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ASYMPTOTICS OF SMOOTHED KERNEL QUANTILE ESTIMATORS

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ABSTRACT

Let $\{X_i, 1 \le i \le n\}$ be a random sample from a continuous *df* $F(x)$ with the $\lambda-$th quantile $q(\lambda) = \inf(x : F(x) > \lambda)$. The present work concerns with the study of the smoothed kernel quantile estimator $\hat{q}(\lambda)$ proposed by Yang (1985) and establish in terms of mean square error its asymptotic viz. LIL, Berry-Esseen's, theorem. By Monte-Carlo study establish, its superiority over all other forms of quantile estimators of $q(\lambda)$ considered so far in the literature.

1. INTRODUCTION

The estimation of population quantiles is of great interest when one is not prepared to assume a parametric form for the underlying distribution F. Besides, quantiles often arise as the natural things to estimate when F is skewed. Quantile estimators may also be used in estimating percentage points of a statistic by simulation techniques. Let $X_1, X_2, \ldots, X_n$ be a sequence of *i.i.d.* random variables with distribution function $F(x)$ and let $q(\lambda)$ be $\lambda-$th quantile defined by $q(\lambda) = \inf\{x : F(x) \ge \lambda\}$, $0 < \lambda < 1$. If $\hat{q}(\lambda)$ is smoothed estimator of $q(\lambda)$ and U_i, $1 \le i \le n$, is a random sample from $U[0,1]$, then $\hat{q}(U_i)$, $1 \le i \le n$, provides a smooth bootstrap sample (Efron 1979, 1981).

The traditional estimator of $q(\lambda)$ is the $\lambda-$th sample quantile given by $X_{(n\lambda+1)}$, $[x]$ denoting the integer part of x. The main drawback here is that it experiences a substantial lack of efficiency caused by the variability of individual order statistics. The obvious way of improving the efficiency of sample quantiles is to reduce this variability by forming a weighted average of all order statistics, using an appropriate weight function. These estimators are called $L-$estimators, a popular class of which is called kernel quantile estimators. Bahadur (1966) considered the almost sure (*a.s.*) representation for the inverse type estimator of $q(\lambda)$ defined by

$$q_n(\lambda) = \inf\{x : F_n(x) \ge \lambda\}, \qquad 0 < \lambda < 1, \tag{1.1}$$

where $F_n(x) = n^{-1}\sum_{i=1}^{n} I(X_i \le x)$ and the asymptotic properties of $q_n(\lambda)$ automatically follow from it. Bahadur-Kiefer almost sure representation for $q_n(\lambda)$ was established in Kiefer (1967).

The first proposal due to Nadaraya (1964) starts with smoothed nonparametric kernel estimator defined by

$$\bar{q}_n(\lambda) = \inf\{x : \hat{F}_n(x) \ge \lambda\} \tag{1.2}$$

where $\hat{F}_n(x) = n^{-1}\sum_{i=1}^{n} K((x - X_i)/a_n)$, $K(x) = \int_{-\infty}^{x} k(t)\,dt$, k being a suitable kernel function and $\{a_n\}$ bandwidth sequence controlling the smoothing of observations, such that $a_n \downarrow 0$ and $n a_n \to \infty$ as $n \to \infty$. The asymptotics of $\bar{q}_n(\lambda)$ have been well explored in a series of papers and we refer to Ralescu (1992) and the references therein.

Parzen (1979) proposed another version of a smoothed nonparametric kernel estimator of $q(\lambda)$ in terms of kernel function k defined by

$$\tilde{q}_n(\lambda) = \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} k_n(\lambda, t)\,dt\, X_{[i]}$$

$$= \int_0^1 F_n^{-1}(t)\, k_n(\lambda, t)\,dt, \quad 0 < \lambda < 1 \tag{1.3}$$

where $k_n(\lambda, t) = a_n^{-1} k((\lambda - t)/a_n)$. The estimator (1.2a) puts most weight on $X_{[i]}$. Bahadur–Kiefer almost sure representation for $\tilde{q}_n(\lambda)$ was recently established by X. Xiang (1994). In practice, the following approximation to the kernel estimator $\tilde{q}_n(\lambda)$ defined in (1.2a) is

$$\tilde{\tilde{q}}_n(\lambda) = n^{-1}\sum_{i=1}^{n} k_n((i/n, \lambda)/a_n) X_{(i)} - n^{-1}\sum_{1}^{n} X_i k_n(F_n(X_i), \lambda) \tag{1.4}$$

The derivatives of $\tilde{q}_n(\lambda)$ and $\tilde{\tilde{q}}_n(\lambda)$ gives smooth estimators for the derivative of $q(\lambda)$. These may be useful in the approach to statistical data analysis proposed in Parzen (1979).

To make the total weight unity, we modify $\tilde{\tilde{q}}_n(\lambda)$ as $\hat{q}_n(\lambda)$ defined as

$$\hat{q}_n(\lambda) = \sum_{n}^{n} W_n(X_i) X_i, W_n(X_i) = k_n(F_n(X_i), \lambda) / \left\{ \sum_{1}^{n} K_n(F_n(X_i), \lambda) + n^{-2} \right\} \tag{1.5}$$

$$= n^{-1}\sum_{1}^{n} X_i\, k_n(F_n(X_i),\lambda)/t_n(\lambda),$$

where the factor n^{-2} is introduced to assure that the denominator does not vanish.

Yang (1985) showed that $\tilde{\tilde{q}}_n(\lambda)$ and $\tilde{q}_n(\lambda)$ as defined in above are asymptotically equivalent in mean square. By Monte Carlo simulation studies he showed that, for normal population the performances of $\tilde{\tilde{q}}_n(\lambda)$ and $\tilde{q}_n(\lambda)$ defined above are roughly the same, while, for exponential distributions, his proposed estimator $\tilde{\tilde{q}}_n(\lambda)$ seemed to be slightly better than $\tilde{q}_n(\lambda)$. For other type of equivalent estimators, we refer to Sheather and Marron (SM, 1990) and references therein. Simulation studies of SM 1990 based on samples from different populations showed that no quantile estimator dominated over the others, nor was any better than the sample quantile $q(\lambda)$ defined in (1.1a), in every case. Fiqure 2 in SM (1990) is plot of efficiency of $\hat{q}_n(\lambda)$ with respect to $q_n(\lambda)$ for samples from log normal population and indicates that there is little difference between them and they concluded that, given the well known distribution inference procedures associated with $q_n(\lambda)$ as well as the ease with which it can be calculated, $q_n(\lambda)$ will often be a reasonable choice as a quantile estimator. Comparing the estimators mentioned above with the estimators $\tilde{q}_n(\lambda)$ and $\tilde{\tilde{q}}_n(\lambda)$ or $\hat{q}_n(\lambda)$, these latter two are computationally simpler and have more easily adjustable smoothing parameters a_n to regulate the amount of smoothness desired.

Objective

The objective of the paper is to establish the superiority of smoothed Kernel estimator $\hat{q}_n(\lambda)$ defined by (1.2c) over all other types of quantile estimators of $q(\lambda)$ considered so far in the literature.

Here we study the asympotics of the proposed smooth quantile estimator $\hat{q}_n(\lambda)$, almost sure (*a.s.*) representation rates of convergence to normality, exact evaluation of *MSE* of $\hat{q}_n(\lambda)$ comparisons of their performances based on their *MSE's*, by Monte Carlo studies.

2. ASSUMPTIONS

The following assumptions are necessary for the main results in the present investigations.

AI i) The population *df* $F(x)$ has a *pdf* $f(x)$, which is absolutely continuous and positive in some *nghd* of $q(\lambda)$.

ii) $f^{(1)}(X)$ exists and is continuous at $q(\lambda)$,

iii) $E|x|^{2p+\delta} < \infty$ for $p \geq 2$, $\delta > 0$,

iv) $q^{(2)}(\lambda)$ is continuous in a *nghd* of λ.

A.II i) $k(x)$ is positive, bounded kernel function with bounded support $[-1,1]$ (say),

ii) $k(x)$ has finite continuous derivatives up to order r, $1 \leq r \leq 7$,

iii) $k(x)$ is symmetric about 0, i.e. $\int t^j k(t)dt = 0$, $j = 1,3$,

A.III i) Let $\{a_n\}$ be bandwidth sequence such that as $n \to \infty$, $a_n \to 0$ and $na_n \to \infty$,

ii) $n a_n^4 / \log\log n = O(1)$ as $n \to \infty$,

The conditions imposed in A.II on k are satisfied by a variety of kernel functions and we give two examples below for k:

a) $k(s) = (2\pi)^{-1/2}(1-s^2)^{-3/2} \exp[-s^2/2(1-s^2)]$, if $|s|<1$,
$= 0$, otherwise,

b) $k(s) = c_r(1-s^2)^r$, if $|s|<1$, for $r \geq 1$,
$= 0$, otherwise,

where c_r is chosen so that $\int_{-1}^{1} k(s)ds = 1$

$k(s) = (2\pi)^{-1/2} e^{-s^2/2}, s \in R$.

3. A.S. REPRESENTATION FOR SMOOTH QUANTILE ESTIMATOR

In order to develop the results in the present paper, we need the *a.s.* representation for the smoothed sample quantile $\hat{q}_n(\lambda)$, $0 < \lambda < 1$, defined in (1.2c) based on the random sample $X_1, X_2, \ldots, X_n$. We need the following results in the present work.

Proposition 3.1:

i) (Stute, 1982): If $\log a_n^{-1} / \log\log a_n^{-1} \to \infty$ and $n a_n / \log n \to \infty$, as $n \to \infty$ then by setting

$U_n(x) = n^{1/2}(F_n(x) - F(x))$ with $F_n(x) = [\# X_i]/n$

$$\underset{\substack{|x-u|\le a_n \\ x,u\in R(1)}}{Sup}\ U_n(x)-U_n(U)=O(a_n \log a_n)^{1/2} \quad a.s.$$

ii) (Stute, 1984): Under the assumed conditions

$$\underset{\substack{|x-u|\le a_n \\ |y-v|\le h_n}}{Sup}\ |\beta_n(u,v)-\beta_n(x,y)|=O(a_n h_n \log n)^{1/2} \quad a.s.,$$

where $\beta_n(x,y)=n^{1/2}(H_n-1+)\circ(x,y)H_n$ being bivariate empirical *df* of bivariate sample drawn from H.

Proposition 3.2: (Drovetzky *et al.* (1956), Lemma 2): For a random sample $X_1,\ldots,X_n$ *i.i.d.* F, there exists some $\in > 0$ such that

$$\sum_{n\ge 1}\Pr(\underset{x}{Sup}\, n^{1/2}|F_n(x)\text{-}F(x)|>\in)<\exp[-\in^2].$$

We first state the following results which are needed in establishing Bahadur type *a.s.* representation for $\hat{q}_n(\lambda)$.

Lemma 3.1: Under the conditions A.I (i), A.II and A.III (i),

$$t_n(t_n(\lambda)=1+O(\log n/na_n)^{1/2} \ a.s. \text{ as } n\to\infty.$$

Lemma 3.2: Under the conditions Lemma 3.1

$$(t_n^{-1}(\lambda)=Op(\log n/na_n)^{1/2}.$$

We now establish a.s. representation for $\hat{q}_n(\lambda)$ in the following:

Theorem 3.1: Under the assumptions A.I, A.II, A.III (i) on k, $\{a_n\}$ and F

$$\sqrt{n}[\hat{q}_n(\lambda)-q(\lambda)]=\sqrt{n}q^1(\lambda)(\lambda-F_{n}{}_0 q(\lambda))+c_2(\lambda)(a_n^2 n^{1/2})$$
$$+O(a_n \log n)^{1/2}v(\log\log n/na_n)^{1/2}, \tag{3.1}$$

where

$$c_2(\lambda)=2^{-1}q^{(2)}(\lambda)\mu_2(K) \text{ and } \mu_2(K)=\int t^2 k(t)\,dt.$$

Proof: From (1.2c), we have

$$[\hat{q}_n(\lambda)-q(\lambda)]=[\upsilon_n(\lambda)/t_n(\lambda)-q(\lambda)]$$
$$=[\upsilon_n(\lambda)-q(\lambda)t_n(\lambda)]/t_n(\lambda)$$
$$=:\bar{\upsilon}_n(\lambda)/t_n(\lambda). \tag{3.2}$$

By Taylor expansion of $k_n(F(x),\lambda)$ at $F(x)-\lambda/a_n$,

$$\bar{v}_n(\lambda)\int(x-q(\lambda))k_n(F(x),\lambda)\,dF_n(x)$$
$$+a_n^{-1}\int(x-q(\lambda))(F_n(x)-F(x))k_n^{(1)}(F(x),\lambda)\,dF_n(x)$$
$$+2^{-1}a_n^{-2}\int(x-q(\lambda))(F_n(x)-F(x))^2k_n^{(2)}(\Delta(x))\,dF_n(x)$$
$$=:I_{n1}+I_{n2}+I_{n3}, \tag{3.3}$$

where

$\Delta(x)$ lies between $(F(x)-\lambda)/a_n$ and $(F_n(x)-\lambda)/a_n$ and $k_n^{(2)}\Delta(x)=a_n^{-1}k^{(2)}(\Delta(x))$.

Let

$ta_n=F(x)-\lambda$, $x=q(\lambda+a_nt)$, then

$$\left|I_{n1}\right|=\left|\int a_n^{-1}[q(\lambda+a_nt)-q(\lambda)]k(t)a_n\,dt\right|$$
$$+\int(x-q(\lambda))k_n(F(x),\lambda)\,d(F_n-F)ox$$
$$=\left|\int(tq^{(1)}(\lambda)+2^{-1}a_nt^2q^{(2)}(\lambda)+\ldots)k(t)a_ndt\right.$$
$$+\left|+\int(x-q(\lambda))k_n(F(x),\lambda)\,d(F_n-F)\circ x\right.$$
$$=2^{-1}a_n{}^2q^{(2)}(\lambda)\mu_2(K)+\in_{n1},\ldots \tag{3.4}$$

and by Proposition 2.1(i) regarding *a.s.* oscillation result,

$$\sqrt{n}\,|\in_{n1}|\le n^{1/2}a_n^{-1}\sup_{|x-q(\lambda)|\le ca_n}|(x-q(\lambda)|\|k\|_\infty(F_n-F)\circ x-(F_n-F)\circ q(\lambda)|$$
$$=a_n{}^{-1}n^{1/2}ca_n\|k\|_\infty(a_n\log n/n)^{1/2}\quad a.s$$
$$=O(a_n\log n)^{1/2}\quad a.s. \tag{3.5}$$

Again, from (3.3)

$$\left|I_{n3}\right|=2^{-1}a_n^{-1}\int(x-q(\lambda))(F_n(x)-F(x))^2k_n{}^{(2)}(\Delta_n)\,dF(x)$$
$$+2^{-1}a_n{}^{-2}\int(x-q(\lambda))(F_n(x)-F(x))^2k_n{}^{(2)}(\Delta_n)$$
$$\times d[(F_n-F)\circ x-(F_n-F)\circ q(\lambda)]$$
$$=:\in_{n31}+\in_{n32} \tag{3.6}$$

$$\sqrt{n}\,|\epsilon_{n31}| = 2^{-1} a_n^{-3} \sup_{|x-q(\lambda)|\le ca_n} \sqrt{n}\,|x-q(\lambda)|\,\|F_n - F\|_\infty^2 \|k^{(2)}\|_\infty \int dF(x)$$

$$= O(a_n^{-3} n^{1/2} ca_n (((\log\log n/n)a_n))$$

$$= O(\log\log^2 n/na_n^2)^{1/2} \quad (3.7)$$

by using Proposition 2.1(i). Similarly,

$$\sqrt{n}\,|\epsilon_{n32}| = 2^{-1} a_n^{-3} n^{1/2} \sup_{|x-q(\lambda)|\le ca_n} |x-q(\lambda)|\,\|F_n - F\|_\infty^2 \|k^{(2)}\|_\infty$$

$$\times |(F_n - F)\circ x - (F_n - F)\circ q(\lambda)|$$

$$= O(a_n^{-3} n^{1/2} ca_n (\log\log n/n)(a_n \log\log n/n)^{1/2}\ a.s$$

$$= O(\log\log^2 n\ \log\log n/na_n^3)^{1/2} \quad (3.8)$$

$$\sqrt{n}\,|I_{n3}|\,O(\log n \log\log n \log n/na_n^2)^{1/2}, \quad (3.9)$$

whereas, from (3.3)

$$I_{n2} = a_n^{-1}\int (x-q(\lambda)(F_n\circ q(\lambda)-\lambda)k_n^{(1)}(F(x),\lambda)\,dF_n(x)$$

$$+ a_n^{-1}\int (x-q(\lambda)[F_n - F)\circ x - (F_n - F)\circ q(\lambda)]k_n^{(1)}(F(x),\lambda)\,dF_n(x)$$

$$=: I_{n21} + I_{n22} \quad (3.10)$$

$$I_{n21} = a_n^{-1}\int (x-q(\lambda))(F_{n^o} q(\lambda)-\lambda)\,k_n^{(1)}(F(x),\lambda)\,dF(x)$$

$$+ a_n^{-1}(F_{n^o} q(\lambda)-\lambda)\int (x-q(\lambda))\,k_n^{(1)}(F(x),\lambda)$$

$$d[(F_n - F)\circ x - (F_n - F)\circ q(\lambda)]$$

$$=: T_n + \epsilon_{n21} \quad (3.11)$$

$$T_n = (F_{n^o} q(\lambda)-\lambda)a_n^{-2}\int (q(\lambda + a_n t) - q(\lambda))\,k^{(1)}(t)\,a_n\,dt$$

$$= (F_n\circ q(\lambda)-\lambda)[q^{(1)}(\lambda)\int tk^{(1)}(t)dt + 2^{-1}(q^{(2)}(\lambda))a_n$$

$$\int t^2 k^{(1)}(t)dt + \ldots +]$$

$$= -q^{(1)}(\lambda)(F_n\circ q(\lambda)-\lambda) + O(a_n^3(F_n\circ q(\lambda)-\lambda)) \quad (3.12)$$

$$= -q^{(1)}(\lambda)(F_n\circ q(\lambda)-\lambda) + O(a_n^6 \log\log n/n)^{1/2}\ a.s$$

in view of assumptions A.I and A.II and Proposition 2.2 whereas

$$[\hat{q}_n(\lambda)-q(\lambda)]=a_n^{-2}\sqrt{n}\,|F_n\circ q(\lambda)-\lambda|\sup_{|x-q(\lambda)|\le ca_n}|x-q(\lambda)|\,\|k^{(1)}\|_\infty$$

$$\times|(F_n-F)\circ x-(F_n-F)\circ q(\lambda)|$$

$$=O(a_n^{-2}ca_n(a_n\log n/n)^{1/2}\ a.s$$

$$=O(\log n/na_a)^{1/2}\ a.s\ \text{as}\ n\to\infty \tag{3.13}$$

by Proposition 2.2. Hence, from (3.3h) – (3.3i)

$$I_{n21}=-q^{(1)}(\lambda)\sqrt{n}(F_n\circ q(\lambda)-\lambda)$$

$$+O(a_n^3(\log\log n)^{1/2}\,v(\log n/na_n)^{1/2}) \tag{3.14}$$

and

$$I_{n22}=a_n^{-1}\int(x-q(\lambda))(F_n-F)\circ X-(F_n-F)\circ q(\lambda)\,k_n^{(1)}(F(x),\lambda)$$

$$\times d[(F(x)+(F_n-F)\circ X-(F_n-F)\circ q(\lambda)]$$

$$=:I_{n221}+I_{n222} \tag{3.15}$$

$$|I_{n222}|\le a_n^{-2}\sup_{|x-q(\lambda)|\le ca_n}|x-q(\lambda)|$$

$$|(F_n-F)\circ X-(F_n-F)\circ q(\lambda)|^2\,\|k^{(1)}\|_\infty$$

$$=a_n^{-2}ca_n(a_n\log n/n)\,\|k^{(1)}\|_\infty$$

$$=O(\log n/n)\ a.s. \tag{3.16}$$

by using Proposition 2.1(i) and

$$|I_{n222}|\le a_n^{-2}ca_n(a_n\log n/n)^{1/2}\,\|k^{(1)}\|_\infty\int_{x-q(\lambda)|\le ca_n}dF(x)$$

$$=O(a_n\log n/n)^{1/2}\ a.s. \tag{3.17}$$

Combining equations (3.3), (3.3b) to (3.3n), we get WPI,

$$\sqrt{n}\,\bar{\upsilon}_n(\lambda)=\sqrt{n}\ q^{(1)}(\lambda)(\lambda-F_n\circ q(\lambda))+c_2(\lambda)((a_n^2n^{1/2})$$

$$+O((a_n\log n/n)^{1/2}) \tag{3.18}$$

$$\bar{v}_n(\lambda) = (\hat{q}_n(\lambda) - q(\lambda))\, t_n(\lambda)$$

$$= -[F_n \circ q(\lambda) - \lambda]\, q^{(1)}(\lambda) + c_2(\lambda) a_n^2 + O((a_n \log n / n)^{1/2})$$

$$\hat{q}_n(\lambda) - q(\lambda) = \frac{-[F_n \circ q(\lambda) - \lambda]\, q^{(1)}(\lambda)}{1 + (t_n(\lambda) - 1)} + c_2(\lambda) a_n^2 + O((a_n \log n / n)^{1/2})$$

$$= -[F_n \circ q(\lambda) - \lambda]\, q^{(1)}(\lambda) + c_2(\lambda) a_n^2 + O((a_n \log n / n)^{1/2})$$

by using Lemma 3.1. Hence the theorem follows.

We now establish LIL for $\hat{q}_n(\lambda)$ in the following:

Theorem 3.2: Under the assumptions of Theorem 3.1,

$$\underset{n\to\infty}{Lim}\sup \pm (2 \log\log n / n)^{1/2} [\hat{q}_n(\lambda) - q(\lambda)]$$

$$= q^{(1)}(\lambda) \sqrt{(\lambda(1-\lambda))} + c_2(\lambda) d^{1/2} \ a.s.,$$

where

$$d = \lim_{n\to\infty} ((n\, a_n^4) \log\log n)^{1/2}.$$

Proof: From Theorem 3.1

$$\hat{q}_n(\lambda) - q(\lambda) = q^{(1)}(\lambda)(\lambda - F_n \circ q(\lambda)) + c_2(\lambda) a_n^2 + O(a_n \log n / n)^{1/2}) \ a.s.$$

$$=: T_n' + c_2(\lambda) a_n^2 + O(\tau_n),$$

where

$$T_n' = -q^{(1)}(\lambda)(F_n \circ q(\lambda) - \lambda)$$

$$= -q^{(1)}(\lambda) n^{-1} \left(\sum_1^n I(X_i \le q(\lambda)) - \lambda \right)$$

$$= -q^{(1)}(\lambda) n^{-1} \left(\sum_1^n I(F(X_i) \le \lambda) - \lambda \right)$$

$$= -q^{(1)}(\lambda) \left(n^{-1} \sum Z_i - \lambda \right), Z_i\ I(U_i \le \lambda), \quad U_i \sim [0,1]$$

$$= -q^{(1)}(\lambda)(S_n - E S_n), \qquad \mathrm{Var}\, S_n = \lambda(1-\lambda)/n = \sigma_n^2 \text{ (say)},$$

so that

$$(2\sigma_n^2 \log\log n)^{-1/2} T_n' = -q^{(1)}(\lambda)(2\sigma_n^2 \log\log n)^{-1/2}(S_n - E S_n)$$

$$\lim_{n\to\infty} \sup \pm (2n^{-1} \log\log n)^{1/2} T_n' = q^{(1)}(\lambda)\sqrt{(\lambda(1-\lambda)} \; a.s.$$

Now

$$\lim_{n\to\infty} \sup \pm (2\log\log n)^{-1/2}(\hat{q}_n(\lambda) - q(\lambda))$$

$$= \lim_{n\to\infty} \sup \pm (2\log\log n)^{-1/2} T_n' + c_2(\lambda) \lim_{n\to\infty} a_n^2$$

$$= q^{(1)}(\lambda)\sqrt{(\lambda(1-\lambda))} + c_2(\lambda) d^{1/2} \; a.s.$$

Hence the result follows.

4. MEAN SQUARE ERROR (*MSE*)

In order to achieve the main objective of the present investigation, we need to evaluate the exact mean square errors of all kernel quantile estimators of $q(\lambda)$), so that the efficiency ratios of MS Errors can be computed for comparison purposes.

We first obtain the mean square error of the smoothed estimator defined in (1.2c) in the following:

Theorem 4.1: Under the conditions

$$MSE\,\hat{q}c_2^2(\lambda) + \sigma_n^2 oq(\lambda)n^{-1} + o(a_n/n)_n(\lambda) = a_n^4,$$

where

$$\sigma_n^2 oq(\lambda) = q^{(1)}(\lambda)^2[\lambda(1-\lambda) - 2a_n\psi(K)],\; c_2(\lambda) = \text{is as defined in (3.1)}$$

and

$$\psi(K) = 2\int s\,K(s)\,dK(s).$$

Proof: From (1.2c),

$$\hat{q}_n(\lambda) = \sum X_i k_n(F_n(X_i),\lambda)/\{k_n(F_n(X_i),\lambda) + n^{-2}\}$$

$$MSE\,\hat{q}_n(\lambda) = E[\hat{q}_n(\lambda) - q(\lambda)]^2$$

$$= E[\sum(X_i - q(\lambda))k_n(F_n(X_i),\lambda)/\{\sum k_n(F_n(X_i),\lambda) + n^{-2}\}]^2 + \epsilon_{n1}$$

$$=: E[L_n(\lambda)/t_n(\lambda)]^2 + \epsilon_{n1}, \tag{4.1}$$

where

$$\epsilon_{n1} = E[q(\lambda)/n^4 t_n^2(\lambda)) - 2q(\lambda)L_n(\lambda)/n^2 t_n(\lambda)] = o(a_n/n)$$

in view of Lemma 3.2 and Theorem 3.1. By Taylor expansion of $k_n(F_n(X_i),\lambda)$ about $(F(X_i)-\lambda)/a_n)$ up to seven terms

$$L_n(\lambda)=n^{-1}\sum_1^n(X_i-q(\lambda))k_n(F_n(X_i,\lambda)$$

$$=n^{-1}\sum_1^n(X_i-q(\lambda))k_n(F(X_i),\lambda)I_{A_n}(X_i)$$

$$+n^{-1}\sum_1^n(X_i-q(\lambda))\sum_{j=1}^6((F_n-F)^j\circ X_i/j!a_n^{j+1})k^{(j)}(F(X_i),\lambda)I_{A_n}(X_i)$$

$$+n^{-1}\sum_1^n(X_i-q(\lambda))(7!a_n^8)^{-1}(F_n-F)^7\circ X_ik^{(7)}(\Delta_n)I_A(X_i)$$

$$=:\sum_1^8 J_{ni}(\lambda), \tag{4.2}$$

where $k^{(j)}(.)$ denotes $j-$th derivative of $k(.)$, $A_n:\{X_i:|F_n(X_i)-\lambda|\le a_n\}$ and Δ_n lies between $F(X_i)-\lambda)/a_n$ and $F_n(X_i)-\lambda)/a_n$.

$$E\,L_n^2(\lambda)=\sum_{i=1}^8 EJ_{nj}^2(\lambda)+2\sum_{i<j}^8 EJ_{ni}(\lambda)J_{nj}(\lambda) \tag{4.3}$$

$$=E\,J_{n1}^2(\lambda)+E\,J_{n2}^2(\lambda)+2\,E\,J_{n1}(\lambda)J_{n2}(\lambda)+\eta_n,$$

where $\eta_n=\sum_{i=3}^8 EJ_{nj}^2(\lambda)+2\sum_{i<j=3}^8 EJ_{ni}(\lambda)J_{nj}(\lambda)$ can be shown, to be of $o(a_n/n)$.

$$EJ_{n1}^2(\lambda)=n^{-2}[nE(X_1-q(\lambda))^2k_n^2(F(X_1),\lambda)I_{A_n}(X_1)$$

$$+n(n-1)E(X_1-q(\lambda)k_n(F(X_1),\lambda)(X_2-q(\lambda))k_n((F\,X_2),\lambda)$$

$$\{1-I_{A_n^c}(X_1)\}-I_{A_n^c}(X_2)+I_{A_n^c}(X_1)\}I_{A_n^c}(X_2)\}]$$

$$=I_{n1}-I_{n2}+I_{n3} \tag{4.4}$$

$$I_{n1}=n^{-1}\int(u-q(\lambda))^2\,k_n^2\,(F(u),\lambda)\,dF(u)$$

$$=n^{-1}\int a_n^{-2}[q(\lambda+a_n\,s)-q(\lambda)]^2\,k^2\,(s)a_nds$$

$$=(n\,a_n)^{-1}\int[a_n\,s\,q^{(1)}(\lambda)+2^{-1}a_n^2\,s^2q^{(2)}(\lambda)+\ldots]^2\,k^2\,(s)ds$$

$$=a_n\,n^{-1}q^{(1)}(\lambda)^2\int s^2\,k^2(s)\,ds+o(a_n^3/n)$$

$$=: a_n n^{-1} c_1(\lambda) + o(a_n / n) \tag{4.5}$$

$$I_{n2} = \iint (u_1 - q(\lambda)) k_n (F(u_1), \lambda)(u_2 - q(\lambda)) k_n (F(u_2), \lambda) dF(u_1) dF(u_2)$$

$$= a_n^{-2} \left[\int (q(\lambda + a_n s) - q(\lambda))\, k(s) a_n ds \right]^2$$

$$= 4^{-1} a_n^4 \mu_2^2(K) q^{(2)}(\lambda)^2 + o(a_n^4)$$

$$=: a_n^4 c_2^2(\lambda) + o(a_n^4) \tag{4.6}$$

$$I_{n3} \le E[| X_1 - q(\lambda)|| X_2 - q(\lambda)| k_n (F(X_1), \lambda) k_n (F(X_2), \lambda)]$$

$$\times [\tilde{P}\{| F_n(X_1) - \lambda| > a_n\} + \tilde{P}\{| F_n(X_2) - \lambda| > a_n\}$$

$$+ \tilde{P}\{| F_n(X_1) - \lambda| > a_n\}^{1/2} \tilde{P}\{| F_n(X_2) - \lambda| > a_n\}^{1/2}],$$

where, by Proposition 2.2

$$\tilde{P}\{| F_n(X_i) - \lambda| > a_n\}$$

$$= \tilde{P}[| F_n(X_1) - F(X_1)| > a_n - | F(X_1) - \lambda|]$$

$$= \tilde{P}\left[\, | F_{n-1}(X_1) - F(X_1)\,| > \frac{n a_n}{n-1} - \frac{n}{n-1} | F(X_1) - \lambda| - \frac{1}{n-1}(1 - F(X_1))\right]$$

$$= \tilde{P}[\sup | F_{n-1}(x) - F(x)| > a_n \varepsilon_n],$$

where

$\varepsilon_n = (n/n-1)(1 - |F(X_1) - \lambda)/a_n| - (1 - F(X_1)/n-1)a_n$, so that the *rhs* is dominated by

$$\exp[-c_1 n a_n^2 \varepsilon_n^2] = \exp[-c_1 n a_n^2 (1 - |s_1| - c_2 \psi_n)^2], \quad \psi_n \downarrow 0 \text{ 0 as } n \to \infty$$

and

$$| I_{n3} | \le c_n^{-2} \int (q(\lambda + a_n s_1) - q(\lambda)) k(s_1) a_n ds_1$$

$$\times \int (q(\lambda + a_n s_2) - q(\lambda)) k(s_2) a_n ds_2$$

$$\times \exp[-c_1 n a_n^2 (1 - |s_1| - c_2 \psi_n)^2]$$

$$\le c a_n^4 \left[\int_{1-\delta_n}^{1} + \int_0^{1-\delta_n} \right] k(s) \exp\{-c_1 n a_n^2 (1-s-a_n^{3/2})^2\} ds,$$

where $\delta_n = a_n^{1/4}$ and by Taylor's expansion of $k(s)$ at $s = 1$, we have

$$a_n^4 \int_{-\delta_n}^{1} (s-1)^8 k^{(8)}(s^*) \exp\{-c_1 n a_n^2 (1-s-a_n^{3/2})\} ds$$

$$\le a_n^6 \int_0^{\delta} n \exp \int_0^{\delta} n \exp\{-c_1\, n\, a_n^2 (s - c_2\, a_n^{3/2})^2\} ds$$

$$\le a_n^6 / \sqrt{n\, a_n^2} = o(a_n / n) \tag{4.7}$$

and using $(1-s) \ge \delta_n$, the other integral can similarly be shown to be of $o(a_n / n)$ as $n \to \infty$.

Thus

$$E J_{n1}^2(\lambda) = a_n n^{-1} c_1(\lambda) + a_n^4 c_2^2(\lambda) + o(a_n / n), \tag{4.8}$$

where $c_1(\lambda) = q^{(1)}(\lambda)^2 \int s^2 k^2(s) ds$ and $c_2(\lambda)$ is as defined in (3.1). Again from (4.1),

$$E J_{n2}^2(\lambda) = (n a_n^3)^{-2} [n E (X_1 - q(\lambda) \lambda^2 B_n(X_1)^2 k^{(1)2}(F(X_1), \lambda) I_{An}(X_1)$$

$$+ n(n-1) E (X_1 - q(\lambda))(X_2 - q(\lambda)) k^{(1)}(F(X_1), \lambda) k^{(1)}(F(X_2), \lambda)$$

$$B_n(X_1) B_n(X_2) \{1 - I_{A_n^c}(X_1) - I_{A_n^c}(X_2) - I_{A_n^c}(X_1) I_{A_n^c}(X_2)\}]$$

$$= I_{n4} + I_{n5} + I_{n6}, \tag{4.9}$$

where

$$B_n(X_1) = F_n(X_1) - F(X_1) = n^{-1} \left[\sum_{j=2}^{n} \{I(X_j \le X_1) - F(X_1)\} + (1 - F(X_1)) \right].$$

Setting $\tilde{E}(.)$ as conditional expectation given X_1, $\tilde{E} B_n(X_1)^2 \le F(X_1)(1 - F(X_1)) / n \le 1/4n$,

$$|I_{n4}| \le n^{-1} a_n^{-6} \int |q(\lambda + a_n s_1) - q(\lambda)|^2 (k^{(1)2}(s)/4n) a_n ds$$

$$= o(a_n / n) \tag{4.10}$$

$$I_{n5} = a_n^{-4} E (X_1 - q(\lambda))(X_2 - q(\lambda)) k^{(1)}(F(X_1), \lambda) k^{(1)}(F(X_2), \lambda)$$

$$\times \tilde{\tilde{E}} B_n(X_1) B_n(X_2),$$

where

$$\widetilde{\widetilde{E}}\, B_n(X_1)B_n(X_2) = ((n-3)/n^2)[F(X_1 \wedge X_2) - F(X_1)F(X_1)] + O(n^{-2}).$$

$$I_{n5} = n^{-1}a_n^{-4}E(X_1 - q(\lambda))(X_2 - q(\lambda))k^{(1)}(F(X_1),\lambda)k^{(1)}(F(X_2),\lambda)$$
$$[F(X_1 \wedge X_2) - F(X_1)F(X_1)] + o(a_n / n)$$
$$= n^{-1}a_n^{-4}\iint [q(\lambda + a_n s_1) - q(\lambda)][q(\lambda + a_n s_2) - q(\lambda)]$$
$$\times k^{(1)}(s_1)k^{(1)}(s_2)[(\lambda + a_n s_1) \wedge (\lambda + a_n s_2)]$$
$$-(\lambda + a_n s_1)(\lambda + a_n s_2)a_n^2 ds_1 ds_2$$
$$= n^{-1}a_n^{-2}\iint [a_n s_1 q^{(1)}(\lambda) + \ldots][a_n s_2 q^{(1)}(\lambda) + \ldots]k^{(1)}(s_1)k^{(1)}(s_2)$$
$$\times [\lambda(1-\lambda) + a_n s_1 \wedge s_2 - \lambda a_n(s_1 + s_2) + a_n^2 s_1 s_2]ds_1\, ds_2$$
$$= n^{-1}q^{(1)}(\lambda)^2 \iint [s_1 k^{(1)}(s_1)s_2 k^{(1)}(s_2) + O(a_n^2)]$$
$$\times [\lambda(1-\lambda) + a_n s_1 \wedge s_2 - \lambda a_n(s_1 + s_2) + o(a_n^2)]ds_1\, ds_2$$
$$= n^{-1}q^{(1)}(\lambda)^2[\lambda(1-\lambda)a_n\psi(K) + o(a_n^2)].$$

I_{n6} can, as for I_{n3}, be similarly shown to be $o(a_n / n)$ so that

$$E\,J_{n2}^2(\lambda) = n^{-1}q^{(1)}(\lambda)^2[\lambda(1-\lambda) - a_n\psi(K) + o(a_n^2)] + o(a_n / n). \tag{4.11}$$

We now consider

$$E\,J_{n1}(\lambda)J_{n5}(\lambda) = (n^2 a_n^2)^{-1}[n\,E(X_1 - q(\lambda))^2 k_n(F(X_1),\lambda)k^{(1)}(F(X_1),\lambda)$$
$$\times \widetilde{E}\, B_n(X_1)I_{A_n}(X_1)$$
$$+ n(n-1)E(X_1 - q(\lambda))(X_2 - q(\lambda))k_n(F(X_1),\lambda)k^{(1)}(F(X_2),\lambda)$$
$$\times \widetilde{E}\, B_n(X_2)\{1 - I_{A_n^c}(X_1) - I_{A_n^c}(X_2) + I_{A_n^c}(X_1)I_{A_n^c}(X_2)\}]$$
$$= I_{n7} + I_{n8} + I_{n9} \tag{4.12}$$

Since $\widetilde{E}\, B_n(X_1) = (1 - F(X_1))/n$,

$$|I_{n7}| = n^{-2}a_n^{-3}\int (q(\lambda + a_n s_1) - q(\lambda))^2 k(s_1)k^{(1)}(s_1)(1 - \lambda - a_n s_1)a_n ds_1$$
$$= O(n^{-2}) = o(a_n / n). \tag{4.13}$$

$$|I_{n8}| = a_n^{-3} \int (q(\lambda + a_n s_1) - q(\lambda)) k(s_1) a_n ds_1$$

$$\times \int (q(\lambda + a_n s_2) - q(\lambda)) k^{(1)}(s_2)[(1 - \lambda - a_n s_2)/n] a_n ds_2$$

$$= a_n^{-1} \int [a_n s_1 q^{(1)}(\lambda) + 2^{-1} s_1^2 a_n^2 q^{(2)}(\lambda) + \ldots] k(s_1) a_n ds_1$$

$$\times [a_n s_1 q^{(1)}(\lambda) + \ldots] k^{(1)}(s_2)[(1 - \lambda - a_n s_2)/n] a_n ds_2$$

$$= 2^{-1} a_n q^{(2)}(\lambda)^2 \mu_2(K)\{-a_n q^{(1)}(\lambda)(1-\lambda)/n\} + o(a_n/n)$$

$$= -(1-\lambda)/2 a_n^2 q^{(2)}(\lambda) q^{(1)}(\lambda) \mu_2(K) + o(a_n/n)$$

$$= o(a_n/n). \quad (4.14)$$

Similarly it can be shown as for I_{n3}, that $|I_{n9}| = o(a_n/n)$, so that,

$$E J_{n1}(\lambda) J_{n2}(\lambda) = o(a_n/n) \quad (4.15)$$

Using Lemma 3.2, it is easy to establish from (4.6), (4.9) and (4.13).

$$E[L_n(\lambda)/t_n(\lambda)]^2 = E L_n^2(\lambda) + \varepsilon_{n2}$$

$$= \sigma_n^2 oq(\lambda) n^{-1} + a_n^4 c_2^2(\lambda) + o(a_n/n) + \varepsilon_{n2} \quad (4.16)$$

$$\varepsilon_{n2} = 2E L_n^2(\lambda) O(\log n/n a_n)^{1/2} + E L_n^2(\lambda) O(\log n/n a_n)$$

$$= O(a_n/n(\log h/n a_n^3)^{1/2}) = o(a_n/n)$$

which establishes the assertion of the theorem.

Optimal bandwidth: We now choose the bandwidth $\{a_n\}$ such that the mean square error of $\hat{q}_n(\lambda)$ is a minimum. Setting

$$M(a_n) = MSE(\hat{q}_n(\lambda)) =: n^{-1} d_1 - n^{-1} a_n d_2 + a_n^4 d_3,$$

where

$$d_1 = q^{(1)}(\lambda)^2 \lambda(1-\lambda) > 0, \quad d_2 = q^{(1)}(\lambda)^2 \psi(K) > 0 \text{ and}$$

$$d_3 = 4^{-1} q^{(2)}(\lambda)^2 \mu_2(K)^2 > 0.$$

Partially differentiating *w.r.t.* a_n and equating to zero, we obtain

$$\frac{\partial M(a_n)}{\partial a_n} = -n^{-1} d_2 + 4 a_n^3 d_3 = 0$$

$$a_{n,opt} = (d_2 / 4 d_3 n)^{1/3}$$

$$= [1/n(f^{(1)2} oq(\lambda) / f^4 oq(\lambda))(\psi(K)) / \mu_2^2(K))]^{1/3}$$

$$= [(\psi(K)) / \mu_2^2(K)) f^4 oq(\lambda) / (f^{(1)2} oq(\lambda)]^{1/3} = a_n' / \alpha\, oq(\lambda), \quad (4.17)$$

where

$a_n' = \{\psi(K) / n\mu_2(K)^2\}^{1/3}$ a completely known constant and

$\alpha(x) = \{f^{(1)}(x)^2 / f^4(x)\}^{1/3}$ involving parameters $f(x)$ and $f^{(1)}(x)$.

If f is unknown, the optimal bandwidth has to be estimated based on a preliminary random sample by

$$\hat{a}_{n,opt} = a_n / \alpha_n\, oq(\lambda) \quad (4.18)$$

$\alpha_n(x)$ being a suitable nonparametric estimator of $\alpha(x)$.

$$MSE(\hat{q}_n(\lambda))_{\min} = n^{-1} d_1 - (d_2^4 / 4 d_3)^{1/3} n^{-4/3} + (d_2 / 4 d_3)^{4/3} n^{-4/3} d_3$$

$$= n^{-1} d_1 - (d_2^4 / 4 d_3)^{1/3} [1 - 1/4] n^{-4/3}$$

$$= n^{-1} [d_1 - 3/4 (d_2^4 / 4 d_3)^{1/3} n^{1/3}]$$

$$= n^{-1} [\lambda(1-\lambda) / f^2 oq(\lambda) - 3/2 \{\psi(K)^4 / n(\mu_2(K) f^{(1)} oq(\lambda) f\, oq(\lambda))^2\}^{1/3}]$$

$$= [n f^{(2)}\, oq(\lambda)]^{-1} [\lambda(1-\lambda) - 3/\psi(K) a_n' / \alpha\, oq(\lambda) =: M_1 \text{ (say)} \quad (4.19)$$

and estimated $MSE(\hat{q}_n(\lambda)_{\min}$ is given by

$$M\hat{S}E(\hat{q}_n(\lambda)_{\min} = n^{-1} [\hat{d}_1 - 3/4 (\hat{d}_2^4 / 4 \hat{d}_3)^{1/3} n^{-1/3}]$$

$$= \hat{q}^{(1)}(\lambda)^2 n^{-1} [\lambda(1-\lambda) - 3/\psi(K) a_n' / \alpha_n\, oq(\lambda)],$$

where $\hat{q}(\lambda)$, $\hat{q}^{(1)}(\lambda)$ and $\alpha_n(.)$ are suitable estimators of $q(\lambda)$, $q^{(1)}(\lambda)$ and $\alpha(.)$ respectively.

Comparisons: The other competing estimators of $q(\lambda)$, as mentioned in introduction, are $q_n(\lambda)$, $\bar{q}_n(\lambda)$ and $\tilde{q}_n(\lambda)$, where Mean Square errors are given as follows:

i) $q_n(\lambda) = \inf\{x : F_n(x) \geq \lambda\}$, the unsmoothed nonparametric estimator of $q(\lambda)$ whose *a.s.* representation was thoroughly studied by Bahadur (1966):

$$q_n(\lambda) - q(\lambda) = q^{(1)}(\lambda)[\lambda - F_n oq(\lambda)] + O(\log\log n / n)^{3/4} \quad a.s.$$

$$M_3 = MSE(q_n(\lambda)) = \text{bias}^2 + \text{Var}\, q_n(\lambda)$$

$$= q^{(1)}(\lambda)^2 \lambda(1-\lambda)/n + o(\log\log n / n)^{3/2}.$$

ii) $\overline{q}_n(\lambda)$ is the smoothed nonparametric kernel estimator of $q(\lambda)$ given by $\widetilde{q}_n(\lambda) = \inf\{x : \hat{F}_n(x) \geq \lambda\}$, where F_n is given by $\hat{F}_n(x) = n^{-1}\sum K((x - X_i)a_n)$ and the Bahadur type representation as studied by Mack (1987) is given by

$$\overline{q}_n(\lambda) - q(\lambda) = q^{(1)}(\lambda)(\lambda - \hat{F}_n oq(\lambda)) + o(\log\log n / n)^{3/4}$$

with $\hat{F}_n(x) = \int_{-\infty}^{\infty} K((x-u)/a_n)dF(u) = \int F(x - a_n t)dK(t)$

$$= F(x) + c_2(\lambda)a_n^2 + O(a_n^4)$$

$$E[\hat{F}_n(x) - F(x)]^2 = E\left[n^{-1}\left\{\sum K((x - X_i)/a_n) - F(x)\right\}\right]^2$$

$$= n^{-1}\int \{K((x-u)/a_n) - F(x)\}^2 dF(u)$$

$$+ E^2\{K((x - X_i)/a_n) - F(x)\}$$

$$= n^{-1}\left[\int K^2((x-u)/a_n)dF(u) - 2F(x)\int K((x-u)/a_n)dF(u)\right.$$

$$\left. + F(x)\}^2\right] + [a_n^4 c_2^2(\lambda)]$$

$$= n^{-1}[F(x)(1-F(x)) - f(x)a_n\psi(K) + o(a_n^2)]$$

$$+ a_n^4 c_2^2(\lambda) + o(a_n/n)$$

and

$$V_n^2(x) = \text{Var}(\hat{F}_n(x))$$

$$= n^{-1}[F(x)(1 - F(x) - f(x)a_n\psi(K)] + o(a_n/n)$$

$$MSE(\overline{q}_n(\lambda)) = \text{bias}^2 + \text{Variance}$$

$$= a_n^4 c_2^2(\lambda) + v_n^2\, oq(\lambda) q^{(1)}(\lambda)^2 + o(a_n / n),$$

where $c_2(\lambda)$ is as defined in (3.1).

iii) $\tilde{q}_n(\lambda)$ is the smoothed nonparametric kernel (Parzen (1979)) estimator of $q(\lambda)$ defined by (1.1b), i.e. $\tilde{q}_n(\lambda) = a_n^{-1} \int_0^1 F_n^{-1}(t) k((t-\lambda)/a_n) dt$. Sheather and Marron (1990) studied the asymptotics of $\tilde{q}_n(\lambda)$ and other estimators, made comparisons by means of their $MSEs$. It was shown that

$$MSE(\tilde{q}_n(\lambda)) = q^{(1)2}(\lambda) n^{-1}[\lambda(1-\lambda) - a_n \psi(K)]$$

$$+ a_n^4 c_2^2(\lambda) + o(a_n / n).$$

The comparison of the four estimators of $q(\lambda)$ are made in the following table in terms of their bias and variance components.

Table 4.1:

Estimator	Bias	Variance
$q(\lambda)$	$O(\log\log n / n)^{3/4}$	$\lambda(1-\lambda) / n f^2 oq(\lambda)$
$\overline{q}_n(\lambda)$	$2^{-1}(f^{(1)} oq(\lambda) / f\, oq(\lambda)$ $\mu_2(K) a_n^2 (1+0(1))$	$\dfrac{\lambda(1-\lambda) - a_n f\, oq(\lambda)\psi(K)}{n f^2 oq(\lambda)}(1+0(1))$
$\tilde{q}_n(\lambda)$	$2^{-1}(f^{(1)} oq(\lambda) / f^3\, oq(\lambda))^2$ $\mu_2(K) a_n^2 (1+0(1))$	$\dfrac{\lambda(1-\lambda) - a_n \psi(K)}{n f^2 oq(\lambda)}(1+0(1))$
$\hat{q}_n(\lambda)$	$2^{-1}(f^{(1)} oq(\lambda) / f^3\, oq(\lambda))$ $\mu_2(K) a_n^2 (1+0(1))$	$\dfrac{\lambda(1-\lambda) - a_n \psi(K)}{n f^2 oq(\lambda)}(1+0(1))$

where $\psi(K) = 2 \int s K(s) dK(s) > 0$.

Note that the unsmoothed inverse type estimator $q(\lambda)$ is very much inferior to all other biased kernel estimators given above when compared in terms of their variance component which is just negation of the conclusions made in Sheather and Marron (1990) based on higher order kernel. Further $\hat{q}_n(\lambda)$ and $\tilde{q}_n(\lambda)$ are asymptotically equal in performance. However as the simulation results appended the $\tilde{q}_n(\lambda)$ is definitely superior to $\bar{q}_n(\lambda)$ for small n.

Minimizing *MSEs*, $MSE(\bar{q}_n, a_n)$ and $MSE(\tilde{q}_n, a_n)$ of $\bar{q}_n(\lambda)$ and $\tilde{q}_n(\lambda)$ *w.r.t.* the bandwidth a_n involved, the optimum bandwidths are respectively shown to be

$$\bar{a}_{n,opt} = [f\, oq(\lambda)\psi(K)/n f^{(1)}\, oq(\lambda)^2 \mu_2^2(K)]^{1/3}$$

$$=: \bar{a}_n / \alpha_1\, oq(\lambda) \tag{4.20}$$

$$\tilde{a}_{n,opt} = [f^4\, oq(\lambda)\psi(K)/n f^{(1)}\, oq(\lambda)^2 \mu_2^2(K)]^{1/3}$$

$$=: \bar{a}_n / \alpha\, oq(\lambda),$$

where $\bar{a}_n = [\psi(K)/n\mu_2(K)^2]^{1/3}$, a known constant and $\alpha_1(x) = [f^{(1)}(x)^2 / f(x)]^{1/3}$ and then the attainable *MSEs* at these optimal bandwidths coincide, i.e.

$$MSE(\bar{q}_n, \bar{a}_{n,opt})_{\min} = MSE(\tilde{q}_n, \tilde{a}_{n,opt})_{\min} = M_2$$

$$M_2 = [n f^4\, oq(\lambda)]^{-1}[\lambda(1-\lambda) - 3/4\psi(K)\bar{a}_n / \alpha\, oq(\lambda) \tag{4.21}$$

$$M_1 = M_2 < MSE(\hat{q}_n(\lambda)) = M_3$$

i.e minimal *MSE* obtained with optimum bandwidth $\hat{a}_{n,opt}$ of $\hat{q}_n(\lambda)$ is the least when compared to the unsmoothed kernel estimators $\bar{q}_n(\lambda)$. Thus, since $\hat{q}_n(\lambda)$ is a linear zed estimator, computation wise $\hat{q}_n(\lambda)$ is preferred over $\bar{q}_n(\lambda)$ and $\tilde{q}_n(\lambda)$, studied in the literature . And further due to the empirical verification done in the following section 6. $\hat{q}_n(\lambda)$ is definitely superior over $\bar{q}_n(\lambda)$ and $\tilde{q}_n(\lambda)$, studied extensively in the literature so far.

5. ASYMPTOTIC NORMALITY

In this section, we establish the rates of convergence of the sampling distribution of $\hat{Q}_n(\lambda) = \sqrt{n}\,(q_n(\lambda) - q(\lambda))$ to the normal distribution. We need the following basic lemma 1 in order to prove the Berry – Essen's Theorems for different statistics in the sequel.

Basic Lemma 5.1: Assume that for $\{Z_n)$ and $\{Y_n)$ satisfying

(i) $\Pr(Z_n \le x) = \Phi(x) + p_n(x)\varphi(x) + O(\tau_{n3})$

(ii) $\Pr(|Y_n| > c_1 \tau_{n1}) \le c_2 \tau_{n2}$,

where $p_n(x)$ is a polynomial in x of order 2, Φ the *df* of $N(0,1)$ *rv* and τ_{n1}, τ_{n2}, $\tau_{n3} \downarrow 0$ as $n \to \infty$.

Then

$$\Pr(Z_n + Y_n \le x) = \Pr(Z_n \le x) + O(\tau_{n1} \vee \tau_{n2} \vee \tau_{n3}) \tag{5.1}$$

We now prove the main result of the present section in the following.

Theorem 5.1: Under the assumptions of Theorem 3.1

$$\Pr[\sqrt{n}\,(q_n(\lambda) - q(\lambda)) \le z\,\sigma_n\, oq(\lambda)] - \Phi(f\,oq(\lambda) z) \le c_b \tau_n, \tag{5.2}$$

where

$c_b = c\rho\sigma^{-3}$, $\sigma^2 = \lambda(1-\lambda)$, $\rho = E|I_{n1} - \lambda|^3$, $I_{n1} = I(F(X_1) \le \lambda)$, $\sigma_n^2\, oq(\lambda) = [\lambda(1-\lambda) - a_n \psi(K)]\, q^{(1)}(\lambda)^2$ and $\tau_n = (\log n / n a_n)^{1/2}$.

Proof: From Theorem 3.1

$$\bar{\upsilon}_n(\lambda)/n\,oq(\lambda) = t_n(\lambda)[-q^{(1)}(\lambda)((F_n\, oq(\lambda) - \lambda)/\sigma_n\, oq(\lambda))] + R_n$$

i.e. $\sqrt{n}\,\bar{\upsilon}_n(\lambda)/\sigma_n\, oq(\lambda) =: Z_n + R_n$, $\quad R_n = O\{\sqrt{n}(\tau_{n1} \vee a_n^2)\}$ *a.s.*

with $\tau_{n1} = (a_n \log n / n)^{1/2}$

Now using Lemma 5.1

$$\begin{aligned}&\Pr(\sqrt{n}\,\bar{\upsilon}_n(\lambda)/\sigma_n\, oq(\lambda) \le z) - \Phi(z\,f\,oq(\lambda))\\ &\quad = \Pr(Z \le z) - \Phi(z\,f\,oq(\lambda)) + O(\tau_n)\end{aligned}$$

and

$$\begin{aligned}&\left\|\Pr(\sqrt{n}\,\bar{\upsilon}_n(\lambda)/\sigma_n\, oq(\lambda) \le z) - \Phi(zf\,oq(\lambda))\right\|_\infty\\ &\quad \le \left\|\Pr(Z \le z) - \Phi(z\,f\,oq(\lambda))\right\| + O(\tau_n)\end{aligned} \tag{5.3}$$

Now using Berry Essen's theorem for $Z_n = n^{-1/2}\sum_1^n (I_i - \lambda)/\sigma_n\, oq(\lambda)\, f\, oq(\lambda)$

with

$$I_i = I(X_i \le q(\lambda))$$

$$\left\| \Pr(Z_n \le z) - \Phi(z f\ oq(\lambda)) \right\| \le c E |I_1 - \lambda|^3 / n^{1/2} \sigma_n^3\ oq(\lambda) = c_b n^{-1/2}. \quad (5.4)$$

Equation (5.3) can now be written as

$$\left\| \Pr(\sqrt{n}\, \bar{\upsilon}_n(\lambda) / \sigma_n\ oq(\lambda) \le z) - \Phi(zf\ oq(\lambda)) \right\| \le c_b n^{-1/2} + O(\tau_n). \quad (5.5)$$

Now

$$\Pr[\sqrt{n}\,(\hat{q}_n(\lambda) - q(\lambda)) / \sigma_n\ oq(\lambda) \le z]$$

$$= \Pr(\sqrt{n}\, \bar{\upsilon}_n(\lambda) / \sigma_n\ oq(\lambda) \le z t_n(\lambda)) \quad (5.6)$$

$$\left\| \Pr[\sqrt{n}\,(\hat{q}_n(\lambda) - q(\lambda)) / \sigma_n\ oq(\lambda) \le z] - \Phi(zf\ oq(\lambda)) \right\|_\infty$$

$$\le \left\| \Pr(\sqrt{n}\, \bar{\upsilon}_n(\lambda) / \sigma_n\ oq(\lambda) \le z t_n(\lambda))\, f\ oq(\lambda) - \Phi(z t_n(\lambda))\, f\ oq(\lambda) \right\|_\infty$$

$$+ \left\| \Phi(z t_n(\lambda)\, f\ oq(\lambda)) - \Phi(z f\ oq(\lambda)) \right\|_\infty$$

$$\le c_b n^{-1/2} + \sup_z |z \varphi(z)|\, E |t_n(\lambda) - 1|$$

Using Lemma 3.1, $\varphi(z) = (2\pi)^{-1/2} e^{-z^2/2}$, $\sup_z |z\varphi(z)| = (2\pi e)^{-1/2}$ (5.6), (5.7), we have

$$\left\| \Pr[\sqrt{n}\,(\hat{q}_n(\lambda) - q(\lambda)) / \sigma_n\ oq(\lambda) \le z] - \Phi(zf\ oq(\lambda)) \right\|_\infty \le c_b n^{-1/2} O(\tau_n)$$

So that

$$\left\| \Pr[\sqrt{n}\,(\hat{q}_n(\lambda) - q(\lambda)) / \sigma_n\ oq(\lambda) \le z] - \Phi(zf\ oq(\lambda)) \right\|_\infty \le c_b \tau_n.$$

Hence the result is established.

6. EMPIRICAL JUSTIFICATION FOR SUPERIORITY OF $\hat{q}_n$ OVER OTHER ESTIMATORS $\tilde{q}_n$ AND Q_N BASED ON SES

The following table provides the standard errors of empirical distributions of various quantile estimators $\hat{q}_n(\lambda)$, $\tilde{q}_n(\lambda)$ and $q_n(\lambda)$ of $q(\lambda)$ generated by simulation study consisting of drawing random samples of size $n = 30$, repeatedly $N = 1000$ times from (i) Standard Normal (ii) Exponential, $\lambda = 1$, (iii) Log Normal $(0,1)$.

Table 6.1:

λ	Normal (0,1)			Exponential (1)			Log normal (0,1)		
	SE $\hat{q}_n(\lambda)$	SE $\tilde{q}_n(\lambda)$	SE $q_n(\lambda)$	SE $\hat{q}_n(\lambda)$	SE $\tilde{q}_n(\lambda)$	SE $q_n(\lambda)$	SE $\hat{q}_n(\lambda)$	SE $\tilde{q}_n(\lambda)$	SE $q_n(\lambda)$
0.05	0.2741	7.5462	0.9022	0.0574	14.466	0.9769	0.780	44.028	2.1446
0.15	0.2432	3.9456	0.8336	0.0704	8.0067	0.9729	0.0952	24.939	2.6426
0.25	0.2138	1.9578	0.7717	0.0891	4.0336	1.0273	0.1126	14.008	2.4600
0.35	0.2029	0.8848	0.7131	0.1154	1.7482	1.0646	0.1405	5.2312	2.1167
0.45	0.2089	0.3778	0.6605	0.1494	0.6294	1.0010	0.1911	2.0868	2.2804
0.55	0.1956	0.3607	0.6107	0.1767	0.2422	1.0486	0.2476	0.5062	2.3905
0.65	0.2100	0.8768	0.5832	0.2133	0.2849	1.0385	0.3267	0.4377	2.6022
0.75	0.2180	1.9424	0.5455	0.2799	0.4769	1.0009	0.5261	0.8315	3.6472
0.85	0.2309	3.9244	0.5407	0.3539	0.9070	1.0670	0.8371	1.4863	3.4387
0.95	0.2594	7.3467	0.5022	0.4697	0.7643	1.1262	1.2681	2.5974	3.890

Table 4.2 Efficiency ratios of $SE\, q(\lambda)$ and $SE\, \tilde{q}_n(\lambda)$ to $SE\, \hat{q}_n(\lambda)$ over various λ – values, $(n = 30,\ N = 1000)$, $r_1 = SE\, q_n(\lambda) / SE\, \hat{q}_n(\lambda)$, $r_2 = SE\, \tilde{q}_n(\lambda) / SE\, \hat{q}_n(\lambda)$, $r_3 = SE\, \tilde{q}_n(\lambda) / SE\, q_n(\lambda)$.

Table 6.2:

λ	Normal (0,1)			Exponential (1)			Log normal (0,1)		
	r_1	r_2	r_3	r_1	r_2	r_3	r_1	r_2	r_3
0.05	3.3123	27.7048	0.1196	17.0144	251.952	0.0675	27.5037	564.5886	6.0487
0.15	3.4278	16.2241	0.2113	13.8289	113.8064	0.1215	27.7619	262.00	0.1060
0.25	3.6090	9.1581	0.3942	11.5279	45.7688	0.2547	21.8450	124.3933	0.1756
0.35	3.5141	4.3604	0.8059	9.2244	15.1484	0.6089	15.4289	37.3292	0.4143
0.45	3.1615	1.8083	1.7484	6.7014	4.2133	1.5904	11.9336	10.9204	1.0428
0.55	3.1776	1.8445	1.6929	5.9344	1.3705	4.3301	9.6563	2.0449	4.7220
0.65	2.7778	4.1757	0.6652	4.8689	1.3358	3.6448	7.9639	1.3395	5.9956
0.75	2.502	8.9095	0.2808	3.5755	1.7019	2.1009	6.9327	1.5805	5.3664
0.85	2.2419	16.9981	0.1378	3.0151	2.563	1.1764	4.1077	1.7755	2.3135
0.95	1.9354	28.3179	0.0684	2.3979	3.7564	0.6384	3.0674	1.9774	1.5514

Comments: Note that (i) $q_n(\lambda)$ performs better in non-normal F over normal F whenever $\lambda \in (0, 0.5)$ and is uniformly best for all λ when compared to $q_n(\lambda)$ and $\tilde{q}_n(\lambda)$ based on samples from any F.

ii) For Normal F, $q_n(\lambda)$ is better than $\tilde{q}_n(\lambda)$ for estimating median of F, whereas $\tilde{q}_n(\lambda)$ is better for estimating percentiles away from centre of F.

For non-normal F, $q_n(\lambda)$ is better than $\tilde{q}_n(\lambda)$ for estimating upper percentiles whereas $\tilde{q}_n(\lambda)$ is better for estimating lower percentiles of F.

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