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BAYES AND MAXIMUM LIKELIHOOD ESTIMATORS FOR WEIBULL PARAMETERS UNDER MULTIPLY TYPE-II CENSORING

Puneet K. Srivastava, Akanksha Gupta, S. K. Upadhyay

ABSTRACT

This paper considers a multiply type II censored sample from the two-parameter Weibull distribution. Using a generalized non-informative prior, Bayes estimators of parameters such as scale, shape, reliability and hazard functions are proposed. The proposed estimators are compared with the corresponding maximum likelihood estimators obtained using EM algorithm. The results are illustrated on the basis of simulated as well as the real data sets.

1. INTRODUCTION

The Weibull distribution is particularly important in the context of lifetime data analyses, and a large body of literature on statistical methods has evolved out of it. Reasons for its popularity can be attributed to its flexibility and capability to accommodate a wide variety of situations and perhaps because of the existence of closed form expressions of its reliability and hazard function. In its simplest form, the Weibull distribution has two parameters with probability density function (pdf) given by,

$$f(x\,|\,{}^{-},\ ) = \frac{\ }{-}\, x^{\ -1} e^{-(x/{}^{-})}\ ,\ x \ge 0 \qquad (1)$$

where the positive parameters $^{-}$ and are referred to as the scale and shape parameters, respectively. The reliability function $R(t)$, the probability of survival until time t, for the model is given by

$$R(t) = e^{-(t/{}^{-})}\ . \qquad (2)$$

Similarly the hazard function of the model, which describes the way in which the instantaneous rate of failure for an item changes with time, is given by

$$H(t) = \frac{\ }{\ }(t/{}^{-})^{\ -1} \qquad (3)$$

The hazard function of the Weibull distribution is monotone increasing for >1, decreasing for <1, and the constant for $=1$. Obviously, the Weibull distribution reduces to the constant hazard rate exponential distribution for $=1$. The increasing hazard rate Rayleigh distribution also becomes an important special case of the model for $=2$ (see also Martz and Waller (1982)).

The Weibull distribution is quite rich as far as its inferential developments are concerned. Both classical and Bayesian inferences for the model are available in bulk although the situation sometimes requires for non-closed form inferential procedures and the major reason can be attributed to the non-existence of nice closed forms of sufficient statistics. Classical maximum likelihood (ML) estimators can be obtained by solving iteratively a non-linear equation for . The MLE of the parameters of Weibull distribution for complete and censored data set are obtained by Cohen (1965), Balakrishnan and Kateri (2008) and Ahmed *et al*. (2010), among others. For other detailed discussions on classical inferences, leading to estimation and testing scenarios, readers are referred to Mann *et al.* (1974) and Lawless (2002), etc. A few such discussions are based on empirical findings and/or asymptotic approximations.

Bayes inferences for the Weibull parameters were considered for the first time in a very systematic form by Martz and Waller (1982) though the authors were mostly confined to point and interval estimation problems. The inferences are, in general, not available in closed forms and one is required to consider one or two dimensional integrals depending on the form of entertained priors and/or likelihoods for the model parameters. An unrestricted complete Bayes analysis of Weibull distribution was given by Upadhyay *et al.* (2001) using sample based approaches to Bayes computation.

Upadhyay *et al.* (2001) also discussed several other important features using sample based approaches (see also Upadhyay and Smith (1994)). A relatively recent reference is due to Singpurwalla (2006) that successfully provides the details of various Bayesian developments related to the Weibull model.

The inferential developments discussed as above are equally well applicable to censoring scenarios such as type II or type I schemes, although the latter not as detailed in the literature as the former. The situation, however, becomes slightly difficult if the available data are compounded with censoring such as progressive or multiply schemes. This is perhaps the reason that most of the descriptions given in the literature are confined to type II schemes only. The sample based approaches to Bayesian computation are, however, exceptions and can deal with most of difficult censoring schemes in a routine manner (see, for example, Upadhyay *et al.* (2001) and Upadhyay and Smith (1994)).

The present paper is an attempt to provide a generalization in the form of multiply type II censoring scheme when the entertained model is a two-parameter Weibull with both parameters treated unknown. The work using multiply type II censored data is limited to exponential and some similar distributions (see, for example, Upadhyay *et al.* (1996), Balasubramanian and Balakrishnan (1992), Singh *et al.* (2005), etc.) but very little appears in the literature on Weibull distribution. An important reason being the mathematical tractability as the corresponding likelihood function (LF) further complicates and numerical or simulation methods appear to be the only alternatives.

Multiply type II censoring scheme is actually the combination of doubly censoring and mid censoring schemes. This occurs when the missing observations belong to the two extremes as well as in the middle ranges. Let us consider, for instance, a situation when a reliability practitioner wishes to test the quality and reliability of a product from some manufacturing process. When the experimenter starts taking observations, he notices some failures (say, l in number) have already occurred. The experimenter thus starts observing the failure times of the remaining items ($(l+1)^{th}$ onwards) but after noting the failure times of a few items, he finds that the process is stopped due to some mechanical and/or the process defect. As such he is forced to stop taking the observations for the duration of mechanical failure. When he resumes he finds that a few failures have already occurred that he failed to notice. He resumes the experimentation and terminates the same after sufficient number of failures has been observed to draw the desired inferential procedures. Thus he leaves a few observations in the right extreme as well. A few important situations especially in the context of medical, social and reliability studies have been discussed by Balasubramanian and Balakrishnan (1992) and Upadhyay and Shastri (1997).

The plan of the paper is as follows. The next section provides the necessary mathematical formulation for obtaining ML and Bayes estimators of model parameters under the entertained censoring scheme. Since the issue is mainly computational, we have advocated for the use of expectation-maximization (EM) algorithm for finding the ML estimators. Similarly for Bayes estimators, we have used Gibbs sampler algorithm for censored data problems as discussed by Upadhyay *et al*. (2001). The essential background material needed to obtain Bayes and ML estimators is also provided for completeness although the discussions have been fully supported with the relevant references throughout. The numerical illustration is provided in Section 3 where ML and Bayes estimates are worked out and their corresponding posterior risks have been provided on the basis of both real and simulated data sets. Finally, a brief conclusion is given at the end.

2. ESTIMATION OF PARAMETERS

Let us consider that n items with failure time distribution given in (1) be subjected to testing and that the observed failure times $x_{l+1} \le, \ldots, \le x_{l+k}$ and $x_{l+k+m+1} \le, \ldots, \le x_{n-r}$ arise according to a multiply type II censoring scheme. Thus the missing data consist of l observations in left $\left(x_i < x_{1+1}, i = 1, \ldots, l\right)$, m

observations in the middle $\left(x_{1+k} < x_i < x_{1+k+m+1}, i = 1,\ldots,m\right)$ and r observations in the right $\left(x_i > x_{n-r}, i = 1,\ldots,r\right)$. The LF based on the available information can, therefore, be written as

$$L = \frac{n!}{l!m!r!} \prod_{i=l+1}^{l+k} f\left(x_i \mid \overline{\ },\ \ \right) \prod_{i=p}^{n-r} f\left(x_i \mid \overline{\ },\ \ \right) \left\{F\left(x_{l+1}\right)\right\}^l \left\{F\left(x_p\right) - F\left(x_{l+k}\right)\right\}^m$$

$$\times\left\{1 - F\left(x_{n-r}\right)\right\}^r \tag{4}$$

where $p = l + k + m + 1$ and $F(.)$ is used to denote the cumulative distribution function associated with the form (1). Substituting the corresponding expressions for $F(.)$ and $f(.)$ in (4), the LF reduces to

$$L = \frac{n!}{l!m!r!} \frac{\ \ ^A}{\overline{\ }\ \ ^A} \left\{1 - e^{-\left(x_{l+1}/\overline{\ }\ \right)}\ \right\}^l \left\{e^{-\left(x_{l+k}/\overline{\ }\ \right)} - e^{-\left(x_{l+k+m+1}/\overline{\ }\ \right)}\ \right\}^m \prod_{i=l+1}^{l+k} x_i \prod_{i=p}^{n-r} x_i$$

$$\times e^{-\frac{1}{\overline{\ }}\left\{r\left\{x_{n-r}\right\} + \sum_{i=l+1}^{l+k} x_i + \sum_{i=l+k+m+1}^{n-r} x_i\right\}}, \tag{5}$$

where $A = n - l - m - r$.

2.1 Maximum Likelihood Estimation

Taking logarithm of (5), we get

$$\log L = Q + A\log\ \ - A\ \ \log\overline{\ } + l\log\left\{1 - e^{-\left(x_{l+1}/\overline{\ }\ \right)}\ \right\}$$

$$+ m\log\left\{e^{-\left(x_{l+k}/\overline{\ }\ \right)} - e^{-\left(x_p/\overline{\ }\ \right)}\ \right\} + \left(\ \ -1\right)\left\{\sum_{i=l+1}^{l+k} \log x_i + \sum_{i=p}^{n-r} \log x_i\right\}$$

$$- \frac{1}{\overline{\ }}\left\{r x_{n-r} + \sum_{i=l+1}^{l+k} x_i + \sum_{i=p}^{n-r} x_i\ \right\}, \tag{6}$$

where $Q = \log\left(n!/\left(l!m!r!\right)\right)$.

The ML estimators of $\overline{\ }$ and   can be obtained by maximizing the log of LF given in (6). This can be done by partially differentiating (6) with respect to $\overline{\ }$ and   , respectively, and equating them to zero giving rise to the two likelihood equations. On solving the two likelihood equations can be written as

$$A\text{¯} \quad + \frac{r e^{-(x_{l+1}/\text{¯})}}{\left\{1-e^{-(x_{l+1}/\text{¯})}\quad\right\}} x_{l+1} + \frac{\left\{e^{-(x_{l+k}/\text{¯})}\quad x_{l+k} - e^{-(x_p/\text{¯})}\quad x_p\right\}}{\left\{e^{-(x_{l+k}/\text{¯})}\quad - e^{-(x_p/\text{¯})}\quad\right\}}$$

$$-\left\{r x_{n-r} + \sum_{i=l+1}^{l+k} x_i \quad + \sum_{i=p}^{n-r} x_i\right\} = 0, \tag{7}$$

and

$$I_1 + I_2 + I_3 - I_4 = 0, \tag{8}$$

where

$$I_1 = \frac{e^{-(x_{l+1}/\text{¯})}\quad x_{l+1} \quad \log(x_{l+1})}{\{1-e^{-(x_{l+1}/\text{¯})}\quad\}} + - \quad \left\{\sum_{i=l+1}^{l+k} \log x_i + \sum_{i=p}^{n-r} \log x_i + \frac{A}{}\right\},$$

$$I_2 = \frac{l\left\{e^{-(x_{l+k}/\text{¯})}\quad x_{l+k} \quad \log(x_{l+k}/\text{¯}) \quad - e^{-(x_p/\text{¯})}\quad x_p \quad \log(x_p/\text{¯})\quad\right\}}{\{e^{-(x_{l+k}/\text{¯})}\quad - e^{-(x_p/\text{¯})}\quad\}},$$

$$I_3 = \text{—}\left\{r x_{n-r} \quad + \sum_{i=l+1}^{l+k} x_i \quad + \sum_{i=p}^{n-r} x_i \quad\right\} -$$

$$\left\{r x_{n-r} \quad \log x_{n-r} + \sum_{i=l+1}^{l+k} x_i \quad \log x_i + \sum_{i=p}^{n-r} x_i \quad \log x_i\right\},$$

$$I_4 = \left\{r x_{n-r} \quad \log x_{n-r} + \sum_{i=l+1}^{l+k} x_i \quad \log x_i + \sum_{i=p}^{n-r} x_i \quad \log x_i\right\}.$$

The solutions of (7) and (8) will provide the *ML* estimators of ¯ and , respectively, provided one makes sure about the sufficient condition, that is, the associated Hessian matrix is negative definite (see, for example, Kendal and Stuart (1967)).

It can be seen that the equations for obtaining *ML* estimators of ¯ and cannot be solved analytically to get closed form solutions and, therefore, one has to rely on numerical approximations, which involve solving a set of two nonlinear equations (7) and (8). A possible approach can be Newton-Raphson method or any other iterative procedure. We, however, advocate the use of comparatively efficient expectation-maximization (EM) algorithm (see, for example, Dempster *et al.* (1977)) for missing data problems. The *EM* algorithm consists of two steps. The first step known as the expectation step $(E-step)$ consists of estimating the unknown censored data on the basis of the current values of the parameters. Suppose the current values of ¯ and are denoted by $^{-}{}_c$ and ${}_c$ and suppose the estimated censored observations on the first implementation of $E-step$ are denoted by $x_1,\ldots,x_l$ (in the left region), $x_{l+k+1},\ldots,x_{l+k+m}$ (in the middle region), and $x_{n-r+1},\ldots,x_n$ (in the right most region). These censored observations can be estimated by generating them from the truncated Weibull distributions in the specified ranges $\left(0,x_{l+1}\right)$, $\left(x_{l+k},x_{l+k+m+1}\right)$ and $\left(x_{n-r},\infty\right)$, respectively, using the current values $^{-}{}_c$ and ${}_c$. The analytical estimation for the censored observations might be difficult in this case but one can always use simulation based strategy for estimating the variate values from the corresponding truncated distributions. Say, for instance, one is interested in estimating $x_1,\ldots,x_l$ in the E-step from the truncated Weibull distribution in the region $\left(0,x_{l+1}\right)$. One can generate several thousand values corresponding to each of these missing observations and retain the corresponding simple arithmetic average as its estimate. So, once these missing data are estimated, the complete data likelihood at this stage can be written as

$$L_c=\prod_{i=1}^{n} f\left(x_i \mid \;,\;\right). \tag{9}$$

Obviously, (9) contains both estimated censored data and the observed failure times.

The second step of EM algorithm, known as the maximization step $(M-step)$, consists of maximizing the likelihood (9) with respect to $^{-}$ and . This task is comparatively simpler and reduces to finding Weibull ML estimators for complete data problem. This maximization can be done by differentiating $\log\left(L_c\right)$ separately with respect to $^{-}$ and and solving the corresponding equations after equating them to zero. Of course, one has to verify the sufficient condition that is the associated Hessian matrix is negative definite. On simplification, the two likelihood equations based on (9) can be written as

$$\hat{}_c=\left\{\frac{1}{n}\sum_{i=1}^{n} x_i^{\hat{}_c}\right\}^{\frac{1}{\hat{}_c}}, \tag{10}$$

and

$$\frac{1}{{}_c}-\frac{1}{\sum_{i=1}^{n} x_i^{\,{}_c}}\sum_{i=1}^{n} x_i^{\,{}_c}\log x_i+\frac{1}{n}\sum_{i=1}^{n}\log x_i=0. \tag{11}$$

The equation (11) can be solved by Newton-Raphson method to get an updated ${}_c$ and using this updated ${}_c$ in (10), a new $^{-}{}_c$ can be obtained. These new updated $^{-}{}_c$ and ${}_c$ can be used again in $E-step$ to get the new estimated censored data values and then in $M - step$ to get a new β_c and α_c from (9). The process can be repeated until some systematic pattern of convergence or desired level of accuracy is achieved.

Once the ML estimates of β and α are obtained, the reliability function and the hazard function can be estimated using the invariance property of the ML estimators. Thus, the corresponding ML estimators of reliability and hazard rate at time t are

$$\hat{R}_{ML}(t) = e^{-\left(t/\hat{\beta}_{ML}\right)^{\hat{\alpha}_{ML}}}, \tag{12}$$

and

$$\hat{H}_{ML}(t) = \frac{\hat{\alpha}_{ML}}{\hat{\beta}_{ML}}\left(t/\hat{\beta}_{ML}\right)^{\hat{\alpha}_{ML}-1}, \tag{13}$$

where $\hat{\beta}_{ML}$ and $\hat{\alpha}_{ML}$ are the ML estimators of the parameters β and α, respectively.

2.2 Bayes estimation of the parameters

Bayesian estimation of Weibull parameters in case of multiply type II censoring can also be attempted in a way similar to what has been done with ML estimation but this time the procedure can be based on Gibbs sampler algorithm for censored data situations. Undoubtedly, the Bayesian estimation can be attempted in several other ways but the Gibbs sampler algorithm is being advocated because of its inherent ease. Before we provide the necessary details of the algorithm, let us discuss the relevant modelling formulation for the implementation of the Bayesian paradigm.

To begin with, consider the form of likelihood given in (9) presuming as if there is no censoring and all the observations are made available although the situation involves a few unknown observations corresponding to missing censored data. Following Upadhyay *et al.* (2001), let us assume a joint non-informative prior for the parameters β and α as

$$g(\beta, \alpha) = \frac{1}{\beta\alpha}. \tag{14}$$

Combining the LF in (9) with the prior in (14) via Bayes theorem, the joint posterior distribution of the parameters β and α up to proportionality can be written as

$$p(\beta, \alpha | x) \propto \frac{\alpha^{n-1}}{\beta^{n\alpha+1}} \prod_{i=1}^{n} x_i^{\alpha-1} \exp\left[-\frac{1}{\beta^{\alpha}}\sum_{i=1}^{n} x_i^{\alpha}\right]. \tag{15}$$

The posterior given in (15) is too complicated for the analytical determination of any closed form inferences. Moreover, it has unknowns in the form of missing censored data as well. We, therefore, recommend the use of Gibbs sampler algorithm for censored data problems so that sample based inferences on the posterior (15) can be easily drawn. The discussion given below provides a brief review of the algorithm for censored data problems but with a focus on posterior (15). The interested readers may refer to Upadhyay *et al.* (2001) for further details and relevant references.

The Gibbs sampler algorithm is a Markovian updating scheme for extracting samples from the posteriors specified only up to proportionality as in (15). The algorithm requires the specification of full conditionals for the concerned unknown variate values from the joint posterior distribution. Once the full conditionals are made available, the algorithm proceeds in a cyclic manner by generating from various full conditionals, in turn, and using the most recent values for all the fixed variates in a conditional structure. It can be shown that after sufficiently large number of iterations, the generated variates converge in distribution to a random sample from the corresponding posterior (see, for example, Upadhyay and Smith (1994)).

To clarify, let us consider the joint posterior (15). It has unknown ¯ and besides having the unknown censored observations. The full conditionals of ¯ and can be written as

$$p_1\left(\bar{}\,\middle|\,\ ,x\right) \propto \frac{1}{\bar{}^{\,n\ +1}} \exp\left[-\frac{1}{\bar{}} \sum_{i=1}^{n} x_i\ \right], \tag{16}$$

$$p_2\left(\ \middle|\ \bar{}, x\right) \propto \frac{^{n-1}}{\bar{}^{\,n}} \prod_{i=1}^{n} x_i^{\ -1} \exp\left[-\frac{1}{\bar{}} \sum_{i=1}^{n} x_i\ \right], \tag{17}$$

whereas the full conditionals for the unknown censored observations can be taken as the truncated Weibull distributions in the corresponding cut-off regions. Since the censored observations are independent, each can be drawn separately from its truncated distribution. Suppose, for instance, we are interested in the censored observations $x_1, \ldots, x_l$ (censored in the left region) then each can be generated independently from the truncated Weibull distribution truncated in the region $\left(0, x_{l+1}\right)$. The ranges for generating other censored observations from the corresponding truncated Weibull distributions can be specified similarly. It is to be noted that we need to consider the current values of ¯ and for generating the censored observations from the corresponding truncated Weibull distributions. Now coming back to conditional structures given in (16) and (17), it can be seen that (16) can be reduced to a gamma distribution with scale parameter $\left(1 \Big/ \sum_{i=1}^{n} x_i\ \right)$ and shape parameter n by taking a transformation ª = ¯ ⁻ . Thus ª can be easily generated from the gamma distribution using any gamma generating routine and transforming back to ¯ , one can get the corresponding samples of θ from its conditional structure. (17) can be shown to be concave on logarithmic scale and, therefore, β can be generated using adaptive rejection sampling algorithm of Gilks and Wild (1992). Thus all the full conditionals can be easily generated giving rise to successful implementation of the Gibbs sampler algorithm.

The Gibbs sampler algorithm so defined can be run for sufficiently large number of iterations until some systematic pattern of convergence is noticed among the generating variates. Once the convergence monitoring is done, one can pick up observations at suitably chosen intervals to form a random sample from the corresponding posterior with components representing the sample from the corresponding marginal posterior. The gaps are chosen so as to minimize the serial correlation among the generating variates.

Once the samples are obtained perhaps any sample based posterior estimates can be easily formed (see, for example, Upadhyay *et al*. (2001)) though our study is confined to Bayes estimates of parameters and a few posterior density estimates. The other apparent advantage of the algorithm is that once the parameter estimates are obtained, one can easily use the same to form the estimates of any linear or non-linear function of the parameters. Say, for example, the Bayes estimates of reliability function and hazard function at time t can be written as

$$\hat{R}_B(t) = e^{-\left(t/\hat{\theta}_B\right)^{\hat{\beta}_B}} \tag{18}$$

and

$$\hat{H}_B(t) = \frac{\hat{\beta}_B}{\hat{\theta}_B}\left(t/\hat{\theta}_B\right)^{\hat{\beta}_B - 1}, \tag{19}$$

where $\hat{\theta}_B$ and $\hat{\beta}_B$ are the corresponding Bayes estimates of the parameters θ and β, respectively.

3. NUMERICAL ILLUSTRATION

The numerical illustration is based on a real as well as a simulated data set from the Weibull model (1). In both the cases we obtained classical ML estimators and Bayes estimators of the model parameters. Our focus has been on multiply censoring scheme though the cases of left, right and mid censoring schemes are also considered, the latter being particular cases of the former.

3.1 Simulated data based study

In order to obtain the ML estimates and the corresponding Bayesian results for the parameters of the Weibull model under multiply type II censoring scheme, we first generated a sample of size 20 from (1) with $\theta = 6.00$ and $\beta = 2.00$. Since the cumulative distribution function of the Weibull model is available in closed form, this generation can be easily done by the inverse transform method.

We next considered as if the 50% of the generated Weibull observations were censored in accordance with multiply censoring scheme and took some arbitrary choices of l, m and r leaving the corresponding generated observations . These choices were done so that exactly 50% of the observations were made available as observed failures (Tables 1-4 for different arbitrary choices of l, m and r).

The ML estimates of Weibull parameters based on the above scheme were obtained using EM algorithm (see Section 2.1) for the considered combinations of l, k, m and r . For E-step, we considered 5000 generated values for each of the missing observations by generating from the corresponding truncated Weibull distributions and evaluated simple arithmetic averages as the estimate in each case. The ML estimates of ¯ and for different combinations of l, k, m and r are shown in Table 1 whereas the corresponding estimates for $R(t)$ and $H(t)$ are given in Tables 3 and 4, respectively. The estimates of $R(t)$ and $H(t)$ were obtained by substituting the corresponding estimates of ¯ and in the expressions of $R(t)$ and $H(t)$. The mission time t was arbitrarily fixed at $5.0, 10.0$ and 15.0 (see Tables 3-4).

We next applied the Gibbs sampler algorithm as per the details given in Section 2.2 for the considered combinations of l, k, m and r and using the ML estimates of ¯ and as the initial values for running the chain. We considered a single long run of the Gibbs chain and the convergence monitoring was done using ergodic averages for both ¯ and . It was assessed at about 50,000 iterations for the considered 50% censored scenario. It can be noted that the stabilized ergodic averages are nothing but the posterior means and, therefore, can be considered as the Bayes estimates corresponding to squared error loss function. Besides, we also monitored the chains corresponding to $R(t)$ and $H(t)$ at three different mission times $t(=5, 10, 15)$. The chains corresponding to $R(t)$ and $H(t)$ were obtained from the chains of ¯ and by substitution. The stabilized ergodic averages for $R(t)$ and $H(t)$ were also obtained in a similar manner which may be considered as the Bayes estimates of $R(t)$ and $H(t)$ corresponding to squared error loss function. Table 1 provides the Bayes estimates of model parameters ¯ and whereas the corresponding estimates for $R(t)$ and $H(t)$ are shown in Tables 3 and 4.

The Gibbs chain was run for another $50,000$ iterations after the convergence was monitored based on ergodic averages. We then picked up equally spaced outcomes (every $10-th$) from the last $50,000$ generated outcomes to form samples of size 5000 from the corresponding marginal posteriors of ¯ and . These gaps were chosen to make serial correlation negligibly small. We also evaluated 5000 posterior samples for each of $R(t)$ and $H(t)$ using the samples of ¯ and at $t = 5,10$ and 15 separately. The estimated posterior risks of *ML* estimates and the corresponding Bayes estimates using the 5000 simulated posterior samples are shown in parentheses for each of the considered parameters. These posterior risks correspond to the squared error loss function. It can be seen from Tables 1-3 that the *ML* estimates and the corresponding Bayes estimates are quite close to each other in terms of their magnitudes but the posterior risks of latter are, in general, smaller than those of former. This conclusion suggests that Bayes estimators, in general, outperform the corresponding *ML* estimators.

Table 1: *ML* and Bayes estimates of ¯ and at different combinations of censoring parameters (results based on simulated data with overall censoring 50%)

Censoring scheme	*l* *k* *m* *r*	ML estimate		Bayes estimate	
		$\bar{\hat{\ }}_{ML}$	$\hat{\ }_{ML}$	$\hat{\theta}_B$	$\hat{\beta}_B$
Multiply	6 4 2 2	7.88	2.56	7.94	2.22
	5 6 2 3	7.77	2.58	7.89	2.19
	7 5 2 1	7.78	2.37	7.76	2.05
	5 4 4 1	7.92	2.46	7.94	2.21
	4 3 4 2	7.92	2.60	8.02	2.33
	3 4 4 3	7.93	2.80	8.06	2.47
	5 5 4 1	7.88	2.44	7.89	2.18
Left	10 10 0 0	7.53	2.20	7.27	1.78
Doubly	8 10 0 2	7.74	2.45	7.68	1.97
Mid	0 5 10 0	8.24	2.78	8.29	2.63
Right	0 10 0 10	6.70	5.67	6.21	9.16

We have also provided the marginal posterior density estimates of ¯ and in the form of histograms (see Figures 1-2) using the simulated posterior samples of size 5000 from each of the two posteriors. In each of the figures, the vertical line corresponds to the *ML* estimates. The histograms are shown for one combination of l, m, r and k although a similar behaviour was noticed for all other considered combinations. For rest of the combinations of l, m, r and k, these estimated densities are, however, shown in the form of boxplot representations (see Figures 3-4). It can be seen that most of the estimated posterior densities are more or less similar in appearance and there is no appreciable change in the estimates for changing combinations of l, m, r and k.

Table 2: Estimated posterior risks of *ML* and Bayes estimators of ¯ and corresponding to squared error loss (results based on simulated data with overall censoring 50%)

Censoring scheme	l	k	m	r	Posterior Risk			
					¯^ $_{ML}$	^ $_{ML}$	$\hat{\theta}_B$	$\hat{\beta}_B$
Multiply	6	4	2	2	0.876	0.253	0.878	0.366
	5	6	2	3	1.002	0.286	1.020	0.429
	7	5	2	1	1.010	0.235	1.011	0.332
	5	4	4	1	0.791	0.220	0.793	0.282
	4	3	4	2	0.862	0.270	0.873	0.338
	3	4	4	3	0.772	0.291	0.787	0.406
	5	5	4	1	0.909	0.216	0.910	0.280
Left	10	10	0	0	1.362	0.241	1.42	0.386
Doubly	8	10	0	2	1.183	0.270	1.187	0.495
Mid	0	5	10	0	0.618	0.201	0.622	0.221
Right	0	10	0	10	0.289	2.959	0.533	15.376

Table 3: *ML* and Bayes estimates of reliability and the corresponding posterior risks (of order 10^{-3}) in parentheses (results based on simulated data with overall censoring 50%)

Censoring scheme	$l\ k\ m\ r$	ML estimate of $R(t)$			Bayes estimate of $R(t)$		
		$\hat{R}_{ML}(5)$	$\hat{R}_{ML}(10)$	$\hat{R}_{ML}(15)$	$\hat{R}_B(5)$	$\hat{R}_B(10)$	$\hat{R}_B(15)$
Multiply	6 4 2 2	0.732 (11.555)	0.159 (7.436)	0.006 (2.798)	0.684 (9.340)	0.198 (6.286)	0.030 (1.862)
	5 6 2 3	0.726 (10.778)	0.147 (9.055)	0.004 (2.348)	0.682 (8.914)	0.193 (6.919)	0.033 (1.481)
	7 5 2 1	0.704 (11.813)	0.163 (6.429)	0.009 (2.103)	0.658 (9.701)	0.193 (5.540)	0.033 (1.481)
	5 4 4 1	0724 (10.210)	0.170 (6.450)	0.008 (1.735)	0.687 (8.859)	0.196 (5.740)	0.028 (1.309)
	4 3 4 2	0.739 (9.089)	0.160 (7.232)	0.005 (1.737)	0.705 (8.015)	0.192 (6.142)	0.025 (1.300)
	3 4 4 3	0.760 (8.656)	0.147 (8.233)	0.003 (1.719)	0.723 (7.250)	0.187 (6.690)	0.023 (1.307)
	5 5 4 1	0.719 (9.823)	0.167 (6.101)	0.008 (1.496)	0.682 (8.620)	0.194 (5.315)	0.029 (1.081)
Left	10 10 0 0	0.666 (18.551)	0.155 (5.183)	0.011 (2.234)	0.592 (12.798)	0.179 (4.600)	0.039 (1.469)
Doubly	8 10 0 2	0.710 (15.330)	0.154 (7.514)	0.006 (2.842)	0.644 (10.990)	0.193 (6.075)	0.038 (1.863)
Mid	0 5 10 0	0.779 (6.758)	0.180 (5.575)	0.005 (0.780)	0.756 (6.214)	0.200 (5.210)	0.016 (0.528)
Right	0 10 0 10	0.872 (10.665)	0.751 (5.419)	0.0000 (0.071)	0.802 (5.673)	0.802 (2.142)	0.001 (0.070)

Table 4: ML and Bayes estimates of hazard rate and the corresponding posterior risks in (results based on simulated data with overall censoring 50%)

Censoring scheme	l k m r	ML estimate of $H(t)$			Bayes estimate of $H(t)$		
		$\hat{H}_{ML}(5)$	$\hat{H}_{ML}(10)$	$\hat{H}_{ML}(15)$	$\hat{H}_{B}(5)$	$\hat{H}_{B}(10)$	$\hat{H}_{B}(15)$
Multiply	6 4 2 2	2.252 (8.211)	0.488 (6.632)	0.017 (5.415)	2.598 (8.067)	0.821 (6.499)	0.183 (5.376)
	5 6 2 3	2.185 (0.875)	0.442 (0.596)	0.012 (0.202)	2.600 (0.712)	0.823 (0.459)	0.186 (0.176)
	7 5 2 1	2.311 (0.679)	0.535 (0.458)	0.028 (0.174)	2.590 (0.602)	0.821 (0.377)	0.174 (0.152)
	5 4 4 1	2.331 (4.766)	0.545 (3.886)	0.026 (3.173)	2.582 (4.698)	0.800 (3.815)	0.154 (3.154)
	4 3 4 2	2.251 (0.546)	0.486 (0.360)	0.015 (0.096)	2.517 (0.472)	0.743 (0.292)	0.123 (0.083)
	3 4 4 3	2.151 (0.668)	0.417 (0.449)	0.007 (0.144)	2.468 (0.567)	0.705 (0.366)	0.112 (0.133)
	5 5 4 1	2.322 (0.420)	0.540 (0.265)	0.026 (0.055)	2.553 (0.365)	0.778 (0.209)	0.139 (0.043)
Left	10 10 0 0	2.280 (0.589)	0.529 (0.366)	0.036 (0.124)	2.516 (0.540)	0.829 (0.283)	0.215 (0.094)
Doubly	8 10 0 2	2.242 (0.779)	0.485 (0.517)	0.020 (0.163)	2.620 (0.646)	0.860 (0.383)	0.207 (0.128)
Mid	0 5 10 0	2.309 (1.350)	0.5345 (1.058)	0.014 (0.784)	2.440 (1.331)	0.679 (1.036)	0.069 (0.780)
Right	0 10 0 10	0.590 (0.409)	0.000 (0.032)	0.000 (0.002)	1.066 (0.182)	0.045 (0.030)	0.004 (0.002)

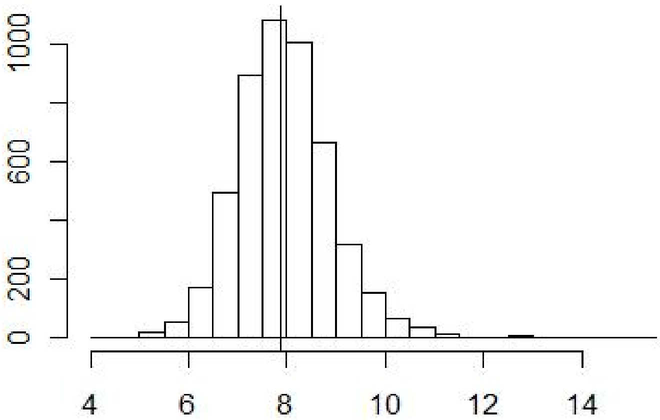

Fig 1: Histogram showing the marginal posterior density estimate of ¯ for $l = 6, k = 4, m = 2, r = 3$, vertical line shows the ML estimate.

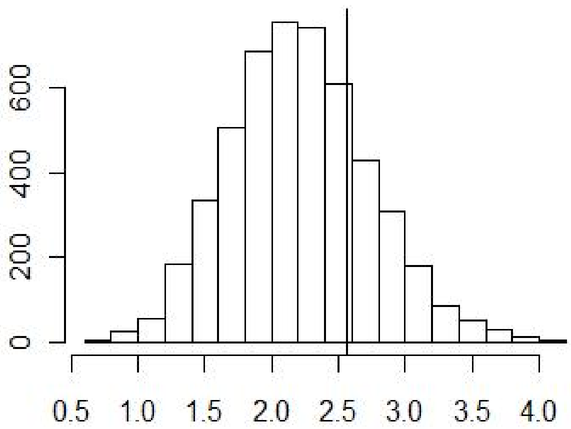

Fig 2: Histogram showing the marginal posterior density estimate of for $l = 6, k = 4, m = 2, r = 2$, vertical line shows the ML estimate.

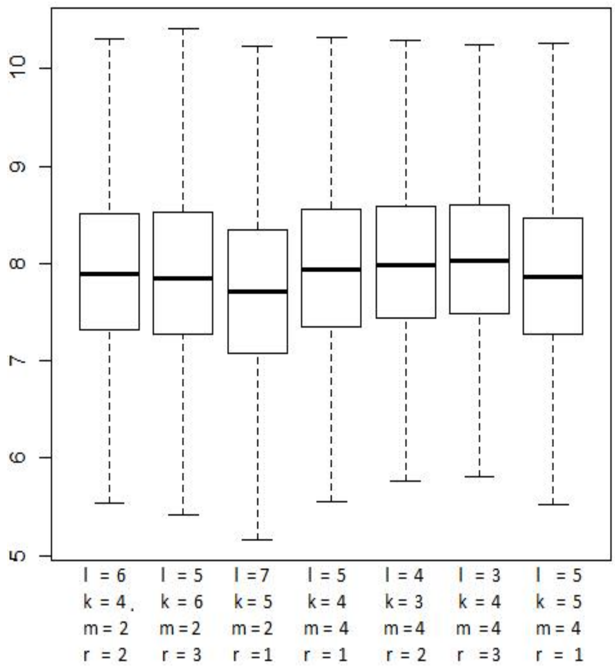

Fig 3: Box plot showing the marginal posterior density estimates of ¯ at different combinations of l, k, m and r based on simulated data.

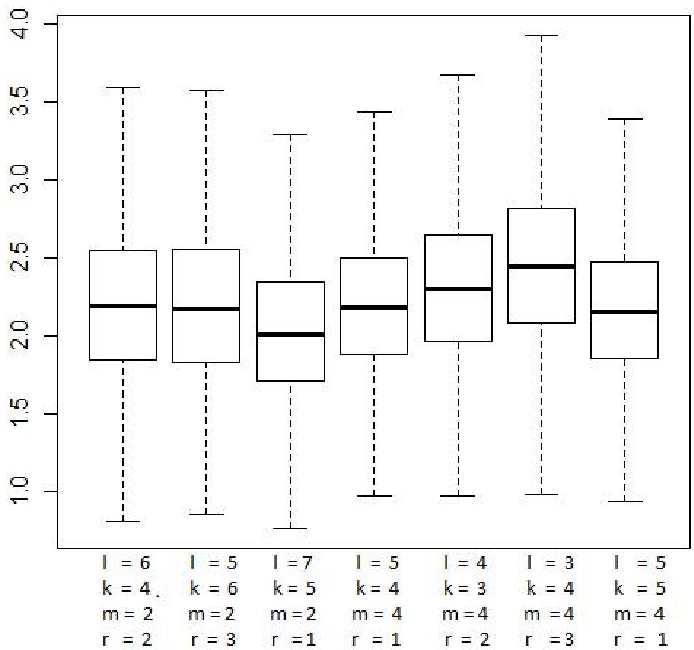

Fig 4: Box plot showing the marginal posterior density estimates of at different combinations of l, k, m and r based on simulated data.

3.2 Real data based study

The illustration in this section is based on a data set reported by Lawless (1982). The data set of size 20 represents the voltage levels at which the failures occurred in the electrical cable insulation when specimens were subjected to an increasing voltage stress in a laboratory experiment. The complete set of observations in an ordered form is given in Table 5.

Table 5: Failure voltages (in kilovolts per millimetre)

39.4	45.3	49.2	49.4	51.3	52.0	53.2	53.2	54.9	55.5
57.1	57.2	57.5	59.2	61.0	62.4	63.8	64.3	67.3	67.7

In order to obtain the ML estimates and the corresponding Bayesian results for the Weibull parameters, we first allowed for multiply censoring scheme by leaving 10 observations from the left, mid and right regions and thereby allowing only 10 observed failures. This was done to have 50% censoring though other levels of censoring percentages can be similarly fixed in advance for obtaining the estimates. The different combinations of l, m, r and k so as to allow 50% censoring from left, right and mid regions are given in Tables 6-8. The ML estimates based on EM algorithm and the Bayes estimates based on ergodic averages are shown in Tables 6-8 for the considered combinations of l, k, m and r. These estimates were obtained exactly similar to what has been discussed for simulated data set and, therefore, we do not feel any further advantage in discussing the details afresh. Once again the convergence in Gibbs sampler algorithm was monitored based on single long run of the chain and it was assessed at about 50,000 iterations.

The posterior density estimates of ¯ and based on a single combination of l, k, m and r are shown in Figures 5-6 with vertical lines in each figure corresponding to ML estimates. It can be seen that the ML estimates also lie in the high probability regions and are quite close to the Bayes estimates in each case, a conclusion that was observed earlier too. The posterior density estimates of ¯ and for other combinations of l, k, m and r are not shown although we have shown here the density estimates of $R(t)$ and $H(t)$ for $t = 50$ at different combinations of l, k, m and r. These are shown by means of box plot representations in Figures 7-8. A word of remark: most of the results given in the paper are meant for illustration only. We feel that once the posterior samples are obtained perhaps any study becomes routine in Bayesian paradigm.

Table 6: *ML* and Bayes estimates of ¯ and at different combinations of censoring parameters (results based on real data with overall censoring 50%)

Censoring Scheme	*l* *k* *m* *r*	ML estimate		Bayes estimate	
		$\hat{\bar{\ }}_{ML}$	$\hat{\ }_{ML}$	$\hat{\bar{\ }}_{B}$	$\hat{\ }_{B}$
Multiply	6 4 2 2	58.94	8.31	58.81	9.43
	5 6 2 3	59.34	8.03	59.07	9.29
	7 5 2 1	59.00	7.35	58.69	8.95
	5 4 4 1	59.26	7.92	59.18	8.65
	4 3 4 2	59.12	8.72	58.96	9.69
	3 4 4 3	59.32	8.32	59.04	9.35
	5 5 4 1	59.33	7.94	59.23	8.68

Table 7: Estimated posterior risks of *ML* and Bayes estimators of ¯ and corresponding to squared error loss (results based on real data with overall censoring 50%)

Censoring Scheme	*l* *k* *m* *r*	Posterior Risk			
		$\hat{\bar{\ }}_{ML}$	$\hat{\ }_{ML}$	$\hat{\bar{\ }}_{B}$	$\hat{\ }_{B}$
Multiply	6 4 2 2	3.976	5.342	3.950	4.115
	5 6 2 3	4.720	5.657	4.641	4.024
	7 5 2 1	4.940	5.696	4.839	3.107
	5 4 4 1	4.014	3.462	4.006	2.934
	4 3 4 2	3.464	4.713	3.433	3.816
	3 4 4 3	3.911	4.669	3.851	3.581
	5 5 4 1	4.310	3.597	4.304	3.055

Table 8: ML and Bayes estimates of $R(50)$, $H(50)$ and the corresponding posterior risks (of order 10^{-3}) in parentheses (results based on real data with overall censoring 50%)

Censoring	l k m r	Reliability and Hazard of ML estimate		Reliability and Hazard of Bayes estimate	
		$\hat{R}_{ML}(50)$	$\hat{H}_{ML}(50)$	$\hat{R}_{B}(50)$	$\hat{H}_{B}(50)$
Multiply	6 4 2 2	0.805 (9.822)	5.022 (1.541)	0.761 (8.058)	5.616 (1.178)
	5 6 2 3	0.808 (10.144)	5.141 (2.168)	0.763 (8.029)	5.893 (1.590)
	7 5 2 1	0.787 (12.263)	5.167 (2.048)	0.732 (9.196)	6.086 (1.188)
	5 4 4 1	0.792 (8.138)	5.421 (1.179)	0.75 (7.0916)	5.849 (0.991)
	4 3 4 2	0.816 (8.026)	4.969 (1.189)	0.780 (6.806)	5.471 (0.941)
	3 4 4 3	0.809 (7.931)	5.111 (1.579)	0.773 (6.536)	5.720 (1.214)
	5 5 4 1	0.794 (8.613)	5.422 (1.240)	0.762 (7.504)	5.863 (1.048)

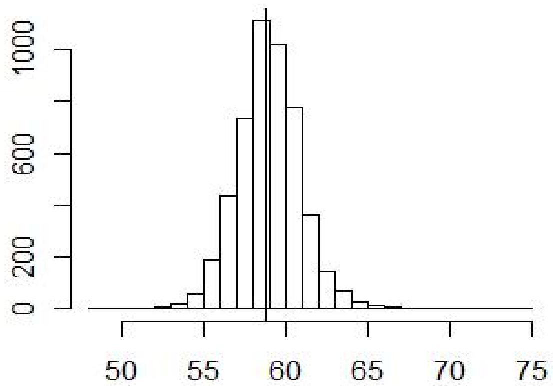

Fig 5: Histogram showing the marginal posterior density estimate of ¯ for $l = 6, k = 4,\ m = 2, r = 2$, vertical line shows the ML estimate.

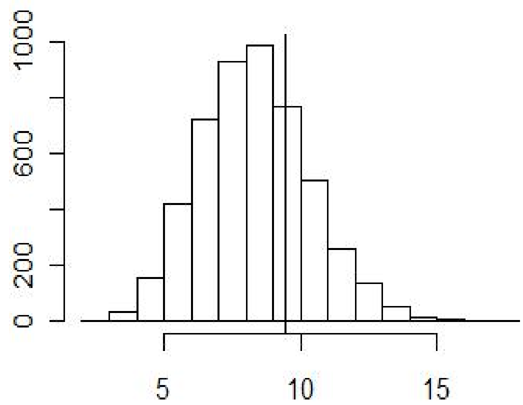

Fig 6: Histogram showing the marginal posterior density estimate of for $l = 6, k = 4,\ m = 2, r = 2$, vertical line shows the ML estimate.

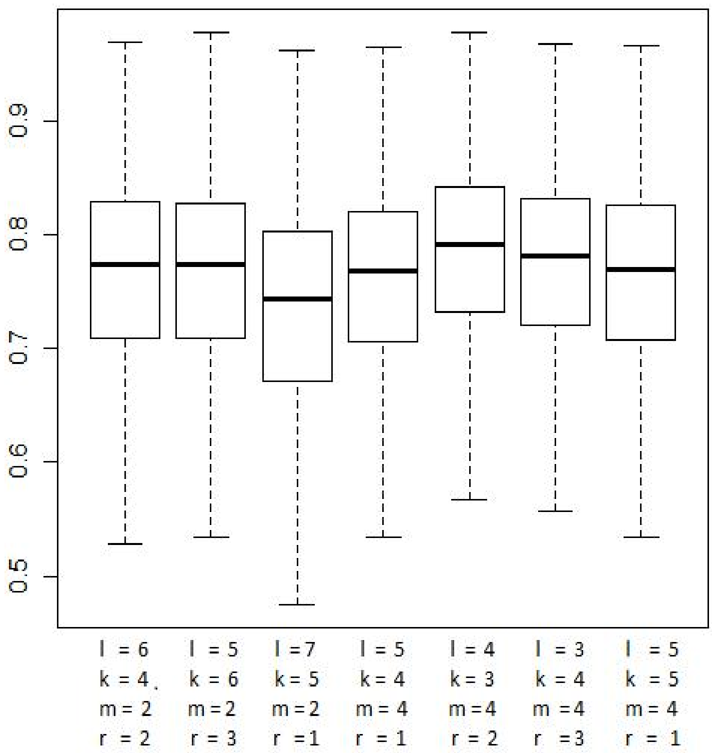

Fig 7: Box plot showing the marginal posterior density estimate of $R(50)$ at different combinations of l, k, m and r (results based on real data with overall censoring 50%).

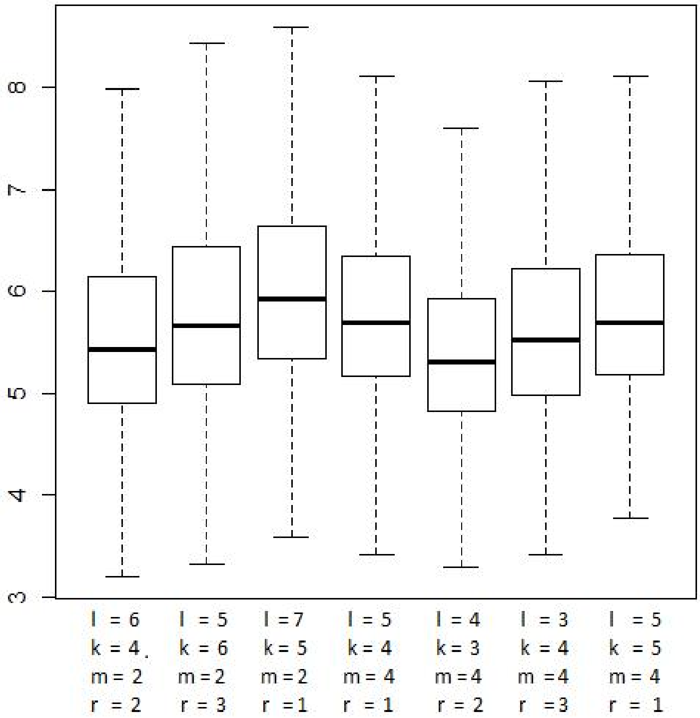

Fig 8: Box plot showing the marginal posterior density estimate of $H(50)$ at different combinations of l, k, m and r (results based on real data with overall censoring 50%).

4. CONCLUSIONS

The present study is an attempt to provide Bayes and ML estimates of Weibull parameters under multiply censoring scheme. Multiply censoring scheme which is a generalized version of type II or item censoring scheme is normally difficult to deal analytically due to complex form of likelihood function. It can be seen that our approaches are not only straightforward but capable of providing routine implementation from the viewpoints of applied reliability practitioners. Besides, it can be seen that although ML estimators are generally close to Bayes estimators, they sometimes lie in the low probability regions on the estimated posterior densities of the corresponding parameters giving a clear message in favour of Bayes estimators.

Our study also reveals an important fact. It suggests that whatever left, right and mid censoring combinations are used, the density estimates of the parameters remain more or less same. This may be because of the fact that missing data are estimated from the two-parameter Weibull distribution and, as such, we have used the complete data set for the final reporting. This, in turn, suggests that our modelling assumption is also appropriate at least for the considered real data set.

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Puneet K. Srivastava and S. K. Upadhyay

DST-Centre for Interdisciplinary
Mathematical Sciences
Banaras Hindu University

Akanksha Gupta

Department of Statistics
Banaras Hindu University