}_{1} \end{bmatrix} dx$$

We know that

$$\int_{-\infty}^{\infty} y h_i(y\,|\,x)dy = «_{2i} + °±_{2i}\left(\frac{x-«_1}{±_1}\right).$$

Hence,

$$E\left[Y_{[r:n]}\right] = \frac{1}{n}\sum_i («_{2i} + °±_{2i}\sim_{r:n}) \tag{4}$$

where we write $\sim_{r:n}$ to denote the expected value of the r^{th} order statistic $U_{r:n}$ arising from a random sample of size n from $N(0,1)$.

$$E\left[Y_{[r:n]}^2\right] = \int_{-\infty}^{\infty} y^2 f_{Y_{[r:n]}}(y)dy$$

$$= \frac{1}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} Per\begin{bmatrix} F(x) & 1-F(x) & f(x)\int_{-\infty}^{\infty} y^2 h_1(y\,|\,x)dy \\ \vdots & \vdots & \vdots \\ \underbrace{F(x)}_{r-1} & \underbrace{1-F(x)}_{n-r} & \underbrace{f(x)\int_{-\infty}^{\infty} y^2 h_n(y\,|\,x)dy}_{1} \end{bmatrix} dx. \tag{5}$$

We know that

$$\int_{-\infty}^{\infty} y^2 h_i(y\,|\,x)dy = ±_{2i}^2(1-∘^2) + \left[«_{2i} + °±_{2i}\left(\frac{x-«_1}{±_1}\right)\right]^2$$

Hence,

$$E\left[Y_{[r:n]}^2\right] = \frac{1}{n}\sum_i «_{2i}^2 + \frac{1}{n}\sum_i ±_{2i}^2(1-∘^2) + 2°\sim_{r:n}\frac{1}{n}\sum_i «_{2i}±_{2i} + \sim_{r,r:n}∘^2\frac{1}{n}\sum_i ±_{2i}^2 \tag{6}$$

where we write $\sim_{r,r:n}$ to denote $E\left[U_{r:n}^2\right]$ and similarly we write

$${}_{r,r:n} = Var(U_{r:n}) = \sim_{r,r:n} - \sim_{r:n}^2 .$$

Now,

$$Var\left[Y_{[r:n]}\right] = E\left[Y_{[r:n]}^2\right] - \left(E\left[Y_{[r:n]}\right]\right)^2$$

$$= (1-∘^2)\frac{1}{n}\sum_i ±_{2i}^2 + \frac{1}{n}∘^2 \;\; {}_{r,r:n}\sum_i ±_{2i}^2 + \frac{1}{n^2}\sum_{i<j}\left[(«_{2i}-«_{2j}) + °(±_{2i}-±_{2j})\sim_{r:n}\right]^2$$

We have, for $r < s$,

$$E\left[Y_{[r:n]}Y_{[s:n]}\right]=\frac{1}{(r-1)!(s-r-1)!(n-s)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}PerIdx_2dx_1,$$

where the matrix I is given by

$$\begin{bmatrix} F(x_1) & F(x_2)-F(x_1) & 1-F(x_2) & f(x_1)\int_{-\infty}^{\infty}y_1f_1(y_1|x_1)dy_1 & f(x_2)\int_{-\infty}^{\infty}y_2f_1(y_2|x_2)dy_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \underbrace{F(x_1)}_{r-1} & \underbrace{F(x_2)-F(x_1)}_{s-r-1} & \underbrace{1-F(x_2)}_{n-s} & \underbrace{f(x_1)\int_{-\infty}^{\infty}y_1f_n(y_1|x_1)dy_1}_{1} & \underbrace{f(x_2)\int_{-\infty}^{\infty}y_2f_n(y_2|x_2)dy_2}_{1} \end{bmatrix}$$

Now if we write $\sim_{r,s:n}=E\left(U_{r:n}U_{s:n}\right)$ then,

$$E\left[Y_{[r:n]}Y_{[s:n]}\right]=\frac{1}{n(n-1)}\sum_{i\neq j}(«_{2i}«_{2j}+°«_{2i}±_{2j}\sim_{r:n}+°«_{2j}±_{2i}\sim_{s:n}+°^2±_{2i}±_{2j}\sim_{r,s:n})$$

$$Cov\left[Y_{[r:n]}Y_{[s:n]}\right]=E\left[Y_{[r:n]}Y_{[s:n]}\right]-E\left[Y_{[r:n]}\right]E\left[Y_{[s:n]}\right]$$

$$=\frac{1}{n(n-1)}°^2\ _{r,s:n}\sum_{i\neq j}±_{2i}±_{2j}-\frac{1}{n^2(n-1)}\sum_{i<j}[(«_{2i}-«_{2j})+°(«_{2i}-«_{2j})\sim_{r,n}]$$

$$\times\left[(«_{2i}-«_{2j})+°(±_{2i}-±_{2j})\sim_{s:n}\right]$$

Consider the units of bivariate sample in which measurement of the X variate can be done easily where as a measurement of Y is not so easy or economic. In this case we order the X observations and make measurements only on the concomitants $Y_{[c+1:n]},...,Y_{[n-c:n]}$. . Now based on this restricted sample we may use estimates based on the available concomitants of order statistics. Suppose $«_{2i}=«_2,\forall i=1,2,...,n$ and consider the transformation $Y_i^*=Y_i-«_2$. Then the corresponding vector of transformed concomitants of record values $Y_{[n,c]}^*$ has expectation and variance-covariance matrix which can be expressed in the form

$$E\left[Y_{[n,c]}^*\right]=°\ \ , \tag{7}$$

where

$$=\frac{1}{n}\sum_{i=1}^{n}±_{2i}(\sim_{c+1:n},\sim_{c+2:n},...,\sim_{n-c:n})'$$

and

$$D\left[Y_{[n,c]}^*\right]=(1-°^2)\frac{1}{n}\sum_{i=1}^{n}±_{2i}^2I+°^2H, \tag{8}$$

where I is the identity matrix of order $(n-c)\times(n-c)$ and $H = \|h_{rs}\|$,

$$h_{rr} = \frac{1}{n}\sum_{i} \pm^{2}_{2i} \quad {}_{r,r:n} + \frac{1}{n^2}\sum_{i<j}(\pm_{2i} - \pm_{2j})^2 \sim_{r:n}{}^{2}$$

and for $r \neq s$,

$$h_{rs} = \frac{1}{n(n-1)}\sum_{i\neq j} \pm_{2i}\pm_{2j} \quad {}_{r,s:n} + \frac{1}{n^2(n-1)}\sum_{i<j}(\pm_{2i} - \pm_{2j})^2 \sim_{r:n}\sim_{s:n},$$

where i and j vary from 1 to n and r and s are such that $c+1 \le r < s \le n-c$.

Clearly (7) and (8) do not provide a general Gauss-Markov set up so as to derive the BLUE of $\circ$. Hence we may obtain two linear unbiased estimators of $\circ$ by minimizing the variance in a restricted sense as done in Chacko and Thomas (2008).

Theorem 2.1 Let R be a column vector of scalars of order n and $R'Y^*_{[n,c]}$ be a linear function of $Y_{[c+1:n]}, Y_{[c+2:n]}, \ldots, Y_{[n-c:n]}$ with variance given by

$$Var\left(R'Y^*_{[n,c]}\right) = (1 - \circ^2)\frac{1}{n}\sum_{i=1}^{n} \pm^2_{2i} R'R + \circ^2 R'HR \qquad (9)$$

Then an estimator $\hat{\ }_1$ obtained by minimizing $R'R$ involved in (9) subject to the condition that $R'Y^*_{[n,c]}$ is unbiased for $\circ$ is given by $\hat{\ }_1 = \frac{'}{'} Y^*_{[n,c]}$ and an estimator $\hat{\ }_2$ obtained by minimizing $R'HR$ involved in (9) subject to the condition that $R'Y^*_{[n,c]}$ is unbiased for $\circ$ is given by $\hat{\ }_2 = \frac{'H^{-1}}{'H^{-1}} Y^*_{[n,c]}$.

The variances of these estimators are given by

$$Var(\hat{\ }_1) = (1 - \circ^2)\frac{1}{n}\sum_{i=1}^{n} \pm^2_{2i} \frac{1}{'} + \circ^2 \frac{'H}{(\ '\ )^2} \qquad (10)$$

and

$$Var(\hat{\ }_2) = (1 - \circ^2)\frac{1}{n}\sum_{i=1}^{n} \pm^2_{2i} \frac{'H^{-2}}{(\ 'H^{-1}\ )^2} + \circ^2 \frac{1}{'H^{-1}} \qquad (11)$$

Proof

Using (7), we have

$$E(R'Y^*_{[n,c]}) = R' \ \circ .$$

Hence, $R'Y^*_{[n,c]}$ will be an unbiased estimator for $\circ$ if

$$R' \quad = 1\,. \tag{12}$$

To minimize $R'R$ subject to the condition that $R'Y^{*}_{[n,c]}$ is unbiased for ° , we have to minimize

$$\cdot_1 = R'R - 2\text{a}_1(R' \quad -1), \tag{13}$$

Where a_1 is the Lagrangian multiplier. Differentiating (13) with respect to R and equating to zero, we get $2R - 2\text{a}_1 \quad = 0.$

That is,

$$R = \text{a}_1 \quad .$$

Substituting the value of R in (12), we get

$$\text{a}_1 = \frac{1}{\prime}$$

Therefore,

$$R = \frac{}{\prime}$$

Thus the required unbiased estimator σ_1 of ° is given by $\sigma_1 = \frac{\prime}{\prime} Y^{*}_{[n,c]}$.

Then by using (8), the variance of σ_1 is given by

$$Var(\sigma_1) = (1 - {\circ}^2)\frac{1}{n}\sum_{i=1}^{n} \pm^2_{2i} \frac{1}{\prime} + {\circ}^2 \frac{\prime H}{(\ \prime\ )^2}$$

Similarly, to minimize $R'HR$ subject to the condition that $R'Y^{*}_{[n,c]}$ is unbiased for ° , we have to minimize

$$\cdot_2 = R'HR - 2\text{a}_2(R' \quad -1), \tag{14}$$

where a_2 is the Lagrangian multiplier. Differererentiating (14) with respect to R and equating to zero, we get

$$2HR - 2\text{a}_2 \quad = 0.$$

That is,

$$R = \text{a}_2 H^{-1} \quad .$$

Substituting the value of R in (12), we get

$$\text{a}_2 = \frac{1}{\prime H^{-1}}.$$

Therefore,

$$R = \frac{H^{-1}}{'H^{-1}}.$$

Thus the required unbiased estimator σ_2 of $\circ$ is given by $\sigma_2 = \frac{'H^{-1}}{'H^{-1}} Y^*_{[n,c]}$.

Then by using (8), the variance of σ_2 is given by

$$Var(\sigma_2) = (1-\circ^2)\frac{1}{n}\sum_{i=1}^{n} \pm^2_{2i} \frac{'H^{-2}}{(\ \ 'H^{-1}\ \ )^2} + \circ^2 \frac{1}{'H^{-1}}$$

Theorem 2.2 *If* $\sigma_1 = \frac{'}{'} Y^*_{[n,c]}$ *and* $\sigma_2 = \frac{'H^{-1}}{'H^{-1}} Y^*_{[n,c]}$ are two unbiased estimators of $\circ$ with variances given by (10) and (11) respectively, then σ_1 is more efficient than σ_2 if $|\circ| < \sqrt{\frac{K_2}{K_1+K_2}}$ where

$$K_1 = \frac{'H}{(\ \ '\ \ )^2} - \frac{1}{'H^{-1}}$$

and

$$K_2 = \left[\frac{'H^{-2}}{(\ \ 'H^{-1}\ \ )^2} - \frac{1}{'}\right]\frac{1}{n}\sum_{i=1}^{n} \pm^2_{2i}\ .$$

Proof

$$Var(\sigma_1) = (1-\circ^2)\frac{1}{n}\sum_{i=1}^{n} \pm^2_{2i} \frac{1}{'} + \circ^2 \frac{1}{'H^{-1}} + K_1 \circ^2, \qquad (15)$$

$$Var(\sigma_2) = (1-\circ^2)\frac{1}{n}\sum_{i=1}^{n} \pm^2_{2i} \frac{1}{'} + \circ^2 \frac{1}{'H^{-1}} + K_2(1-\circ^2) \qquad (16)$$

From (15) and (16), we have

$Var(\sigma_1) < Var(\sigma_2)$ if $K_1 \circ^2 < K_2(1-\circ^2)$.

Thus σ_1 is more efficient than σ_2 if

$$\circ^2 < \frac{K_2}{K_1+K_2}\ .$$

That is,

$$|\circ| < \sqrt{\frac{K_2}{K_1+K_2}}\ .$$

3. ESTIMATION OF COMMON PARAMETERS μ_2 AND σ_2

Let $(X_i, Y_i), i = 1, 2, ..., n$ be independent bivariate random variables with (X_i, Y_i) having *pdf* $f_i(x, y)$ of the form

$$\frac{(2\pi\sigma_1\sigma_2)^{-1}}{\sqrt{1-\rho_i^2}}\exp\left\{\frac{-2^{-1}}{1-\rho_i^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}-2\rho_i\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

for $i = 1, 2, ..., n$. In this section we estimate the common parameters μ_2 and σ_2 under the assumption that $\rho_i, i = 1, 2, ..., n$ are known using concomitants of order statistics of *inid* random variables.

Clearly the marginal distributions of X_i and Y_i are $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$ respectively for $i = 1, 2, ..., n$. Let $f(x)$ and $F(x)$ denote the marginal *pdf* and distribution function respectively of each of the X_i 's. Let $f_i(y \mid x)$ denote the conditional *pdf* of Y_i given $X_i = x,\ i = 1, 2, ..., n$.

Then we have,

$$E\left[Y_{[r:n]}\right]=\mu_2+\frac{1}{n}\sum_i\rho_i\sigma_2\alpha_{r:n} \tag{17}$$

$$E\left[Y_{[r:n]}^2\right]=\frac{1}{n}\sum_i[\mu_2^2+\sigma_2^2(1-\rho_i^2)+2\rho_i\mu_2\sigma_2\alpha_{r:n}+\rho_i^2\sigma_2^2\alpha_{r,r:n}]$$

$$=\mu_2^2+\sigma_2^2(1-\frac{1}{n}\sum_i\rho_i^2)+2\mu_2\sigma_2\alpha_{r:n}\frac{1}{n}\sum_i\rho_i+\sigma_2^2\alpha_{r,r:n}\frac{1}{n}\sum_i\rho_i^2$$

Now,

$$Var(Y_{[r:n]})=\sigma_2^2+(\ _{r,r:n}-1)\frac{1}{n}\sum_i\rho_i^2\sigma_2^2+\frac{1}{n^2}\sigma_2^2\alpha_{r:n}^2\sum_{i<j}(\rho_i-\rho_j)^2 \tag{18}$$

We have, for $r < s$,

$$E\left[Y_{[r:n]}Y_{[s:n]}\right]=\frac{1}{n(n-1)}\sum_{i\neq j}\left(\mu_2^2+\rho_i\mu_2\sigma_2\alpha_{r:n}+\rho_j\mu_2\sigma_2\alpha_{s:n}+\rho_i\rho_j\sigma_2^2\alpha_{r,s:n}\right)$$

$$=\mu_2^2+\frac{1}{n}\sum_i\rho_i\mu_2\sigma_2(\alpha_{r:n}+\alpha_{s:n})+\frac{1}{n(n-1)}\sigma_2^2\alpha_{r,s:n}\sum_{i\neq j}\rho_i\rho_j.$$

Hence,

$$Cov\left[Y_{[r:n]}Y_{[s:n]}\right]=E\left[Y_{[r:n]}Y_{[s:n]}\right]-E\left[Y_{[r:n]}\right]E\left[Y_{[s:n]}\right]$$

$$=\frac{1}{n(n-1)}\sigma_2^2\ _{r,s:n}\sum_{i\neq j}\rho_i\rho_j-\frac{1}{n^2(n-1)}\sigma_2^2\alpha_{r:n}\alpha_{s:n}\sum_{i<j}(\rho_i-\rho_j)^2. \tag{19}$$

Consider the units of bivariate sample in which measurement of the X variate can be done easily where as a measurement of Y is not so easy or economic. In this case we order the X observations and make measurements only on the concomitants $Y_{[c+1:n]},\ldots,Y_{[n-c:n]}$. Now based on this restricted sample we may use estimates based on the available concomitants of order statistics.

Let

$$Y_{[n,c]} = (Y_{[c+1:n]}, Y_{[c+2:n]}, \ldots, Y_{[n-c:n]})' .$$

Then

$$E[Y_{[n,c]}] = \text{«}_2 1 + \pm_2 \sim \qquad (20)$$

where 1 is a column vector of $n-2c$ ones,

$$\sim\, = \frac{1}{n}\sum_{i=1}^{n} \circ_i (\sim_{c+1:n}, \sim_{c+2:n}, \ldots, \sim_{n-c:n})'$$

and the variance covariance matrix of $Y_{[n,c]}$ can be written in the form

$$D\left[Y_{[n,c]}\right] = \pm_2^2 G, \qquad (21)$$

where

$G = \| g_{rs} \|$ given by

$$g_{rr} = 1 + (\;\; _{r,r:n} - 1)\frac{1}{n}\sum_{i} \circ_i^2 + \frac{1}{n^2}\sim_{r:n}^2 \sum_{i<j} (\circ_i - \circ_j)^2$$

and for $r \neq s$,

$$g_{rs} = \frac{1}{n(n-1)} \;\; _{r,s:n}\sum_{i\neq j} \circ_i \circ_j - \frac{1}{n^2(n-1)}\sum_{i<j} (\circ_i - \circ_j)^2 \sim_{r:n} \sim_{s:n} ,$$

where i and j vary from 1 to n and r and s are such that $c+1 \le r < s \le n-c$.

Equations (20) and (21) together defines a generalized Gauss-Markov set up when the $\circ_i$'s are known and then the *BLUEs* $\hat{\text{«}}_2$ and $\hat{\pm}_2$ are given by

$$\hat{\text{«}}_2 = \frac{\sim' G^{-1}(\sim 1' - 1 \sim')G^{-1}}{\Delta} Y_{[n,c]}$$

and

$$\hat{\pm}_2 = \frac{1' G^{-1}(1 \sim' - \sim 1')G^{-1}}{\Delta} Y_{[n,c]},$$

where

$$\Delta = (\sim' G^{-1} \sim)(1' G^{-1} 1) - (\sim' G^{-1} 1)^2 .$$

The variances of the above estimators are given by

$$Var(\hat{«}_2) = \frac{\sim' G^{-1} \sim}{\Delta} \pm_2^2$$

and

$$Var(\hat{\pm}_2) = \frac{1' G^{-1} 1}{\Delta} \pm_2^2 .$$

It is clear that $\hat{«}_2$ and $\hat{\pm}_2$ are linear functions of the concomitants and hence we can write

$$\hat{«}_2 = \sum_{r=c+1}^{n-c} a_r Y_{[r:n]}$$

and

$$\hat{\pm}_2 = \sum_{r=c+1}^{n-c} b_r Y_{[r:n]} \; ,$$

where $a_r b_r, r = c+1, c+2, ..., n-c$ are constants which can be determined.

The values of $\sim_{r:n}$ for different values of r and n are given in Harter (1961) and those of $_{r,s:n}$ for different values of r , s and n are given in Sarhan and Greenberg (1956). Hence, the constants $a_r b_r, r = c+1, c+2, ..., n-c$ can be easily determined using *MATHCAD* software. For an illustration, let us assume that k of the observations are from a bivariate normal population with correlation coefficient $°_1$ and the remaining $n-k$ observations are from a bivariate normal population with correlation coefficient $°_2$. Then we have computed the values of the constants $a_r b_r, r = c+1, c+2, ..., n-c$ and the variances of the estimators $\hat{«}_2$ and $\hat{\pm}_2$ for $(°_1, °_2) = (0.8, 0.7), (0.6, 0.5)$ for $n = 5$. The values are given in the following tables. From tables numbered 1 to 4, we observe that both $«_2$ and $\pm_2$ are estimated with more precision for larger values of $°_i$'s than for the cases with smaller values of $°_i$'s. Since $Var(\hat{«}_2)$ is always smaller than $Var(\hat{\pm}_2)$ for the concomitants of order statistics for a given collection of random variables we infer that concomitants of order statistics of *inid* n normal random variables can be more profitably used for estimating $«_2$ than for estimating $\pm_2$. Very rarely in problems of estimation of common parameters of several distributions, explicit expression for the values of the proposed estimator exists. However in our method it can be explicitly expressed and hence we can determine the quality of our estimator as well.

Table 1: Coefficients a_i's in the *BLUE* $\hat{«}_2 = \sum_{r=c+1}^{n-c} a_r Y_{[r:n]}$ and $Var(\hat{«}_2)/±_2^2$ for $°_1 = 0.8, °_2 = 0.7$

n	c	k	a_1	a_2	a_3	a_4	a_5	$Var(«_2)/±_2^2$
5	0	1	0.1995	0.2003	0.2004	0.2003	0.1995	0.2000
		2	0.1991	0.2005	0.2008	0.2005	0.1991	0.2000
		3	0.1991	0.2005	0.2008	0.2005	0.1991	0.2000
		4	0.1994	0.2003	0.2006	0.2003	0.1994	0.2000
	1	1		0.3370	0.3260	0.3370		0.2779
		2		0.3374	0.3253	0.3374		0.2744
		3		0.3379	0.3242	0.3379		0.2713
		4		0.3385	0.3230	0.3385		0.2685

Table 2: Coefficients b_i's in the *BLUE* $±_2 = \sum_{r=c+1}^{n-c} b_r Y_{[r:n]}$ and $Var(±_2)/±_2^2$ for $°_1 = 0.8, °_2 = 0.7$.

n	c	k	b_1	b_2	b_3	b_4	b_5	$Var(±_2)/±_2^2$
5	0	1	-0.5064	-0.2132	0.0000	0.2132	0.5064	0.4243
		2	-0.4927	-0.2076	0.0000	0.2076	0.4927	0.3921
		3	-0.4798	-0.2018	0.0000	0.2018	0.4798	0.3625
		4	-0.4678	-0.1959	0.0000	0.1959	0.4678	0.3350
		1		-1.4033	0.0000	1.4033		2.2230
	1	2		-1.3644	0.0000	1.3644		2.0105
		3		-1.3291	0.0000	1.3291		1.8175
		4		-1.2949	0.0000	1.2949		1.6404

Table 3: Coefficients a_i's in the BLUE $\hat{«}_2 = \sum_{r=c+1}^{n-c} a_r Y_{[r:n]}$ and $Var(\hat{«}_2)/±_2^2$ for $°_1 = 0.6, °_2 = 0.5$

n	c	k	a_1	a_2	a_3	a_4	a_5	$Var(\hat{«}_2)/±_2^2$
5	0	1	0.1995	0.2003	0.2004	0.2003	0.1995	0.2001
		2	0.1994	0.2003	0.2006	0.2003	0.1994	0.2001
		3	0.1994	0.2003	0.2006	0.2003	0.1994	0.2001
		4	0.1996	0.2002	0.2004	0.2002	0.1996	0.2001
		1		0.3351	0.3298	0.3351		0.3050
	2	2		0.3350	0.3300	0.3350		0.3010
		3		0.3350	0.3300	0.3350		0.2999
		4		0.3351	0.3298	0.3351		0.2975

Table 4: Coefficients b_i's in the *BLUE* $±_2 = \sum_{r=c+1}^{n-c} b_r Y_{[r:n]}$ and $Var(±_2)/±_2^2$ for $°_1 = 0.6, °_2 = 0.5$.

n	c	k	b_1	b_2	b_3	b_4	b_5	$Var(±_2)/±_2^2$
5	0	1	-0.7003	-0.2973	0.0000	0.2973	0.7003	0.9781
		2	-0.6741	-0.2866	0.0000	0.2866	0.6741	0.8940
		3	-0.6502	-0.2762	0.0000	0.2762	0.6502	0.8187
		4	-0.6280	-0.2661	0.0000	0.2661	0.6280	0.7510
		1		-1.9415	0.0000	1.9415		5.8262
	3	2		-1.8694	0.0000	1.8694		5.2755
		3		-1.8049	0.0000	1.8049		4.7905
		4		-1.7426	0.0000	1.7426		4.3519

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