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NEW OPTIMAL DESIGNS FOR COMPARING TEST TREATMENTS WITH A CONTROL

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ABSTRACT

In this paper, new block designs for comparing test treatments with a control for k (block size) $\leq v$ (number of treatments) have been proposed by using the method of cyclic shifts. The construction method helps in assessing important properties of the designs without constructing the actual blocks of the designs. These newly proposed designs possess the property of A-optimality for some specific values of b (number of blocks), v and the block size $k\,(\leq v)$.

1. INTRODUCTION

In problems such as screening experiments or in the beginning of a long-term experimental investigation, it is desirable to determine the relative performance of new test treatments with respect to the control or standard treatment (Hedayat, Jacroux and Majumdar, 1988). Experiments to compare certain test treatments with a control treatment were first considered by Hoblyn, Pearce and Freeman (1954). Cox (1958) suggested augmenting an incomplete block design in test treatments with one or more replications of the control in each block to obtain a good design. Pearce (1960) developed a systematic approach for designing such type of comparative experiments and Pearce (1983) made two suggestions for such experiments; one is supplementation and the other is reinforcement (following Das, 1958). Pesek (1974) compared a balanced incomplete block design ($BIBD$) with an augmented $BIBD$ suggested by Cox (1958) for estimating control-test treatment contrasts and noticed that the latter design was more efficient.

Bechhofer and Tamhane (1981) developed the theory of incomplete block designs for comparing several treatments with a control. They did not consider the $A-$ or $MV-$optimality of a design but obtained optimal simultaneous confidence intervals. Their developments led to the concept of *balanced treatment incomplete block* ($BTIB$) designs; Notz and Tamhane (1983) studied their construction. Constantine (1983) showed that a $BIBD$ in test treatments augmented by a replication of control in each block is $A-$optimal in the class of designs with exactly one replication of the control in each block. Jacroux (1984) showed that Constantine's (1983) conclusion remains valid even when the *BIBDs* are replaced by some divisible designs.

2. NOTATIONS AND THE FORMULATION OF THE PROBLEM

Majumdar and Notz (1983) gave a method of obtaining $A-$ and $MV-$optimal designs among all designs for block designs. Hedayat and Majumdar (1984) gave an algorithm and a catalogue of $A-$ and $MV-$optimal designs. Ture (1982, 1985) also studied $A-$optimal designs and suggested their construction. He constructed $A-$ optimal designs when the control treatment replication size, r_0, is a multiple of b for fixed v and k. Hedayat and Majumdar (1985) gave families of $A-$ and $MV-$optimal designs. Notz (1985) proposed optimal row-columns designs for comparing test treatments with a control. Majumdar (1986) and Hedayat, Jacroux and Majumdar (1988) considered the problem of finding optimal designs for comparing the test treatments with two or more controls

Jacroux (1987a, 1987b, 1988) gave new methods for obtaining $MV-$optimal design, and also gave catalogues for such designs. Jacroux (1986) also studied optimal two-column designs for comparing treatments with a control by utilizing techniques of Hall (1935) and Agrawal (1966). Hedayat and Majumdar (1988) studied designs simultaneously optimal under the set up of both block designs and row-column designs. Jacroux (1989) generalized the Hedayat and Majumdar's (1984) algorithm for finding $A-$optimal designs. Cheng, Majumdar, Stufken and Ture (1988) introduced new families of A- and MV-optimal block designs. Stufken (1986, 1987, 1988) also studied $A-$ and $MV-$optimal block designs. Mandal, Shah and Sinha (2000) considered distance optimality criterion introduced by Sinha (1970) for comparing a test treatment with control treatments. The matter of comparing test treatments with two or more controls has been discussed in detail by Majumdar (1986) and Hedayat, Jacroux and Majumdar (1988) and Majumdar (1996). Jacroux (2000, 2001, 2002) also constructed $A-$optimal designs for comparing a set of test treatments to a set of standard (control) treatments.

Suppose in an experiment we are interested in comparing test treatments $1,2,...,v$ with a control treatment denoted by '0'. For example we have a *BTIB* design with $v=4$ treatments in $b=6$ blocks each of size $k=3$.

$$\begin{array}{ccc@{\qquad}ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 3 & 4 & 4 \end{array}$$

Here test treatments (1,2,3,4) are replicated $r_1=3$ times and are compared with a control treatment '0', which is replicated $r_0=6$ times. Each treatment pair excluding the control treatment '0' appears together within blocks $\lambda_1=1$ time and the control treatment '0' appears with each test treatment within blocks $\lambda_0=3$ times.

Our objective in this paper is the construction of new designs for comparing test treatments with a control when such *BTIB* designs do not exist for some specific values of v, b and k. The organization of the paper is as follows. The *BTIB* designs are briefly described in Section 3. The method of cyclic shift is described in detail in Section 4, and the construction of *BTIB* designs using cyclic shifts is explained in Section 5. The newly proposed designs and their comparison is made in Section 6. The concluding remarks appear in Section 7.

3. THE BALANCE TEST-TREATMENTS INCOMPLETE BLOCK (BTIB) DESIGNS

Suppose v test treatments and a control is to be compared in b blocks of size k each. The test treatments will be labeled as $1, 2, ..., v$ and the control as '0'. Then the model for the response Y_{ijq} obtained by applying $i-$th treatment to the $q-$th unit in $j-$th block is

$$Y_{ijq} = \mu + \tau_i + \beta_j + \varepsilon_{ijq} \quad , \quad i = 0,1,2,...,v\,; \;\; j = 1,2,...,b\,;$$

$$q = 1, 2, ..., n_{ij} \; (n_{ij} = 0,1,2,...)\,,$$

where n_{ij} denotes the number of experimental units in block j assigned to treatment i. There is no observation Y_{ijq} if $n_{ij} = 0$. The unknown constants μ, τ_i and β_j represent the general mean, the effect of treatment *i*, the effect of block j and ε_{ijq}'s are uncorrelated random variables having mean zero and variance σ^2.

Let $D(v,b,k)$ be the set of all possible designs and let $(\hat{\tau}_0 - \hat{\tau}_i)$; $i = 1, 2, ..., v$ be the best linear unbiased estimator of $(\tau_0 - \tau_i)$. Our objective here is to allocate the treatments $0,1,2,...,v$ to the blocks in a way that allows the best possible inference on the vector of control-test treatment contrasts $(\tau_0 - \tau_1, \tau_0 - \tau_2, ..., \tau_0 - \tau_v)$ using the criteria of $A-$optimality. A design is called $A-$optimal if it minimizes

$$\sum_{i=1}^{v} \mathrm{var}\,(\hat{\tau}_0 - \hat{\tau}_i)$$

Bechhofer and Tamhane (1981) defined a class of designs, known as *BTIB* designs and discussed some optimal properties of these designs for setting simultaneous confidence bounds for the set of control-test treatment contrasts. A *BTIB design* is an incomplete block design in which each test treatment appears in the same block with the control the same number of times $(= \lambda_0)$ and any pair of test treatment appears together in the same block the same number of times $(=\lambda_1)$). Formally, we may define a *BTIB* design by the relation

$$\sum_{j=1}^{b} n_{oj}\, n_{ij} = \lambda_0 \quad \text{for} \quad i = 1, 2, \ldots, v$$

and $$\sum_{j=1}^{b} n_{ij}\, n_{i^* j} = \lambda_1 \quad \text{for} \quad i \neq i^*; \quad i, \quad i^* = 1, 2, \ldots, v.$$

Note that a *BTIB* is a *BTIB* design with $n_{ij} = 0,1$ and $\lambda_0 = \lambda_1$.

Cheng, Majumdar, Stufken and Ture (1988), Hedayat and Majumdar (1984), and Hedayat, Jacroux and Majumdar (1988) defined two types of *BTIB* designs, rectangular $(R-)$ type and step $(S-)$ type. Let *BTIB* (v,b,k,t,s) denote a *BTIB* having $bt+s$ replicates of the control. It is called $R-$ type design when $s=0$ and a $S-$ type design otherwise, where s is the number of replicates of the control treatment in addition to the obligatory bt replicates of the control treatment. Here each block has t replicates of the control. The layouts of these *BTIB* designs are pictured in Figure 1 and Figure 2, with columns as blocks.

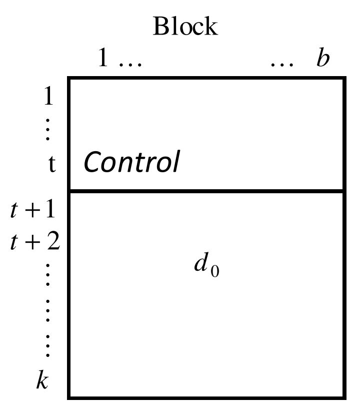

Figure 1. R-type BTIB design

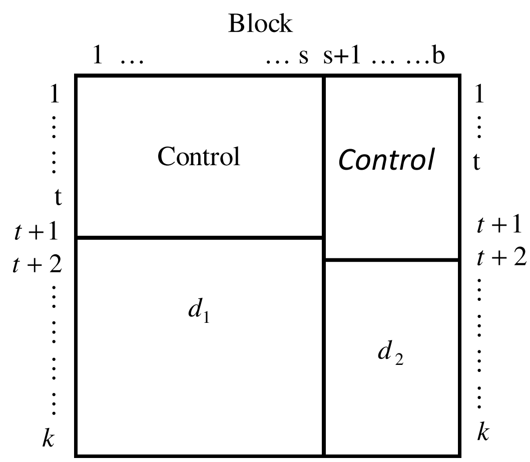

Figure 2. S-type BTIB design

4. THE METHOD OF CYCLIC SHIFTS

The method of cyclic shifts is a particular way of constructing test treatments versus control block designs. Here the v treatments are labeled as $0,1,2,\ldots,v-1$ and we consider the equi-replicate binary design for v treatments in $b=v$ blocks of size k. The method of construction is to allocate to the first plot in the

$i-$th block the treatment i; $i=0,1,2,...,v-1$. We denote this using the vector $u_1=[0,1,2,\ ...,v-1]^{\mathrm{T}}$, which holds the treatments allocated to the first plot in each of the blocks $1,2,...,v$ respectively. To obtain the treatment allocation of the remaining plots in each block, we cyclically shift the treatments allocated to the first plot. In order to define a cyclic shift, let u_i denote the $1\times v$ *vector*, which defines the allocation of treatments to the $i-$th plot in each block. That is, the $j-th$ element of u_i is the treatment allocated to plot i of block j. A cyclic shift of size q_i, when applied to plot i, is such that $u_{i+1}=[u_i+q_i\ 1]^{\mathrm{T}}$, where addition is mod v, 1 is a $1\times v$ vector of ones, $1\le i\le \mathrm{k}-1$ and $1\le q_i\le \mathrm{v}-1$. Assuming that we always start with u_1 as defined above, a design is completely defined by a set of $k-1$ shifts, Q, say, where $Q=[q_1,q_2,...,q_{k-1}]$. To avoid a treatment occurring more than once in a block, one must ensure that sum of any two successive shifts, the sum of any three successive shifts, … , the sum of any $k-1$ successive shifts is not equal to zero mod v. Subject to this constraint, Q may consist of any combination of shifts including repeats. Also the shifts need only range from 1 to $[v/2]$ inclusive, where $[v/2]$ is the greatest integer less than or equal to $v/2$. This is because a shift of size q is equivalent to one of size $[v-q]$ mod v.

Let us consider the construction of a design for $v=6$ and $k=4$ to illustrate the above method of construction. The set of shifts are defined by $Q=[q_1,q_2,q_3]$, where $q_i\in[1,2,3,4,5]$; $i=1,2,3$. Suppose that $Q=[1,2,5]$, then $u_1=[0,1,2,3,4,5]^{\mathrm{T}}$, $u_2=[1,2,3,4,5,0]^{\mathrm{T}}$, $u_3=[3,4,5,0,1,2]^{\mathrm{T}}$ and $u_4=[2,3,4,5,0,1]^{\mathrm{T}}$. Then the complete design will be

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 0 & 1 \end{matrix}$$

The properties of a design depend on the number of concurrences between the pairs of treatments. A concurrence between two treatments occurs when both treatments are in the same block. Because of the cyclic nature of the construction, the number of concurrences between any treatment and the remainder can be obtained from the number of concurrences between treatment 0 and the remainder. Also the number of concurrences between 0 and the remainder can easily be obtained from Q (the set of shifts used to construct the design). If shifts q_1 and q_2, for example, are applied successively to treatment 0, the result is a concurrence between treatment 0 and treatment q_1 and q_2, and a concurrence between treatment 0 and treatment q_1+q_2. If a third shift,

q_3 say, is applied after q_1 and q_2, then the following treatments will also concur with treatment 0: q_3, q_2+q_3 and $q_1+q_2+q_3$. This adding of shifts to get the treatments, which concur with 0 works for the general case and so enables the number of concurrences of a design to be obtained directly from the shifts, which defines it. In general, if shifts $q_1, q_2, ..., q_{i-1}$ have been applied successively to treatment 0, then the additional concurrences, which results when shift q_i is applied are between treatment 0 and treatments $q_i, q_i+q_{i-1}, ..., q_2+q_3+..+q_i, ..., q_1+q_2+...+q_i$ when addition is mod v. It can also be noted that any shift of size q that results in concurrence between treatment 0 and treatments q also results in a concurrence between treatment 0 and treatment $(v-q)$ mod v.

In the above design, $q_1=1$, $q_2=2$ and $q_3=5$ were used. For this one obtains $q_1+q_2=3$, $q_2+q_3=7=1(\text{mod } 6)$ and $q_1+q_2+q_3=8=2(\text{mod } 6)$. In the above design 1 appears twice (i.e. $q_1=1$ and $q_2+q_3=1$) and since 1 is symmetric to 5 and 5 appears once (i.e. $q_3=5$), therefore the concurrence between treatment 0 and treatment 1 is 3 and between treatment 0 and treatment 5 is also 3. Similarly 2 appears twice (i.e. $q_2=2$ and $q_1+q_2+q_3=2$) and since 2 is symmetric to 4, therefore, the concurrence between treatment 0 and treatment 2 is 2 and also between treatment 0 and treatment 4 is 2. And 3 appears once (i.e.). $q_1+q_2=3$ Since 3 is symmetric to itself, therefore the concurrence between treatment 0 and treatment 3 is 2. Therefore, the concurrences between treatment 0 and treatments 1,2,3,4,5 are 3,2,2,2,3 respectively. The concurrences between treatment 1 and treatments 2,3,4,5 2 follow the same pattern. i.e. the concurrences are 3,2,2,2. Similarly the concurrences between treatment 2 and treatments 3,4,5 are 3,2,2 and so on.

By using certain combinations of shifts we can construct designs that are made up of complete replicates of smaller designs. When v and k are not relatively prime, then partial sets of v/d blocks can also be obtained, where 'd' is any common divisor of v and k. The shifts producing such partial sets of blocks can be obtained as follows. The smallest integer 'a' is found where $(a \times v)$ $1/k=n$ and 'n' is an integer. Then the set of shifts used to construct the design is such that the sum of every 'a' successive shifts is equal to n. The design will contain v/n blocks. Designs which are constructed using such shifts are referred to as *fractional designs*. For even v and k, the fractional designs can be constructed by setting the middle shift ($q_{k/2}$) equal to $v/2$ and ensuring that shifts q_1 and q_{k-i}; $i=1,2,...,(\frac{k}{2}-1)$ are complement of each other. For example, a fractional design for $v=6$ and $k=4$ by using the set of cyclic shifts $[2,3,4](1/2)$ is given by

0	1	2	3	4	5
2	3	4	5	0	1
5	0	1	2	3	4
3	4	5	0	1	2

In order to construct a design with more than v blocks, we combine the blocks obtained from more than one sets of shifts. As an illustration, given below is a design for $v = 6$ treatments in 15 blocks of size 4 which has been constructed by combining together the blocks which are obtained from the three sets of shifts $[1,1,2]$, $[1,1,3]$ and $[2,3,4](1/2)$.

0	1	2	3	4	5	0	1	2	3	4	5	0	1	2
1	2	3	4	5	0	1	2	3	4	5	0	5	0	1
2	3	4	5	0	1	2	3	4	5	0	1	5	0	1
4	5	0	1	2	3	5	0	1	2	3	4	3	4	5

The above design has been constructed by using shifts $[1,1,2]+[1,1,3]+[2,3,4](1/2)$, where the "+" signs indicate that the blocks constructed from the separate sets of shifts must be combined together.

5. CONSTRUCTION OF TEST-CONTROL TREATMENT BLOCK DESIGNS

Method 1. If D_1 is a $BITB$ design that contains the control treatment t times in each block ($t = 1,2,...$), then the design D_2 in the test treatments obtained by deleting the control from each block of D_1 satisfies the definition of a BIB design. Thus one easy way of constructing a $BITB$ design is to start with a BIB design D_2 in the test treatments and to augment each block of D_2 with the control t times for some value of $t = 1,2,...$. $BTIB$ designs that are constructed using this augmentation process are called augmented BIB designs ($(BIBD's)$ as defined by Majumdar and Notz (1983).

In our method, we first construct a design in which each treatment pair appears together within blocks an equal number of times. The block sizes may or may not be equal. If the block sizes are equal, we have an $R-$type design, and then each block of the design can be augmented by t replicates of the control treatments, where $t \geq 1$. If the block sizes are not equal, then the design is an $S-$type. In this case, each block is augmented by a possibly different value of $t \geq 0$, so that the blocks of the augmented design also become equal sized. As we have our own catalogues of BIB designs and methods of constructing such designs, we therefore have a large number of designs for different values of v, k

and b to choose from. Therefore, we can construct designs for many different values of r_0 and r_1.

Example 1. Let $v=8$, $t=1$, $b=40$, $k=3$, $r_0=32$ and , $r_1=1$. The set of shifts used to construct the required design are $[12]+[1+2+3+4]C$, where [12] means the set of shifts $[12],[1+2+3+4]$ means the sets of shifts $[1]+[2]+[3]+[4]$ and C means to augment each block of this part of the design with a control treatment once. The complete design is given below:

Design 1

1 2 3 4 5 6 7 8	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0
2 3 4 5 6 7 8 1	1 2 3 4 5 6 7 8	1 2 3 4 5 6 7 8	1 2 3 4 5 6 7 8	1 2 3 4 5 6 7 8
4 5 6 7 8 1 2 3	2 3 4 5 6 7 8 1	3 4 5 6 7 8 1 2	4 5 6 7 8 1 2 3	5 6 7 8 1 2 3 4

Example 2. Let $v=8$, $k=4$, $b=12$, $r_0=16$ and $r_1=4$. The set of shifts used to construct the required design are $[12]C$ and $[4(1/2)]2C$. The C is defined in the above example and $2C$ means augment each block of this part of design with the control treatment two times, i.e. each block of this part of the design contains two replicates of the control treatment.

Design 2

0	0	0	0	0	0	0	0	0	0	0	0
1	2	3	4	5	6	7	8	0	0	0	0
4	5	6	7	8	1	2	3	5	6	7	8

Method 2. Bechhofer and Tamhane (1981) defined another method of design construction, which is as follows. Starting with a BIB design containing $t>v$ treatments in b blocks, one can replace the treatments $v+1, v+2, ..., t$ by zero to obtain a new $BTIB$ design with possibly an additional block or blocks, each one of the latter containing only one test treatment or only the control treatment. After deleting all of these one-treatment blocks and identifying the support of the resulting $BTIB$ design, we obtain the derived generator design (s). A generator design is defined by Bechhofer and Tamhane (1981) as $BTIB$ design, which is such that no proper subset of its blocks forms a $BTIB$ design, and no block of which contains only one of the ($v+1$) treatments. Bechhofer and Tamhane (1981) pointed out in their design (3.7a) by their method II; that every $BIBD$ involving t treatment yields a $BTIB$ design with $v=t-1$ test treatments.

In our second method of construction, we construct a design for ($v+1$) test treatments. If the design is a BIB, then each treatment appears together an equal number of times, and if we consider the 'zero' treatment as the control, then we have a *BTIB* design for v test treatments and one control. In this case $r_{01} = r_1$, (where r_{01} is number of replicates of the control treatment before augmentation). Here we can also augment each block by a control treatment t times. Then we have $r_0 = r_{01} + bt$. In this case we can also take designs with different values of k, and then augment the blocks with a control treatment to make all the blocks of equal size. We have also obtained a *BTIB* design by combining method 1 and method 2 (see example 4).

Example 3. Let $v = 8$, $k = 5$, $b = 18$, $r_0 = 26$ and $r_1 = 8$. The set of shifts used to construct the required design are: $\{[112 + 132](t+1)\}C$.

In this case we have constructed the design by using the sets of shifts $[1,1,2]$ and $[1,3,2]$ for $v+1$, i.e. 9 treatments (denoted in the sets of shifts by adding $t+1$ at the end) and then augmented the full design by the control treatment. The final design is given below.

Design 3

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8	0
2	3	4	5	6	7	8	0	1	4	5	6	7	8	0	1	2	3
4	5	6	7	8	0	1	2	3	6	7	8	0	1	2	3	4	5

Example 4. Let $v = 8$, $k = 4$, $b = 30$, $r_0 = 24$ and , $r_1 = 12$. The set of shifts used to construct the required design are: $[112 + 132](t+1) + [12]C + [4(1/2)]2C$.

Design 4

0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8	0	0	0	0	0	0	0	0	0	0	0	0
1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8	0	0	0	0
2	3	4	5	6	7	8	0	1	4	5	6	7	8	0	1	2	3	2	3	4	5	6	7	8	1	1	2	3	4
4	5	6	7	8	0	1	2	3	6	7	8	0	1	2	3	4	5	4	5	6	7	8	1	2	3	5	6	7	8

6. SUGGESTED DESIGNS AND COMPARISONS

Since $A-$optimal design exists for each set of values of v, b and k, we can find that for fixed value of r_0, which is a multiple of b as given by Ture (1985) and also for r_1, so that the design is $A-$optimal. But the question arises, what we should do if we do not have the required value of r_0; and if we have, we do not have the required number of replications? Then the obvious choice will be to use another design with different value of r_0. Therefore, for particular sets of (v,b,k), we give all possible designs for different values of r_0 and r_1. In all cases, the $A-$optimal design is in this class. If we compare our designs with the designs given in Appendix A, we will find that we have developed many new designs. For example, we give the designs for $v=4$, $k=3$ and $b=14$. There is no such design available for the value of v, k and $b=14$ given in their designs. However, we have proposed new designs for different values of r_0 and r_1 (see example 5 to example 7).

Example 5. Let $v=4$, $k=3$, $b=14$, $r_0=6$, $r_1=9$, $\lambda_0=3$, $\lambda_1=5$ and set of shifts $[1\,1(2)]+[1+2(1/2)]C$.

Design 8

1	2	3	4	1	2	3	4	0	0	0	0	0	0
2	3	4	1	2	3	4	1	1	2	3	4	1	2
3	4	1	2	3	4	1	2	2	3	4	1	3	4

Example 6. Let $v=4$, $k=3$, $b=14$, $r_0=10$, $r_1=8$, $\lambda_0=4$, $\lambda_1=4$ and set of shifts $[1\,1]+[1\,1](t+1)+\{[2](t+1)\}C$.

Design 9

1	2	3	4	0	1	2	3	4	0	0	0	0	0
2	3	4	1	1	2	3	4	0	0	1	2	3	4
3	4	1	2	2	3	4	0	1	2	3	4	0	1

Example 7. Let $v=4$, $k=3$, $b=14$, $r_0=14$, $r_1=7$, $\lambda_0=5$, $\lambda_1=3$ and set of shifts $[1\,1]+\{[1+2](t+1)\}C$.

Design 10

1	2	3	4	0	0	0	0	0	0	0	0	0	0
2	3	4	1	0	1	2	3	4	0	1	2	3	4
3	4	1	2	1	2	3	4	0	2	3	4	0	1

We have developed our designs using proposed methods (defined in Section 4) based on shifts. In this paper, we have also proposed a class of new designs for different values of (v,k,b). The class of these new designs is available for the following sets of $\boldsymbol{v}$, $\boldsymbol{k}$, different values of b, r_0 and r_1.

v	3	4	5	6	7	8	9	10
k	3	3, 4	3, 4, 5	3, 4, 5,6	3, 4, 5, 6, 7	3, 4, 5, 6, 7	3, 4, 5, 6, 7	3, 4, 5, 6, 7

v	11	12	13	14	15
k	3, 4, 5, 6,7, 8, 9	3, 4, 5,6	3, 4, 5, 6, 7	4, 5, 6	3, 4, 5, 6, 7, 8

But we will give only one specific case with $v = 8$ and $k = 4$ presented in Table

- **Table 1:** Suggested designs for one representative case $v = 8$ and $k = 4$.

ℓ	t	s	r_0	r_1	**Sets of Shifts**
12	1	4	14	4	$[12]C+[4(1/2)]2C$
16	0	16	16	6	$[122]+[1]2C$
18	0	32	32	5	$[222(1/4)]+[1+3]2C$
20	0	24	24	7	$[131(1/2)]+[12]C+[2]2C$
21	0	28	28	7	$[112](t+1)+\{[33(1/3)](t+1)\}C+\{[4](t+1)\}2C$
22	0	16	16	9	$[131](1/2)+222(1/4)]+[12(2)]C$
24	1	8	32	8	$[12(2)]C+[4]2C$
26	0	24	24	10	$[222](1/4)+[12(2)+13]C$
	0	16	16	11	$[123+131(1/2)+222(1/4)]+[11]C+[4(1/2)]2C$
27	0	44	44	8	$[112](t+1)+\{[3+4](t+1)\}2C$
	0	28	28	10	$[122](t+1)+\{[12+13](t+1)\}C$
28	2	0	56	7	$[1+2+3+4(1/2)]2C$
	0	32	32	10	$[112]+[12]C+[3+4(1/2)2C$
30	0	16	16	13	$[111+123+131(1/2)+222(1/4)]+[4]2C$
	0	48	48	9	$[222(1/4)]+[12]C+[1+3+4(1/2)]2C$
	0	24	24	12	$[113+222(1/4)]+[12(2)]C+[4(1/2)]2C$
	1	34	64	7	$\{[33(1/3)](t+1)\}C+\{[1+2+4](t+1)\}2C$
32	0	32	32	12	$[112+113]+[1+2]2C$
	0	8	8	15	$[112+113+122]+[12]C$
	0	40	40	11	$[131(1/2)]+[12(2)]C+[2+4(1/2)]2C$
33	0	20	20	14	$[112+113+122](t+1)+\{[33(2/3)](t+1)\}C$
	0	44	44	11	$[113](t+1)+\{[22+33(2/3)](t+1)\}C+\{[1](t+1)\}2C$
34	0	48	48	11	$[113+222(1/4)]+[1+2+3]2C$
	0	24	24	14	$[113+122+222(1/4)]+[12]C+[1]2C$
	0	32	32	13	$[131(1/2)+222(1/4)]+[12(3)]C+[4(1/2)]2C$
36	1	12	48	12	$\{[12+22+13(2)](t+1)\}C$
	2	8	80	8	$\{[1+2+3+4](t+1)\}2C$
	0	40	40	13	$[112+132](t+1)+\{[12](t+1)\}C+\{[4](t+1)\}2C$

37	0	44	44	7	[122]+[1]2C+{[4](t+1)}2C+[112](t+1) +{[33(1/3)](t+1}C
38	0	56	56	12	[131(1/2)+222(1/4)]+[12]C+[1+2+3]3C
	0	32	32	15	[113+131(1/2)+222(1/4)]+[12(2)]C+[2]2C
	0	40	40	14	[113+131(1/2)+222(1/4)]+[12(2)]C+[2]2C
39	0	60	60	12	[112](t+1)+{[3+4](t+1}2C+[12]C+[4(1/2)]2C
	0	44	44	14	[121](t+1)+{[12+14+22+33(1/3)](t+1)}C
	0	36	36	15	[112(2)+132](t+1)+{33(1/3)](t+1)}C+{[4](t+1)]2C
40	1	32	72	11	[12]C+[1+2+3+4]2C
	0	48	48	14	[112]+[12(2)]C+[3+4]2C
	0	24	24	17	[113+131(1/2)+222(1/2)]+[12(3)]C
41	0	52	52	14	[131(1/2)]+[12]C+[2]2C+{[4](t+1)}2C+[112](t+1) +{[33)1/3)](t+1)C+[112](t+1)+{[33(1/3)](t+1)}C
42	1	38	80	11	{[14+33(2/3)](t+1)}C+{[1+2(2)](t+1)}2C
	0	40	40	16	[112+113](t+1)+{[13+14+33(2/3)](t+1)}C
	0	56	56	14	[112+122](t+1)+{[33(2/3)](t+1)}C+{[1+2](t+1)}2C
	0	32	32	17	[111+123+131(1/2)+222(1/4)]+[12]C+[4(2/3)]2C
	0	64	64	13	[222(1/4)]+[12(2)]C+[1+3+4]2C
	0	40	40	18	[113+222(1/4)]+[12(3)]C+[4]2C
43	1	60	60	14	[112](t+1)+{[3+4](t+1)}2C+[122]+[1]2C
	0	44	44	16	[122](t+1)+{[12+13](t+1)}C+[122]+[1]2C
44	0	48	48	16	[112+113]+[12]C+[1+2+4(1/2)]2C
	0	24	24	19	[112+113+122]+[12(2)]C+[4(1/2)]2C
	0	56	56	15	[131(1/2)]+[12(3)]C+[2+4]2C
	0	72	72	13	[122]+[1(2)+2+3+4(1/2)]2C
	0	32	32	18	[222(1/4)]+[12(2)+13]C+[112+132](t+1)
45	0	36	36	18	[112+122+132](t+1)+{[12+13](t+1)}C
	0	52	52	16	[112(2)+132](t+1)+{[3+4](t+1)}2C
	0	60	60	15	[112](t+1)+{[12+13+23](t+1)}C+{[4](t+1)}2C
	0	76	76	13	[113](t+1)+{[12](t+1)}C+{[2+3+4](t+1)}2C
46	0	64	64	15	[12]C+[4(1/2)]2C+[113+222(1/4)]+[1+2+3]3C
	0	40	40	18	[12]C+[4(1/2)]2C+[113+122+222(1/4)]+[12]C +[12]2C
	0	48	48	17	[12]C+[4(1/2)]2C+[131(1/2)+222(1/4)]+[12(3)]C +[4(1/2)]2C
	0	32	32	19	[111+122+123+131(1/2)+222 (1/4)]+[1+4]2C
	0	80	80	13	[122]+[1]2C+{[33(1/3)](t+1)}+{[1+2+4](t+1)}2C
47	0	68	68	15	[112](t+1)+{[3+4](t+1)}2C+[131(1/2)]+[12]C

					+[2]2C
	0	52	52	17	[122](t+1)+{[12+13](t+1)}C+[131(1/2)]+[12]C+[2]2C
	0	44	44	18	[123+131(1/2)+222(1/4)]+[11]C+[4(1/2)]2C+ {[4](t+1)}2C+[112](t+1)+[33(1/3)](t+1)}C
48	1	16	64	16	[4(1/2)]2C+{[12+22+13(2)](t+1)}C+[12]C
	1	48	96	12	[4(1/2)]2C+{[1+2+3+4](t+1)}C+[12]C
	0	56	56	17	[12]C+[4(1/2)]2C+[112+132](t+1)+{[12](t+1)}C +{[4](t+1)}2C
	1	32	80	14	{[12]+13+22+33(1/3)](t+1)}C+{[1+4](t+1)}2C
	0	32	32	20	{[14+33(1/3)](t+1)}C+[112(2)+132+124](t+1)
	0	72	72	15	[112+132](t+1)+{[33(1/3)](t+1)}C+{[1+2+4] (t+1)}2C
	0	48	48	18	[112+113+122]+[1(2)+2]2C
	0	24	24	21	[112+113+122(2)]+[12]C+[1]2C
	0	40	40	19	[131(1/2)+222(1/2)]+[12(4)+13]C
49	0	36	36	20	[112+113+122](t+1)+{(33(2/3)](t+1)}C+[122] +[1]2C
	0	60	60	17	[113](t+1)+{[22+33(2/3)](t+1)}C+{[1](t+1)}2C +[122]+[1]2C
	0	84	84	14	[112](t+1)+{[33(1/3)](t+1)}C+{[4](t+1)}2C+ [1+2+3+4(1/2)]2C
	0	44	44	19	[131(1/2)+222(1/4)]+[12(2)]C+[122](t+1) +{[12+13](t+1)}C
50	0	64	64	17	[113+122+222(1/4)]+[1(2)+2+3]2C
	0	40	40	20	[113+122(2)+222(1/4)]+[1(2)]2C+[12]C
	0	48	48	19	[122+131(1/2)+222(1/4)]+[12(3)]C+[1+4(1/2)]2C
	0	72	72	16	[131(1/2)+222(1/4)]+[12(2)]C+[1+2+3+4(1/2)]2C
	0	88	88	14	[131(1/2)]+[12]C+{[33(1/2)](t+1)}C+[2]2C+ {[1+2+4](t+1)}2C
	0	56	56	18	[222(1/4)]+[12(4)+13]C+[4]2C
	0	16	16	23	[112+113(2)+122+222(1/4)]+[12(2)]C

7. CONCLUDING REMARKS

In this paper, we have proposed new methods for the construction of block designs for comparing test treatments with a control by using cyclic shifts. These new designs also provide a flexible family of *BTIB* designs but in this paper we have only given one representative case for $v = 8$ and $k = 4$. The complete set of designs is available from the first author on request. These new designs can be considered for the sets of v, k and different values of b, r_0 and r_1.

An important feature of these designs is that, for all cases the A-optimal design may be present in this class. We can use a different rotation to construct the tables of these designs. In cases where *BIBD* does not exist, we can fill the gap by using a regular graph designs (see Iqbal and Jones, 1999).

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Appendix A

Majumdar and Notz (1983) gave designs for the following sets of parameters.

v	***k***	***b***
5	3	10
6	4	30
7	4	14
10	5	30
4	3	30
25	6	30
5	4	30

Hedayat and Majumdar (1984) gave designs for the following sets of parameters.

		b	***v***		***b***	***v***		***b***
		3, 6,…	4		6, 12, …	5		10
		37	8		40	9		24
		4, 8,…	5		10, 20, …	7		7, 14, …
		12, 24, …	6		18, 36, …	7		35
		18,	1		15,	1		33

		36, …	0		30, …	2		
6		20, 40,…,	1 6		48	2 1		21, 42, …
		48	1 4		35	1 5		16
		28	9		12, 24, …			

Ture (1985) gave designs for the following sets of parameters.

	k	***b***
	3	3, 4, 5,…, 25.
	3	7, 10, 14, 17, 20, 24.
	3	7, 10, 11, 15, 18, 22, 25.
	3	12, 18, 23, 28.
	4	6, 7, 10, 11, 12, 14,…, 25.
	4	7, 11, …, 14, 17, …, 21, 24, 25.
	5	5, 6, 7, …, 25.
	5	7, 10, 11, 13, 14, 17, 18, 20, 21, 24, 25.
	6	6, 7, 25.
	6	10, 11, 12, 14, 15, 17, 18, …, 22.
	3	2, 6, 10, 12, 13, 16, 18, 19, 22, 24, 25.
	3	7, 12, 17, 21, 24, 28.
	4	4, 5,…, 25.
	4	7, 9, 10, 12, 14, 16, 17, 19, 20, 22, 24.
	4	12, 15, 19, 22, 24, 25, 26, 29.
	5	6, 7, 9… 25.
	5	11, 14, 17, 18, 20, 21, 22, 24, 25.
	6	7, 10, …, 19.

Hedayat and Majumdar (1985) gave designs for the following sets of parameters.

v	***k***	***b***
4	4	4, 8, …, 20.
9	5	18, 36, …, 90.
16	6	48, 96, …, 288.
8, 9,10	8	
10	9	

Stufken (1987) gave designs for the following sets of parameters.

v k b	***v k b***	***v k b***	***v k b***	***v k b***
3 3 3	5 4 10	4 3 6	6 4 10	7 4 7
8 8 28	8 4 56	9 4 12	9 8 12	10 5 15
10 9 30	11 5 55	11 9 55	12 5 33	12 9 132
13 5 13	13 10 39	14 5 91	14 10 91	15 10 15
15 10 105	16 5 20	16 10 30	17 6 68	18 6 306
19 6 171	20 6 76	21 6 21	22 6 462	23 6 253
24 6 552	25 6 30	26 7 65	27 7 117	28 7 126
29 7 406	30 7 145			

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