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FINITE MIXTURES OF INTERVENED POISSON DISTRIBUTION AND ITS APPLICATIONS

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ABSTRACT

In this paper, we consider some finite mixtures of intervened Poisson distribution and study some of their properties. The identifiability conditions of the mixture models are derived and parameters of these mixtures are estimated by method of factorial moments, method of mixed moments and method of maximum likelihood. Further, this mixture distribution is fitted to some real life data-sets and compared with mixtures of positive Poisson distribution.

1. INTRODUCTION

Finite mixtures of distributions have provided a mathematical based approach to the statistical modeling of a wide variety of random phenomena. Because of their usefulness as an extremely flexible method of modeling, finite mixture models have an increasing attention over the years from practical and theoretical point of view. Indeed, in the past decade the extent and the potential of the application of finite mixture models have widened considerably. Application of mixture models spread over astronomy, biology, genetics, medicine, psychiatry, economics, engineering, marketing and other fields in the biological, physical and social sciences. For details see McLachlan and Peel (2000). In many of these applications, finite mixture models support a variety of techniques in major areas of statistics including cluster and latent class analysis, discriminant analysis, image analysis and survival analysis.

Shanmugam (1985) introduced the intervened Poisson distribution (IPD) as a replacement for the positive Poisson distribution in situations when some intervention process may alter the mean of the rare event generating process under observation. An advantage of the *IPD* is that it provides information on how effective various preventive actions taken by health service agents where positive Poisson fails. The *IPD* is applicable in several areas such as reliability analysis, queuing problems, epidemiological studies etc. For example, see Shanmugam (1985, 1992), Huang and Fung (1989) and Kumar and Shibu (2011, 2012). During the observational period, the failed units are either replaced by new units or rebuilt. This kind of replacement changes the reliability of a system as only some of its components have longer life.

Let V be the number of instances of some rare events distributed according to a Poisson distribution with parameter λ. Assume that the observational process is such that only positive values of V are observed. Let U_1 denote the positive observed number of occurrences of this rare event. Then U_1 has a positive Poisson distribution with probability mass function (*pmf*).

$$h(w) = P(U_1 = w) = \frac{(e^{\lambda} - 1)^{-1} \lambda^w}{w!} \tag{1.1}$$

where $\lambda > 0$ and $w = 1,2,\ldots$.In particular, let V the number of cholera cases in a household. $V = 0$ is not observable since the observational apparatus is activated only when $V > 0$.Thus after U_1 is generated, some intervention mechanism changes λ to $\lambda\rho$ where $\rho \geq 0$. Let U_2 be the number of occurrences generated after this intervention. The random variable U_2 is Poisson with mean $\rho \geq 0$ and is statistically independent of U_1. Assume that our observational apparatus has a record of only the random variable $U = U_1 + U_2$, the total number of rare events occurred altogether is an intervened Poisson distribution with parameters λ and $\lambda\rho$. The *pmf* of *IPD* is given by

$$h_1(u) = P(U = u) = \frac{[(1+\rho)^u - \rho^u]}{(e^{\lambda} - 1)e^{\lambda\rho}} \frac{\lambda^u}{u!}, \tag{1.2}$$

with $\lambda > 0$ and $\rho \geq 0$ for those values of u on the positive integers, and zero elsewhere.

Throughout this paper, we assume that g is a positive integer greater than one. Let X be a discrete random variable with *pmf* $p(x) = P(X = x)$ of the form $p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \ldots + \alpha_g p_g(x)$ where for each $j = 1,2,\ldots, g$, $\alpha_j > 0$

such that $\sum_{j=1}^{g}\alpha_g = 1$ and $p_j(x) \geq 0$ such that $\sum_x p_j(x) = 1$. Then we say that X has a mixture distribution and $p(x)$ is a finite mixture distribution. The parameters $\alpha_1, \alpha_2, ..., \alpha_g$ are known as mixing weights and $p_1, p_2, ..., p_g$, the components of the mixture. We denote Θ as the collection of all distinct parameters occuring in the components and ψ as the complete collection of all distinct parameters occuring in the mixture model.

Let $\Delta = \{F(x,\theta_j) : \theta_j \in \Theta, x \in R\}$ be the class of distribution functions from which mixtures are to be formed. We identify the class of finite mixtures of Δ with the appropriate class of distribution functions, defined by $\hat{H} = \{H(x) : H(x) = \sum_{j=1}^{g}\alpha_j F(x,\theta_j), \alpha_j > 0, F(;\theta_j) \in \Delta, j = 1,2,...,g\}$ so that $\hat{H}$ is the convex hull of Δ, we denote $F(x,\theta_j)$ by $F_j(x)$ or simply F_j and the mixture by $H = \sum_{j=1}^{g}\alpha_j F_j$.

We need the following theorem from Titteringtonet.al. (1985) in order to establish the identifiability condition of the mixture models considered in this paper.

Theorem 1.1 (Titterington et.al., 1985) A necessary and sufficient condition for H to be identifiable is that Δ is linearly independent over the field of real numbers.

There is a great interest in the finite mixture models of Poisson distribution and its application in several areas of research including social and applied sciences. For a detailed account of this, see Mc Lachlan and Peel (2000) and references therein. But, Poisson mixture models are not suitable for situations where intervention arises. As such, through this paper, we introduced finite mixtures of *IPD* and study some of its important aspects. In section 2, we present the definition and some properties of the g component mixture of *IPDs* and in section 3, we discuss the estimation of the parameters of the IPD mixture model. Two real life data sets are also considered in section 3 to illustrate the usefulness of *IPD* mixture models. Data illustration and concluding remarks are given in section 4.

2. MIXTURES OF g COMPONENT INTERVENED POISSON DISTRIBUTION

Definition 2.1 A random variable Y is said to have a g component mixture of *IPDs* if it has the following *pmf* $f(x)=P(Y=x)$, in which $0 \le \alpha_i \le 1$ for $i=1,2,...,g$ with $\sum_{i=1}^{g}\alpha_i = 1$ and $x=1,2,...$

$$f(x)=\sum_{i=1}^{g}\alpha_i f_i(x) \tag{2.1}$$

where

$$f_i(x)=\frac{[(1+\rho_i)^x-\rho_i{}^x]}{e^{\lambda_i\rho_i}(e^{\lambda_i}-1)}\frac{\lambda_i{}^x}{x!} \tag{2.2}$$

with $\lambda_i > 0$, $\rho_i \ge 0$ for each $i=1,2,...,g$. Here after, we denote the distribution with pmf (2.1) by $MIPD_g$. Now we present the identifiability condition of the $MIPD_g$ through the following proposition.

Proposition 2.1

The identifiability condition for $MIPD_g$ with *pmf* $f(x)$ given in (2.1) is $\lambda_i \ne \lambda_j$, $\rho_i \ne \rho_j$ for $i \ne j$ taking values from $1,2,...,g$ and $x=1,2,...$

Proof Assuming $g=2$ and consider the equation

$$b_1F_1(x)+b_2F_2(x)=0 \tag{2.3}$$

where b_1 and b_2 are any two arbitrary real numbers, $F_1(x)=\sum_{j=1}^{x}f(j)$ and $F_2(x)=\sum_{j=1}^{x}\phi(j)$ for $x=1,2,...$ in which $\phi(j)$ obtained from $f(j)$ by replacing λ_j by μ_j, ρ_j by δ_j. Assume that for each $i=1,2$, $\lambda_i \ne \mu_i$ and $\rho_i \ne \delta_i$.

Thus

$$F_1(x) = \frac{1}{j!}\sum_{j=1}^{x}[\alpha \frac{\{(1+\rho_1)^j - \rho_1{}^j\}}{e^{\lambda_1\rho_1}(e^{\lambda_1}-1)}\lambda_1{}^j + (1-\alpha)\frac{\{(1+\rho_2)^j - \rho_2{}^j\}}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)}\lambda_2{}^j] \tag{2.4}$$

$$F_2(x) = \frac{1}{j!}\sum_{j=1}^{x}[\alpha \frac{\{(1+\delta_1)^j - \delta_1{}^j\}}{e^{\mu_1\delta_1}(e^{\mu_1}-1)}\mu_1{}^j + (1-\alpha)\frac{\{(1+\delta_2)^j - \delta_2{}^j\}}{e^{\mu_2\delta_2}(e^{\mu_2}-1)}\mu_2{}^j] \tag{2.5}$$

Now from equations (2.3), (2.4) and (2.5), we have the following:

$$\frac{1}{j!}\sum_{j=1}^{x}[b_1 \frac{\{(1+\rho_1)^j - \rho_1{}^j\}}{e^{\lambda_1\rho_1}(e^{\lambda_1}-1)}\lambda_1{}^j + b_2\frac{\{(1+\rho_2)^j - \rho_2{}^j\}}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)}\mu_1{}^j] = 0 \tag{2.6}$$

$$\frac{1}{j!}\sum_{j=1}^{x}[b_1 \frac{\{(1+\rho_2)^j - \rho_2{}^j\}}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)}\lambda_2{}^j + b_2\frac{\{(1+\delta_2)^j - \delta_2{}^j\}}{e^{\lambda_2\mu_2}(e^{\mu_2}-1)}\mu_2{}^j] = 0 \tag{2.7}$$

Equations (2.6) and (2.7) yield the following.

$$\frac{b_2}{j!}\sum_{j=1}^{x}[\frac{\{(1+\delta_2)^j - \rho_2{}^j\}}{e^{\mu_2\delta_2}(e^{\mu_2}-1)}\mu_2{}^j - \frac{\{(1+\rho_2)^j - \rho_2{}^j\}}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)}\lambda_2{}^j] = 0 \tag{2.8}$$

which implies that $b_2 = 0$ and then from (2.3), we get $b_1 = 0$. This shows that F_1 and F_2 are linearly independent. Hence by theorem 1.1, the proof follows.

Proposition 2.2

The mean and variance of $MIPD_g$ with *pmf* $f(x)$ given in (2.1) are the following.

$$E(X) = \sum_{i=1}^{g}\alpha_i\lambda_i(q_i+\rho_i) = \sum_{i=1}^{g}\alpha_i\eta_i = \eta \tag{2.9}$$

where for $i = 1,2,..., g$.

$$q_i = e^{\lambda_i}(e^{\lambda_i}-1)^{-1} \tag{2.10}$$

$$\eta_i = \lambda_i(\rho_i + q_i) \tag{2.11}$$

$$Var(X) = \sum_{i=1}^{g} \alpha_i [(1+\lambda_i)\eta_i + 2\lambda_i \rho_i q_i] - \eta^2 \tag{2.12}$$

The proof is simple and hence omitted.

Proposition 2.3

For $i = 1,2,..., g$, the $MIPD_g$ is under-dispersed for all values of α_i such that $0 \le \alpha_i \le 1, \lambda_i > 0$ and $\rho_i \ge 0$ such that $\sum_{i=1}^{g} \alpha_i [(1+\lambda_i)\eta_i + 2\lambda_i \rho_i q_i] < \eta^2 + \eta$ and it is over dispersed otherwise.

Proof From (2.9) and (2.12), the proof is obvious.

Proposition 2.4

The probability generating function (pgf) of $MIPD_g$ with pmf (2.1) is the following.

$$P(s) = \sum_{i=1}^{g} \alpha_i \left[\frac{(e^{\lambda_i s} - 1)}{(e^{\lambda_i} - 1)} e^{\lambda_i \rho_i (s-1)} \right] \tag{2.13}$$

Proof By definition, the pgf of $MIPD_g$ with pmf (2.1) is

$$P(s) = E(s^X)$$

$$= \sum_{x=1}^{\infty} s^x f(x)$$

$$= \sum_{x=1}^{\infty} s^x \alpha_i \left[\frac{[(1+\rho_i)^x - \rho_i^x]}{e^{\lambda_i \rho_i} (e^{\lambda_i} - 1)} \frac{\lambda_i^x}{x!} \right] \tag{2.14}$$

On simplifying (2.14) and equating the coefficient of s^x, we get (2.13).

Proposition 2.5

The $r-th$ factorial moment $\mu_{[r]}$ of $MIPD_g$ is the following for $r = 1,2,...$

$$\mu_{[r]} = \sum_{i=1}^{g} \alpha_i \lambda_i^{\,r} \left[\frac{(1+\rho_i)^r e^{\lambda_i} - \rho_i^{\,r}}{(e^{\lambda_i} - 1)} \right] \tag{2.15}$$

Proof By definition, the $r-th$ factorial moment $\mu_{[r]}$ of $MIPD_g$ is

$$\mu_{[r]} = E[X(X-1)...(X-r+1)]$$

$$= \sum_{x=r}^{\infty} \frac{x!}{(x-r)!} f(x) .$$

Now the proof follows from (2.1).

Proposition 2.6

The $r-th$ raw moment m_r of the $MIPD_g$ is the following, for $r = 1,2,...$ in which $S(r, j)'s$ are Stirling numbers of the second kind (Johnson et.al., 2005, p.12)

$$m_r = \sum_{j=0}^{r} \sum_{i=1}^{g} \alpha_i \Lambda_{ij}(\lambda_i, \rho_i) S(r, j) \tag{2.16}$$

where for $i = 1,2,..., g$ and $j = 0,1,...,$

$$\Lambda_{ij}(\lambda_i, \rho_j) = \lambda_i^{\,j} \left[\frac{(1+\rho_i)^j e^{\lambda_i} - \rho_i^{\,j}}{(e^{\lambda_i} - 1)} \right] \tag{2.17}$$

Proof From (2.13), we have

$$P(s) = \sum_{i=1}^{g} \alpha_i \left[\frac{(e^{\lambda_i s} - 1)}{(e^{\lambda_i} - 1)} e^{\lambda_i \rho_i (s-1)} \right]$$

$$= \sum_{i=1}^{g} \sum_{u=0}^{\infty} \frac{\alpha_i}{(e^{\lambda_1} - 1) e^{\lambda_i \rho_i}} (\lambda_i s)^u \left[\frac{(1+\rho_i)^u - \rho_i^{\,u}}{u!} \right] \tag{2.18}$$

Replacing s by e^{it} in (2.18), we get the characteristic function $\phi(t) = P(e^{it})$ of the $MIPD_g$ as

$$\phi(t)=\sum_{i=1}^{g}\sum_{u=0}^{\infty}\sum_{r=0}^{\infty}\left[\frac{\alpha_i\lambda_i^{u}\{(1+\rho_i)^u-\rho_i^{u}\}}{(e^{\lambda_1}-1)e^{\lambda_i\rho_i}}\frac{(itu)^r}{u!r!}\right] \tag{2.19}$$

Equating coefficient of $\frac{t^r}{r!}$ from (2.19), we obtain

$$m_r=\sum_{i=1}^{g}\sum_{u=0}^{\infty}\frac{\alpha_i}{(e^{\lambda_i}-1)e^{\lambda_i\rho_i}}\left[\frac{\lambda_i^{u}\{(1+\rho_i)^u-\rho_i^{u}\}}{u!}u^r\right].$$

On replacing u^r in terms of Stirling numbers of the second kind, we get (2.16).

3. ESTIMATION

In this section we discuss the estimation of the parameters of the mixture model $MIPD_g$ for $g=2,3$ by method of mixed moments, method of factorial moments and method of maximum likelihood. In the method of factorial moments, the first five population factorial moments of $MIPD_g$ are equated to the corresponding sample factorial moments $\tau_{[r]}$ for $r=1,2,3,4,5$ and obtain the following system of equations in which $\Lambda_{ij}=\Lambda_{ij}(\lambda_i,\rho_j)$.

$$\alpha\,\Lambda_{ij}+(1-\alpha)\,\Lambda_{2j}=\tau_{[j]} \tag{3.1}$$

for $j=1,2,3,4,5$.

In method of mixed moments, the parameters $\alpha,\lambda_1,\lambda_2,\rho_1$ and ρ_2 of the $MIPD_2$ are estimated by using the first four sample factorial moments and the first observed frequency of the distribution. Thus the estimates are obtained by solving the equation (3.1) for $j=1,2,3,4$ along with the following equation.

$$\alpha\frac{\lambda_1}{(e^{\lambda_1}-1)e^{\lambda_1\rho_1}}+(1-\alpha)\frac{\lambda_2}{(e^{\lambda_2}-1)e^{\lambda_{12}\rho_2}}=\frac{O_1}{N} \tag{3.2}$$

where O_1 is the observed frequency corresponding to the first observed value and N, the observed total frequency.

In method of maximum likelihood estimation, the parameters of the mixture models are estimated by maximizing the following log likelihood function with respect to the parameters.

$$\log L = \sum_{x=1}^{z} n_x \log f(x) \tag{3.3}$$

where $f(x)$ is the probability model of the mixture, n_x is the observed frequency of x events and z is the highest value of x observed. Thus the maximum likelihood estimates of the parameters of the $MIPD_g$ are obtained by solving the following system of non-linear equations in which Δ_{j1} is given in (3.4) for $j = 1,2$.

$$\Delta_{j1} = (1+\rho_j)e^{\lambda_j} - \rho_j \tag{3.4}$$

$$\sum_{x=1}^{z} \frac{n_x f_1(x)}{f(x)} = \sum_{x=1}^{z} \frac{n_x f_2(x)}{f(x)} \tag{3.5}$$

$$\frac{(e^{\lambda_1} - 1)}{\lambda_1} \sum_{x=1}^{z} x \frac{n_x f_1(x)}{f(x)} = \Delta_{11} \sum_{x=1}^{z} \frac{n_x}{f(x)} \tag{3.6}$$

$$\frac{(e^{\lambda_2} - 1)}{\lambda_2} \sum_{x=1}^{z} x \frac{n_x f_1(x)}{f(x)} = \Delta_{21} \sum_{x=1}^{z} \frac{n_x}{f(x)} \tag{3.7}$$

$$\sum_{x=1}^{z} \frac{n_x f_1(x-1)}{f(x)} = \sum_{x=1}^{z} \frac{n_x f_1(x-1)}{f(x)} \tag{3.8}$$

$$\sum_{x=1}^{z} \frac{n_x f_2(x-1)}{f(x)} = \sum_{x=1}^{z} \frac{n_x f_2(x-1)}{f(x)} \tag{3.9}$$

On solving these normal equations using mathematical soft-wares, we can obtain estimators of parameters of $MIPD_2$. All these procedures can be extended to any value of $g > 2$. The normal equations for obtaining the estimates of parameters of the $MIPD_3$ and that of mixtures of positive Poisson distributions $(MPPD)$ in case of method of factorial moments, method of mixed moments and method of maximum likelihood are obtained and included in Appendix.

4. DATA ILLUSTRATION AND CONCLUDING REMARKS

For numerical illustration, we consider two real life data sets as given in Tables 1 and 2. The first data set given in Table l indicates the distribution of number of articles on theoretical Statistics and Probability for years 1940-49 and initial letter N-R of the author's name. For details, see Kendall (1961). The second data set given in Table 2 represents the distribution of 633 biologists according to the number of research papers to their credit in the review of applied entomology, volume 24, 1936. For details, see Williams (1944). In both data sets, there is some sort of intervention problems and that may be the reason for the spread of probability concentration to higher values of the observations. We have fitted the $MPPD$ and the $MIPD_g$ for particular values of g to both datasets by the method of factorial moments, method of mixed moments and method of maximum likelihood.

The results obtained are included in Tables 1 and 2. Based on the computed $Chi-square$ values and $P-$values in respective cases, it can be observed that the $MIPD_3$ gives the best fit compared to the $MIPD_2$ and the $MPPD$ to both datasets.

For the sake of retaining minimum degrees of freedom, in case of first data-set, we assume that $\lambda_1 = \lambda_2$, $\rho_1 = \rho_2$ and $\alpha_1 = \alpha_2$ while fitting $MIPD_3$. For higher values of g , it can be possible to obtain better fits even for complicated data sets with intervention. Similar studies are possible in case of other probability models subject to identifiability conditions. Several inferential aspects of $MIPD_g$ remains for further investigation.

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Table 1: Comparison of fit of $MIPDg$ for various methods of estimation for the first data set

count	Observed 'f'	Expected frequency by factorial moments			Expected frequency by mixed moments			Expected frequency by maximum likelihood		
		MPPD	$MIPD_2$	$MIPD_3$	MPPD	$MIPD_2$	$MIPD_3$	MPPD	$MIPD_2$	$MIPD_3$
1	83	71	73	71	83	83	83	80	78	80
2	18	24	22	21	12	15	16	16	19	20
3	13	20	16	17	17	15	14	15	14	14
4	9	10	10	12	10	12	13	14	11	10
5	7	8	10	10	9	6	6	6	8	7
6	7	6	6	6	6	6	5	6	6	6
7	2	2	2	2	2	2	2	2	3	2
8	5	3	5	5	5	5	5	5	5	5
Total	144	144	144	144	144	144	144	144	144	144
Degrees of freedom		4	2	1	4	2	1	4	2	1
Chi-square value		5.36	4.20	4.01	4.25	3.96	2.25	2.13	2.04	1.96
P-value		0.52	0.63	0.69	0.60	0.51	0.67	0.75	0.78	0.85

Table 2: Comparison of fit of MIPDg for various methods of estimation for the second data set

count	Observed 'f'	Expected frequency by factorial moments			Expected frequency by mixed moments			Expected frequency by maximum likelihood		
		MPPD	$MIPD_2$	$MIPD_3$	MPPD	$MIPD_2$	$MIPD_3$	MPPD	$MIPD_2$	$MIPD_3$
1	365	335	350	360	365	365	365	375	372	359
2	95	105	107	89	100	105	103	89	88	100
3	50	65	56	45	45	42	46	43	46	46
4	36	45	42	42	43	32	30	43	43	40
5	15	17	11	19	17	19	19	19	18	17
6	11	9	14	14	9	12	12	9	9	13
7	12	14	9	13	14	12	12	12	13	14
8	8	6	6	6	6	6	6	6	6	6
9	5	6	5	5	6	6	6	6	6	6
10	36	31	33	40	28	33	34	31	32	32
Total	633	633	633	633	633	633	633	633	633	633
d.f.		6	4	1	6	4	1	6	4	1
Chi-square value		11.5	7.5	4.52	6.03	5	4.05	5.87	4.5	3.26
P-value		0.24	0.58	0.87	0.74	0.83	0.9	0.75	0.87	0.95

Table 3: Estimated vales of parameters of MIPDg for the first data set

	Expected frequency by factorial moments			Expected frequency by mixed moments			Expected frequency by maximum likelihood		
	MPPD	$MIPD_2$	$MIPD_3$	MPPD	$MIPD_2$	$MIPD_3$	MPPD	$MIPD_2$	$MIPD_3$
Estimated values of parameters	$\hat{\alpha}=0.28$	$\hat{\alpha}=0.84$	$\hat{\alpha}=0.74$	$\hat{\alpha}=0.45$	$\hat{\alpha}=0.28$	$\hat{\alpha}=0.54$	$\hat{\alpha}=0.30$	$\hat{\alpha}=0.35$	$\hat{\alpha}=0.32$
	$\hat{\lambda}=0.84$	$\hat{\lambda}=0.84$	$\hat{\alpha}=0.26$	$\hat{\lambda}=1.12$	$\hat{\lambda}=0.92$	$\hat{\alpha}=0.46$	$\hat{\lambda}=0.86$	$\hat{\lambda}=0.92$	$\hat{\alpha}=0.68$
	$\hat{\lambda}=0.36$	$\hat{\lambda}=0.29$	$\hat{\lambda}=0.80$	$\hat{\lambda}=0.26$	$\hat{\lambda}=0.35$	$\hat{\lambda}=0.66$	$\hat{\lambda}=0.32$	$\hat{\lambda}=0.44$	$\hat{\lambda}=0.90$
		$\hat{\rho}=0.39$	$\hat{\lambda}=0.45$		$\hat{\rho}=0.37$	$\hat{\lambda}=0.49$		$\hat{\rho}=0.32$	$\hat{\lambda}=0.41$
		$\hat{\rho}=0.62$	$\hat{\rho}=0.62$		$\hat{\rho}=0.35$	$\hat{\rho}=0.54$		$\hat{\rho}=0.35$	$\hat{\rho}=0.65$
						$\hat{\rho}=0.19$			$\hat{\rho}=0.28$

Table 4: Estimated vales of parameters of MIPDg for the second data set

	Expected frequency by factorial moments			Expected frequency by mixed moments			Expected frequency maximum likelihood		
	MPPD	$MIPD_2$	$MIPD_3$	MPPD	$MIPD_2$	$MIPD_3$	MPPD	$MIPD_2$	$MIPD_3$
Estimated values of parameters	$\hat{\alpha} = 0.58$	$\hat{\alpha} = 0.49$	$\hat{\alpha} = 0.49$	$\hat{\alpha} = 0.45$	$\hat{\alpha} = 0.34$	$\hat{\alpha} = 0.35$	$\hat{\alpha} = 0.65$	$\hat{\alpha} = 0.54$	$\hat{\alpha} = 0.32$
	$\hat{\lambda} = 0.84$	$\hat{\lambda} = 0.28$	$\hat{\alpha} = 0.28$	$\hat{\lambda} = 0.45$	$\hat{\lambda} = 0.15$	$\hat{\lambda} = 0.45$	$\hat{\lambda} = 0.84$	$\hat{\lambda} = 0.32$	$\hat{\lambda} = 0.54$
	$\hat{\lambda} = 0.18$	$\hat{\lambda} = 0.34$	$\hat{\lambda} = 0.21$	$\hat{\lambda} = 0.33$	$\hat{\lambda} = 0.48$	$\hat{\lambda} = 0.17$	$\hat{\lambda} = 0.42$	$\hat{\lambda} = 0.61$	$\hat{\lambda} = 0.51$
		$\hat{\rho} = 0.62$	$\hat{\lambda} = 0.37$		$\hat{\rho} = 0.51$	$\hat{\lambda} = 0.27$		$\hat{\rho} = 0.54$	$\hat{\lambda} = 0.44$
		$\hat{\rho} = 0.52$	$\hat{\lambda} = 0.41$		$\hat{\rho} = 0.61$	$\hat{\lambda} = 0.55$		$\hat{\rho} = 0.84$	$\hat{\lambda} = 0.72$
			$\hat{\rho} = 0.56$			$\hat{\rho} = 0.55$			$\hat{\rho} = 0.55$
			$\hat{\rho} = 0.48$			$\hat{\rho} = 0.65$			$\hat{\rho} = 0.38$
			$\hat{\rho} = 0.48$			$\hat{\rho} = 0.48$			$\hat{\rho} = 0.85$

Appendix

Here we present the non-linear system of equations for estimating the parameters of $MIPD_3$ and $MPPD$ by method of factorial moments, method of mixed moments and method of maximum likelihood.

Method of factorial moments The first six population factorial moments of $MIPD_3$ are equated to the corresponding sample factorial moments to obtain the following system of equations in which $\tau_{[r]}$ denote the $r-$th sample factorial moment, for $r = 1,2,...,7$.

$$\alpha_1 \Lambda_{1j} + \alpha_2 \Lambda_{2j} + (1-\alpha_1-\alpha_2)\Lambda_{3j} = \tau_{[j]}, \text{ for } j = 1,2,...,7. \quad \text{(A.1)}$$

In a similar way, we have the following system of equations in case of $MPPD$ in which $\upsilon_{[k]}$ denote the $k-$th sample factorial moment in which $k = 1,2,3,4,5$.

$$\alpha_1 \lambda_1^{\ k} q_1 + \alpha_2 \lambda_2^{\ k} q_2 + (1-\alpha_1-\alpha_2)\lambda_3^{\ k} q_3 = \upsilon_{[k]} \quad \text{(A.2)}$$

Method of mixed moments Let O_1 be the observed frequency corresponding to the first observed value and N, the observed total frequency. In method of moments, the parameters of the $MIPD_3$ are estimated by using the first four sample factorial moments and the first observed frequency of the distribution. Thus the estimates are obtained by solving the equation (A.1) for $j = 1,2,3,4,5$ along with the following equation.

$$\alpha_1 \frac{\lambda_1}{(e^{\lambda_1}-1)e^{\lambda_1\rho_1}} + \alpha_2 \frac{\lambda_2}{(e^{\lambda_2}-1)e^{\lambda_{12}\rho_2}} + (1-\alpha_1-\alpha_2)\frac{\lambda_2}{(e^{\lambda_3}-1)e^{\lambda_3\rho_3}} = \frac{O_1}{N}. \quad \text{(A.3)}$$

In the case of $MPPD$, the estimates are obtained by solving the equations (A.2) for $k = 1,2,3,4$ along with the following equation

$$\alpha_1 \frac{\lambda_1}{(e^{\lambda_1}-1)} + \alpha_2 \frac{\lambda_2}{(e^{\lambda_2}-1)} + (1-\alpha_1-\alpha_2)\frac{\lambda_2}{(e^{\lambda_3}-1)e^{\lambda_3\rho_3}} = \frac{O_1}{N}. \quad \text{(A.4)}$$

Method of maximum likelihood Let $q_1(x), A_1(x), m_1$ be the probability of the $MIPD_3$ mixture model, observed frequency of x observed and the largest value of x observed. The maximum likelihood estimators of the $MIPD_3$ can be obtained from the following system of normal equations in which Δ_{j1} for $j = 1,2,3$.

$$\sum_{x=1}^{m_1} \frac{A_1(x) f_1(x)}{q_1(x)} = \sum_{x=1}^{m_1} \frac{A_1(x)}{q_1(x)} f_3(x) \tag{A.5}$$

$$\sum_{x=1}^{m_1} \frac{A_1(x) f_2(x)}{q_1(x)} = \sum_{x=1}^{m_1} \frac{A_1(x)}{q_1(x)} f_3(x) \tag{A.6}$$

$$(e^{\lambda_1} - 1) \sum_{x=1}^{m_1} \frac{x A_1(x) f_1(x)}{q_1(x)} = \lambda_1 \sum_{x=1}^{m_1} \frac{A_1(x) f_1(x)}{q_1(x)} \Delta_{11} \tag{A.7}$$

$$(e^{\lambda_2} - 1) \sum_{x=1}^{m_1} \frac{x A_1(x) f_2(x)}{q_1(x)} = \lambda_2 \sum_{x=1}^{m_1} \frac{A_1(x) f_2(x)}{q_1(x)} \Delta_{21} \tag{A.8}$$

$$(e^{\lambda_2} - 1) \sum_{x=1}^{m_1} \frac{x A_1(x) f_3(x)}{q_1(x)} = \lambda_3 \sum_{x=1}^{m_1} \frac{A_1(x) f_3(x)}{q_1(x)} \Delta_{31} \tag{A.9}$$

$$\sum_{x=1}^{m_1} \frac{A_1(x)}{q_1(x)} f_1(x-1) = \sum_{x=1}^{m_1} \frac{A_1(x)}{q_1(x)} f_1(x) \tag{A.10}$$

$$\sum_{x=1}^{m_1} \frac{A_1(x)}{q_1(x)} f_2(x-1) = \sum_{x=1}^{m_1} \frac{A_1(x)}{q_1(x)} f_2(x) \tag{A.11}$$

$$\sum_{x=1}^{m_1} \frac{A_1(x)}{q_1(x)} f_3(x-1) = \sum_{x=1}^{m_1} \frac{A_1(x)}{q_1(x)} f_3(x) \tag{A.12}$$

In a similar way, we have the following system of equations in case of $MPPD$. Let $q_2(x), A_2(x), m_2$ be the probability of the $MPPD$ mixture model, observed frequency of x observed and the largest value of x observed.

$$\sum_{x=1}^{m_2} \frac{\lambda_1^{\,x}}{x!(e^{\lambda_1} - 1)} = \sum_{x=1}^{m_2} \frac{\lambda_1^{\,x}}{x!(e^{\lambda_1} - 1)} \tag{A.13}$$

$$(e^{\lambda_1} - 1) \sum_{x=1}^{m_2} \frac{A_2(x)}{q_2(x)} \frac{\lambda_1^{\,x-1}}{(x-1)!} = e^{\lambda_1} \sum_{x=1}^{m_2} \frac{A_2(x)}{q_2(x)} \frac{\lambda_1^{\,x}}{x!} \tag{A.14}$$

$$(e^{\lambda_2} - 1) \sum_{x=1}^{m_2} \frac{A_2(x)}{q_2(x)} \frac{\lambda_2^{\,x-1}}{(x-1)!} = e^{\lambda_2} \sum_{x=1}^{m_2} \frac{A_2(x)}{q_2(x)} \frac{\lambda_2^{\,x}}{x!} \tag{A.15}$$

On solving the normal equations $(A.5)$ to $(A.12)$ and $(A.13)$ to $(A.15)$, we can obtain the estimators of the parameters of $MIPD_3$ and $MPPD$ respectively.

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