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ON CHARACTERIZATION OF CONTINUOUS DISTRIBUTIONS CONDITIONED ON A PAIR OF NON-ADJACENT DUAL GENERALIZED ORDER STATISTICS

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ABSTRACT

A generalized family of continuous distributions have been characterized through conditional expectation of dual generalized order statistics, conditioned on a pair of non-adjacent dual generalized order statistics. Further, some of its important deductions are discussed.

1. INTRODUCTION

Kamps (1995) introduced the concept of the generalized order statistics (gos). Using the concept of gos, Burkschat *et al.* (2003) introduced the concept of the dual generalized order statistics $(dgos)$ as follows:

Let X be a continuous random variable with the distribution function (df) $F(x)$ and the probability density function (pdf) $f(x)$, $x \in (\alpha, \beta)$. Further, let $n \in N$, $n \ge 2$, $k \ge 1$, $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r \ge 1$, for all $r \in \{1, \cdots, n-1\}$. Then $X^*(r, n, \tilde{m}, k)$ $r = 1, 2, \cdots, n$ are called $dgos$ if their joint pdf is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i)\right)[F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

for $F^{-1}(1) > x_1 \ge x_2 \ge \cdots \ge x_n > F^{-1}(0)$.

Here we will assume two cases:

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$

Case II: $\gamma_i \ne \gamma_j$, $i \ne j$ for all $i, j \in (1, \cdots, n)$

For **Case I**, the pdf of $dgos$ $X^*(r, n, m, k)$ is given by (Burkschat *et al.*, 2003)

$$f_r(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} g_m^{r-1}(F(x)) f(x). \tag{1.2}$$

The joint density function of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$ is

$$f_{r,s}(x,y)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}[F(x)]$$

$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1} f(y), \quad \alpha<y<x<\beta \tag{1.3}$$

The joint pdf of $X^*(r,n,m,k)$, $X^*(j,n,m,k)$ and $X^*(s,n,m,k)$, $1\le r<j<s\le n$, can similarly be given as

$$f_{r,j,s}(x,t,y)=C_{r,j,s:n}[F(x)]^m g_m^{r-1}(F(x))\,[h_m(F(t))-h_m(F(x)]^{j-r-1}$$

$$\times[h_m(F(y))-h_m(F(t)]^{s-j-1}[F(t)]^m\,[F(y)]^{\gamma_s-1} f(x) f(t) f(y),$$

$$\alpha<y<t<x<\beta, \tag{1.4}$$

where

$$h_m(x)=\begin{cases}-\dfrac{1}{m+1}x^{m+1}, & m\ne -1\\ -\log x, & m=1\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(1),\ x\in[0,1).$$

Therefore conditional distribution of $X^*(j,n,m,k)$ given $X^*(r,n,m,k)=x$ and $X^*(s,n,m,k)=y$ is given by

$$f_{j|r,s}(t\,|\,x,y)=\frac{(s-r-1)!\,(m+1)}{(j-r-1)!(s-j-1)!}$$

$$\frac{[\{F(x)\}^{m+1}-\{F(t)\}^{m+1}]^{j-r-1}\,[\{F(t)\}^{m+1}-\{F(y)\}^{m+1}]^{s-j-1}}{[\{F(x)\}^{m+1}-\{F(y)\}^{m+1}]^{s-r-1}}[F(t)]^m f(t),$$

$$\alpha<y<t<x<\beta. \tag{1.5}$$

For **Case II,** the pdf of $dgos$ $X^*(r,n,\tilde{m},k)$ is given by (Burkschat *et al.*, 2003)

$$f_r(x)=C_{r-1}f(x)\sum_{i=1}^{r}a_i(r)[F(x)]^{\gamma_i-1} \tag{1.6}$$

The joint density function of $X^*(r,n,\tilde{m},k)$ and $X^*(s,n,\tilde{m},k)$ is

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = C_{s-1} f(x) \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\frac{F(y)}{F(x)}\right]^{\gamma_i}$$

$$\times \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i} \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} \qquad (1.7)$$

The joint *pdf* of $X^*(r,n,\tilde{m},k)$, $X^*(j,n,\tilde{m},k)$ and $X^*(s,n,\tilde{m},k)$, $1 \le r < j < s \le n$, may similarly be given as

$$f_{r,\,j,s}(x,t,y) = C_{s-1}\left(\sum_{i=1}^{r} a_i\,(r)\,\,(F(x))^{\gamma_i}\right)\left(\sum_{i=r+1}^{j} a_i^{(r)}(j)\left[\frac{F(t)}{F(x)}\right]^{\gamma_i}\right)$$

$$\times\left(\sum_{i=j+1}^{s} a_i^{(j)}(s)\left[\frac{F(y)}{F(t)}\right]^{\gamma_i}\right)\frac{f(x)}{F(x)}\frac{f(t)}{F(t)}\frac{f(y)}{F(y)},\ \alpha < y < t < x < \beta. \qquad (1.8)$$

Hence the conditional *pdf* of $X^*(j,n,\tilde{m},k)$ given $X^*(r,n,\tilde{m},k) = x$ and $X^*(s,n,\tilde{m},k) = y$, $1 \le r < j < s \le n$, is given by

$$f_{j|r,s}(t|x,y) = \frac{\left[\sum_{i=r+1}^{j} a_i^{(r)}(j)\left\{\frac{F(t)}{F(x)}\right\}^{\gamma_i}\right]\left[\sum_{i=j+1}^{s} a_i^{(j)}(s)\left\{\frac{F(y)}{F(t)}\right\}^{\gamma_i}\right]\frac{f(t)}{F(t)}}{\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left\{\frac{F(y)}{F(x)}\right\}^{\gamma_i}\right]},$$

$$(1.9)$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i\,, \quad \gamma_i = k + n - i + M_i \qquad (1.10)$$

$$a_i(r) = \prod_{\substack{j=1 \\ j \ne i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \quad \gamma_i \ne \gamma_j, \quad 1 \le i \le r \le n \qquad (1.11)$$

and

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \ne i}}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \quad r+1 \le i \le s \le n \qquad (1.12)$$

Khan *et al.* (2009) have characterized family of continuous distributions when conditioned on a non-adjacent single *dgos*. We, in this paper, have extended the result of Khan *et al.* (2009) when conditioned on a pair of *dgos*.

2. CHARACTERIZATION OF DISTRIBUTIONS

The result will first be proved for $\gamma_j \neq \gamma_i$ and then it will be deduced to the case when $m_i = m_j = m$, $i, j = 1, \ldots, n-1$.

Theorem 2.1: Let $X^*(i, n, \tilde{m}, k)$, $i = 1, \ldots, n$ be the *dgos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support (α, β), and $h(t)$ be a monotonic and differentiable function of t. If for two consecutive values r and $r+1$, $2 \le r+1 < j < s \le n$

$$g_{j|l,s}(x, y) = E[h(X^*(j, n, \tilde{m}, k)) | X^*(l, n, \tilde{m}, k) = x,\ X^*(s, n, \tilde{m}, k) = y],$$

$$l = r, r+1 \tag{2.1}$$

exist and $g(x, y)$ is a finite and differentiable function of x, then

$$\gamma_{r+1} \frac{f(x)}{F(x)} + \frac{\frac{\partial}{\partial x} \mathrm{B}_r^s(x, y)}{\mathrm{B}_r^s(x, y)} = \frac{\frac{\partial}{\partial x} g_{j\,|\,r,s}(x, y)}{[g_{j\,|\,r+1,s}(x, y) - g_{j\,|\,r,s}(x, y)]} \tag{2.2}$$

and

$$\frac{[F(x)]^{\gamma_{r+1}} \mathrm{B}_r^s(x, y)}{\mathrm{B}_r^s(\beta, y)} = \exp\left[-\int_x^\beta A_1(t, y)\, dt\right], \tag{2.3}$$

where

$$\mathrm{B}_r^s(x, y) = \left[\sum_{i=r+1}^{s} a_i^{(r)}(s) \left\{\frac{F(y)}{F(x)}\right\}^{\gamma_i}\right], \tag{2.4}$$

and

$$A_1(x, y) = \frac{\frac{\partial}{\partial x} g_{j\,|\,r,s}(x, y)}{[g_{j\,|\,r+1,s}(x, y) - g_{j\,|\,r,s}(x, y)]}. \tag{2.5}$$

Proof: We have, in view of (1.9) and (2.1)

$$g_{j\,|\,r,s}(x, y)\ \mathrm{B}_r^s(x, y) = \int_y^x h(t)\ \mathrm{B}_r^j(x, t)\ \mathrm{B}_j^s(t, y)\ \frac{f(t)}{F(t)} dt \tag{2.6}$$

Differentiate both the sides *w.r.t.* x, to get

$$\frac{\partial}{\partial x} g_{j\,|\,r,s}(x, y) \mathrm{B}_r^s(x, y) + g_{j\,|\,r,s}(x, y)\left[\frac{\partial}{\partial x} \mathrm{B}_r^s(x, y)\right] = \int_y^x h(t)\left[\frac{\partial}{\partial x} \mathrm{B}_r^j(x, t)\right][\mathrm{B}_j^s(t, y)]\,\frac{f(t)}{F(t)} dt \tag{2.7}$$

after noting that $\mathrm{B}_r^s(x,x)=0$, as $\sum_{i=r+1}^{s} a_i^{(r)}(s)=0$ (Khan *et al.*, 2006).

Since $a_i^{(r+1)}(s)=(\gamma_{r+1}-\gamma_i)\,a_i^{(r)}(s)$, $i=r+2,\ldots,s$.

Hence,

$$\mathrm{B}_{r+1}^s(x,y)=\left[\sum_{i=r+2}^{s} a_i^{(r+1)}(s)\left\{\frac{F(y)}{F(x)}\right\}^{\gamma_i}\right]=\gamma_{r+1}\,B_r^s(x,y)-\frac{F(x)}{f(x)}\left[\frac{\partial}{\partial x}B_r^s(x,y)\right] \tag{2.8}$$

Thus (2.7) reduces to

$$\frac{\partial}{\partial x}g_{j\,|\,r,s}(x,y)\,\mathrm{B}_r^s(x,y)=[g_{j\,|\,r+1,s}(x,y)-g_{j\,|\,r,s}(x,y)]\,\mathrm{B}_{r+1}^s(x,y)\,\frac{f(x)}{F(x)}$$

or,

$$\frac{f(x)}{F(x)}\,\frac{\mathrm{B}_{r+1}^s(x,y)}{\mathrm{B}_r^s(x,y)}=\frac{\frac{\partial}{\partial x}g_{j\,|\,r,s}(x,y)}{[g_{j\,|\,r+1,s}(x,y)-g_{j\,|\,r,s}(x,y)]}=A_1(x,y) \tag{2.9}$$

implying that

$$\gamma_{r+1}\frac{f(x)}{F(x)}+\frac{\frac{\partial}{\partial x}\mathrm{B}_r^s(x,y)}{\mathrm{B}_r^s(x,y)}=\frac{\frac{\partial}{\partial x}g_{j\,|\,r,s}(x,y)}{[g_{j\,|\,r+1,s}(x,y)-g_{j\,|\,r,s}(x,y)]} \tag{2.10}$$

Integrating both the sides *w.r.t.* x over (x,β), we get (2.3).

Corollary 2.1: It may be noted that for $\gamma_i\neq\gamma_j$ and $m_1=\cdots=m_{n-1}=m\neq-1$.

$$a_i^{(r)}(s)=\frac{1}{\prod\limits_{\substack{j=r+1\\ i\neq j}}^{s}(\gamma_j-\gamma_i)}=(-1)^{s-i}\,\frac{1}{(m+1)^{s-r-1}}\,\frac{1}{(i-r-1)!\,(s-i)!} \tag{2.11}$$

Thus,

$$\frac{[F(x)]^{\gamma_{r+1}}\,\mathrm{B}_r^s(x,y)}{\mathrm{B}_r^s(\beta,y)}$$

$$=\frac{[F(x)]^{\gamma_{r+1}}\left(\frac{F(y)}{F(x)}\right)^{\gamma_s}\frac{1}{(m+1)^{s-r-1}\,(s-r-1)!}\sum_{i=r+1}^{s}(-1)^{s-i}\frac{(s-r-1)!}{(i-r-1)!(s-i)!}\left(\frac{F(y)}{F(x)}\right)^{\gamma_i-\gamma_s}}{[F(y)]^{\gamma_s}\frac{1}{(m+1)^{s-r-1}\,(s-r-1)!}\sum_{i=r+1}^{s}(-1)^{s-i}\frac{(s-r-1)!}{(i-r-1)!(s-i)!}F(y)^{\gamma_i-\gamma_s}}$$

$$= \frac{[F(x)^{m+1}]^{(s-r-1)}\left[1-\dfrac{F(y)^{m+1}}{F(x)^{m+1}}\right]^{(s-r-1)}}{[1-F(y)^{m+1}]^{(s-r-1)}}$$

implying that

$$\frac{1-\{F(x)\}^{m+1}}{1-\{F(y)\}^{m+1}} = 1-\exp\left[-\frac{1}{(s-r-1)}\int_x^\beta A_1(t,y)\,dt\right],\ m>-1 \tag{2.12}$$

and

$$\frac{\log F(x)}{\log F(y)} = 1-\exp\left[-\frac{1}{(s-r-1)}\int_x^\beta A_1(t,y)\,dt\right],\quad m=-1 \tag{2.13}$$

as $\frac{\partial}{\partial m}\{F(x)\}^{m+1} = \{F(x)\}^{m+1}\log F(x)$, which tends to $\log F(x)$ as $m \to -1$.

Remark 2.1: In the limiting case as $y \to \alpha$, at $\gamma_s = 0$, Theorem 2.1 reduces to

$$F(x) = \exp\left(-\frac{1}{\gamma_{r+1}}\int_x^\beta \frac{g'_{j\,|\,r}(t)}{[g_{j\,|\,r+1}(t)-g_{j\,|\,r}(t)]}\,dt\right),$$

where

$$g_{j\,|\,r}(x) = E[h(X(j,n,\tilde{m},k)) \mid X(r,n,\tilde{m},k)=x]$$

as given by Khan *et al.* (2009). This is true for both the cases $\gamma_j \neq \gamma_i$ and $m_i = m_j = m > -1$.

Theorem 2.2: Let $X^*(i,\,n,\,\tilde{m},\,k)$, $i=1,\ldots,n$ be the $dgos$ from a continuous population with the df $F(x)$ and the pdf $f(x)$ over the support (α,β) and $h(t)$ be a monotonic and differentiable function of t. If for two consecutive values $s-1$ and s, $1 \le r < j < s-1 < n$,

$$g_{j|r,l}(x,y) = E[h(X^*(j,n,\tilde{m},k)) \mid X^*(r,n,\tilde{m},k)=x,\ X^*(s,n,\tilde{m},k)=y],$$

$$l = s-1, s$$

exist, then

$$\gamma_s\frac{f(y)}{F(y)} - \frac{\frac{\partial}{\partial y}B_r^s(x,y)}{B_r^s(x,y)} = \frac{\frac{\partial}{\partial y}g_{j\,|\,r,s}(x,y)}{[g_{j\,|\,r,s}(x,y)-g_{j\,|\,r,s-1}(x,y)]} \tag{2.14}$$

and

$$\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i-\gamma_s} = a_s^{(r)}(s)\ \exp\left[-\int_{\alpha}^{y} A_2(x,t)\,dt\right], \tag{2.15}$$

where $\mathrm{B}_r^s(x,y)$ is as defined in (2.4),

and

$$A_2(x,y) = \frac{\frac{\partial}{\partial y} g_{j\,|\,r,s}(x,y)}{[g_{j\,|\,r,s}(x,y) - g_{j\,|\,r,s-1}(x,y)]}\,. \tag{2.16}$$

Proof: Differentiating both the sides of (2.6) *w.r.t.* y and proceeding as in the Theorem 2.1, we get

$$\frac{f(y)}{F(y)}\ \frac{\mathrm{B}_r^{s-1}(x,y)}{\mathrm{B}_r^{s}(x,y)} = \frac{\frac{\partial}{\partial y} g_{j\,|\,r,s}(x,y)}{[g_{j\,|\,r,s}(x,y) - g_{j\,|\,r,s-1}(x,y)]}, \tag{2.17}$$

where $B_r^{s-1}(x,y) = \left(\gamma_s\,\mathrm{B}_r^s(x,y) - \frac{F(y)}{f(y)}\frac{\partial}{\partial y}\mathrm{B}_r^s(x,y)\right)$

as $a_i^{(r)}(s-1) = (\gamma_s - \gamma_i)\ a_i^{(r)}(s)$.

That is,

$$\gamma_s\ \frac{f(y)}{F(y)} - \frac{\frac{\partial}{\partial y}\mathrm{B}_r^s(x,y)}{\mathrm{B}_r^s(x,y)} = \frac{\frac{\partial}{\partial y} g_{j\,|\,r,\,s}(x,y)}{[g_{j\,|\,r,\,s}(x,y) - g_{j\,|\,r,\,s-1}(x,y)]}\,.$$

Therefore,

$$\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i-\gamma_s} = a_s^{(r)}(s)\ \exp\left[-\int_{\alpha}^{y} A_2(x,t)\,dt\right]$$

and hence the result.

Corollary 2.2: It may be noted that at $\gamma_i \neq \gamma_j$ but $m_1 = \cdots = m_{n-1} = m > -1$.

$$\frac{\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i-\gamma_s}}{a_s^{(r)}(s)} = \left[1 - \frac{F(y)^{m+1}}{F(x)^{m+1}}\right]^{s-r-1}$$

implying that

$$\frac{\{F(y)\}^{m+1}}{\{F(x)\}^{m+1}} = 1 - \exp\left(-\frac{1}{(s-r-1)} \int_{\alpha}^{y} A_2(x,t)\,dt \right), \qquad m > -1. \tag{2.18}$$

Remark 2.2: In the limiting case as $x \to \beta$, at $r = 0$, Theorem 2.2 reduces to

$$\sum_{i=1}^{s} a_i(s)\,[F(y)]^{\gamma_i - \gamma_s} = a_s(s)\,\exp\left[-\int_{\alpha}^{y} \frac{g'_{j\,|\,s}(t)}{[g_{j\,|\,s}(t) - g_{j\,|\,s-1}(t)]}\,dt \right],$$

where $g_{j\,|\,s}(y) = E[h(X(j,n,\tilde{m},k)) \mid X(s,n,\tilde{m},k) = y]$, for $\gamma_i \neq \gamma_j$ and for $m_1 = \cdots = m_{n-1} = m > -1$

$$[F(y)]^{m+1} = 1 - \exp\left[-\frac{1}{(s-1)} \int_{\alpha}^{y} \frac{g'_{j\,|\,s}(t)}{[g_{j\,|\,s}(t) - g_{j\,|\,s-1}(t)]}\,dt \right]$$

as given by Khan *et al.* (2009).

3. EXAMPLES

For adjacent gos at $j = r+1$, $s = r+2$ and $m_{r+1} > -1$, it can be seen that (1.11) reduces to

$$f_{r+1|r,r+2}(t \mid x, y) = \frac{(m_{r+1}+1)\,\{F(t)\}^{m_{r+1}}\,f(t)}{[\{F(x)\}^{m_{r+1}+1} - \{F(y)\}^{m_{r+1}+1}]}, \tag{3.1}$$

and therefore, corresponding to (2.12) and (2.18), we have respectively

$$\frac{1 - \{F(x)\}^{m_{r+1}+1}}{1 - \{F(y)\}^{m_{r+1}+1}} = 1 - e^{-I_1} \tag{3.2}$$

and

$$\frac{\{F(y)\}^{m_{r+1}+1}}{\{F(x)\}^{m_{r+1}+1}} = 1 - e^{-I_2}, \tag{3.3}$$

where

$$I_1 = \int_{x}^{\beta} A_1(t, y)\,dt, \quad I_2 = \int_{\alpha}^{y} A_2(x, t)\,dt.$$

Thus we have,

$$g_{r|r,s}(x,y) = E[h(X^*(r,n,\tilde{m},k)) \mid X^*(r,n,\tilde{m},k) = x,\; X^*(s,n,\tilde{m},k) = y] = h(x)$$

$$g_{s|r,s}(x,y) = E[h(X^*(s,n,\tilde{m},k)) \mid X^*(r,n,\tilde{m},k) = x,\; X^*(s,n,\tilde{m},k) = y] = h(y)$$

and

$$g_{r+1|r,r+2}(x,y)$$

$$= E[h(X^*(r+1,n,\tilde{m},k)) \,|\, X^*(r,n,\tilde{m},k) = x,\ X^*(r+2,n,\tilde{m},k) = y] = g(x,y)\,. \tag{3.4}$$

Therefore,

$$A_1(x,y) = \frac{\frac{\partial}{\partial x} g(x,y)}{[h(x) - g(x,y)]} \tag{3.5}$$

and

$$A_2(x,y) = \frac{\frac{\partial}{\partial y} g(x,y)}{[g(x,y) - h(y)]} \tag{3.6}$$

i) $$g(x,y) = \frac{c}{a(c+1)} \frac{[ah(y)+b]^{c+1} - [ah(x)+b]^{c+1}}{[ah(y)+b]^c - [ah(x)+b]^c} - \frac{b}{a},\ c \neq -1 \tag{3.7}$$

if and only if

$$1 - \{F(x)\}^{m_{r+1}+1} = [ah(x)+b]^c\,, \tag{3.8}$$

where a, b, c and $h(x)$ are so chosen that $F(x)$ is a df.

Proof: To prove (3.8) implies (3.7), we have

$$g(x,y) = (m_{r+1}+1)\int_y^x \frac{h(t)[F(t)]^{m_{r+1}} f(t)}{[1-\{F(y)\}^{m_{r+1}+1}] - [1-\{F(x)\}^{m_{r+1}+1}]}\,dt$$

$$= \frac{ac}{B(x,y)} \int_y^x [ah(t)+b]^{c-1} h(t)\, h'(t)\; dt\,,$$

where

$$B(x,y) = [1 - \{F(y)\}^{m_{r+1}+1} - [1 - \{F(x)\}^{m_{r+1}+1}]]$$

$$= [a\,h(y)+b]^c - [a\,h(x)+b]^c$$

$$= g(x,y) = \frac{c}{B(x,y)} \int_{ah(y)+b}^{ah(x)+b} u^{c-1} \left(\frac{u-b}{a}\right) du$$

implying that

$$g(x,y)=\frac{c}{a(c+1)}\,\frac{[ah(y)+b]^{c+1}-[ah(x)+b]^{c+1}}{[ah(y)+b]^{c}-[ah(x)+b]^{c}}-\frac{b}{a}.$$

Now to prove (3.7) implies (3.8), we have

$$A_1(x,y)=-\frac{ac\,h'(x)\,[ah(x)+b]^{c-1}}{[ah(x)+b]^{c}-[ah(y)+b]^{c}}.$$

Integrating both sides w.r.to x

$$-\int_x^{\beta}A_1(t,y)\,dt=\log\left[1-\frac{\{ah(x)+b\}^{c}}{\{ah(y)+b\}^{c}}\right].$$

Therefore in view of (3.2)

$$\frac{\{ah(x)+b\}^{c}}{\{ah(y)+b\}^{c}}=\frac{1-\{F(x)\}^{m_{r+1}+1}}{1-\{F(y)\}^{m_{r+1}+1}}$$

implying that

$$1-\{F(x)\}^{m_{r+1}+1}=K[ah(x)+b]^{c},$$

where K is a normalizing constant.

But $F(\alpha)=0$. Thus,

$$1-\{F(x)\}^{m_{r+1}+1}=[ah(x)+b]^{c}$$

and hence the result.

ii) $$g(x,y)=\frac{c}{(c-1)}\,\frac{[[h(x)]^{c-1}-[h(y)]^{c-1}]\;h(x)\,h(y)}{[h(x)]^{c}-[h(y)]^{c}},\quad c\neq 1 \tag{3.9}$$

if and only if

$$1-\{F(x)\}^{m_{r+1}+1}=a[h(x)]^{-c}+b, \tag{3.10}$$

where a, b, c and $h(x)$ are so chosen that $F(x)$ is a df.

Proof: For $1-\{F(x)\}^{m_{r+1}+1}=a[h(x)]^{-c}+b$, it is easy to show that

$$g(x,y)=\frac{c}{(c-1)}\,\frac{[[h(x)]^{c-1}-[h(y)]^{c-1}]\;h(x)\,h(y)}{[h(x)]^{c}-[h(y)]^{c}}.$$

Now to prove (3.9) implies (3.10), we have

$$A_1(x,y)=\frac{c\,h'(x)\,\{h(y)\}^c}{h(x)\,[\{h(x)\}^c-\{h(y)\}^c\,]}\,,\qquad I_1=-\log\left(1-\frac{\{a\,(h(x))^{-c}+b\}}{\{a\,(h(y))^{-c}+b\}}\right)$$

and hence the result.

Remark 3.1: For $1-\{F(x)\}^{m_{r+1}+1}=a[h(x)]^{-c}+b$, we have

a) At $c=-1$, $g\,(x,y)=\dfrac{h(x)+h(y)}{2}=A.M.$ (3.11)

b) At $c=2$, $g\,(x,y)=\dfrac{1}{\frac{1}{2}\left(\frac{1}{h(x)}+\frac{1}{h(y)}\right)}=H.M.$ (3.12)

c) At $c=\dfrac{1}{2}$, $g\,(x,y)=\sqrt{h(x)\,h(y)}=G.M.$ (3.13)

iii) $g(x,y)=\dfrac{c}{a\,(c+1)}\;\dfrac{[ah(x)+b]^{c+1}-[ah(y)+b]^{c+1}}{[ah(x)+b]^c-[ah(y)+b]^c}-\dfrac{b}{a}\,,\quad c\neq-1$ (3.14)

if and only if

$$\{F(x)\}^{m_{r+1}+1}=[ah(x)+b]^c\,, \tag{3.15}$$

where a, b, c and $h(x)$ are so chosen that $F(x)$ is a df.

Proof: Proceeding as in example (i), we get

$$g\,(x,y)=\frac{c}{a\,(c+1)}\;\frac{[ah(x)+b]^{c+1}-[ah(y)+b]^{c+1}}{[ah(x)+b]^c-[ah(y)+b]^c}-\frac{b}{a}\,,\quad c\neq-1.$$

Now to prove (3.14) implies (3.15), we have

$$A_2\,(x,y)=\frac{ac\,h'(y)\,\,[ah(y)+b]^{c-1}}{[ah(x)+b]^c-[ah(y)+b]^c}\,.$$

Integrating both the sides *w.r.t* y

$$-\int_{\alpha}^{y}A_2\,(x,t)\,dt=\log\left[1-\frac{\{ah(y)+b\}^c}{\{ah(x)+b\}^c}\right].$$

In view of (3.3), we have

$$\frac{\{F(y)\}^{m_{r+1}+1}}{\{F(x)\}^{m_{r+1}+1}}=\frac{\{ah(y)+b\}^c}{\{ah(x)+b\}^c}$$

implying that

$$\{F(x)\}^{m_{r+1}+1} = K\ [ah(x)+b]^c$$

and hence the result.

iv) For $m_1 = \cdots = m_{n-1} = m > -1$

$$g_{j\,|\,r,\,s}(x,y) = \frac{(s-j)\,h(x) + (j-r)\,h(y)}{(s-r)} \tag{3.16}$$

if and only if

$$1-\{F(x)\}^{m+1} = ah(x)+b\,, \quad m > -1, \tag{3.17}$$

where a, b, c and $h(x)$ are so chosen that $F(x)$ is a df.

Proof: It is easy to prove (3.17) implies (3.16). To see that (3.16) implies (3.17), we have

$$A_1(x,y) = \frac{h'(x)}{h(x)-h(y)}, \qquad I_1 = -\log\left(1-\frac{\{ah(x)+b\}}{\{ah(y)+b\}}\right)$$

$$A_2(x,y) = \frac{h'(y)}{h(x)-h(y)}, \qquad I_2 = -\log\left(1-\frac{1-\{ah(y)+b\}}{1-\{ah(x)+b\}}\right).$$

Also from (3.2),

$$\frac{1-\{F(x)\}^{m+1}}{1-\{F(y)\}^{m+1}} = 1-e^{-I_1} = \frac{[ah(x)+b]}{[ah(y)+b]}, \quad m > -1$$

and from (3.3)

$$\frac{\{F(y)\}^{m+1}}{\{F(x)\}^{m+1}} = 1-e^{-I_2} = \frac{1-[ah(y)+b]}{1-[ah(x)+b]}, \quad m > -1,$$

implying that

$$1-\{F(x)\}^{m+1} = ah(x)+b\,, \quad m > -1.$$

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