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RESISTANT IRREDUCIBLE *BIB* DESIGNS

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ABSTRACT

Resistance of variance balancedness property of irreducible *BIB* designs (both symmetric as well asymmetric) is studied. Also, result on 'Resistant *SBIB* designs' by Hedayat and John (1974) is reviewed.

1. INTRODUCTION

Hedayat and John (1974) introduced resistant *BIB* designs. Shah and Gujarathi (1977) have generalized one of the results by Hedayat and John (1974). Doshi and Gujarathi (2005) provided mathematical proof of the result by Shah and Gujarathi (1977) and studied the unnoticed strength of their result. In the present paper, we study resistance of variance balancedness property for the class of irreducible *BIB* designs. We divide this study into the following two categories: Symmetric irreducible *BIB* designs and Asymmetric irreducible *BIB* designs. We compare our results with that of Hedayat and John (1974) related to variance balancedness property of symmetric BIB designs and gives our additional inferences with illustrations. For precise appreciation, we discuss below, in brief, the related concepts and notations.

Let D be a *BIB* (v,b,r,k,λ) design on a set Ω of v treatments. Let the v treatments be denoted by $\Omega=\{t_1,t_2,\cdots,t_n,t_{n+1},\cdots,t_v\}$.

When a subset consisting of n $(n<v)$ treatments is removed form D, the remaining design is denoted by $\overline{D}$. As per Hedayat and John (1974), Most (1975) and Raghavarao (1971) we define the following:

Definition 1.1: D is said to be locally resistant of degree n $(LR(n))$ if $\overline{D}$ is variance balance (VB) when some subsets of treatments of cardinality n is deleted.

Definition 1.2: D is said to be globally resistant of degree n $(GR(n))$ if $\overline{D}$ is VB when any subset L of treatments of cardinality n is deleted.

Definition 1.3: D is said to be fully locally resistant of degree n $(FLR(n))$ if it is LR to all subset of $\{x_1,x_2,\cdots,x_n\}$ of size less than or equal to n, $n<v$.

Definition 1.4: D is said to be fully globally resistant of degree n $(FGR(n))$ if D is FLR for all subsets of Ω of cardinality less than or equal to n.

Let $x \in \Omega$. Upon deletion of x from D, the remaining structure $\overline{D}$ can be partitioned as follows:

$D_{\overline{X}}$: Subdesign of D with blocks not containing treatment x,

D_X : Subdesign of D with blocks containing treatment x,

D'_X : Subdesign obtained from D_X upon deleting treatment x from each block of D_X.

Evidently $\overline{D} = D'_X \cup D_{\overline{X}}$.

Further, when D is resistant with respect to x, the parameters of sub designs D'_X and $D_{\overline{X}}$ are as follows:

D'_X : $(v-1, r, \lambda, k-1, \lambda_1)$, $\lambda_1 = \lambda(k-2)/(v-2)$.

$D_{\overline{X}}$: $(v-1, b-r, r-\lambda, k, \lambda_2)$, $\lambda_2 = (r-\lambda)(k-1)/(v-2)$.

Definition 1.5: An irreducible block design for v treatments and blocks of size k $(k < v)$ is obtained by taking as blocks, all possible combinations of k out of v treatments. The resultant block design is a *BIBD* having parameters

$$\left[v, \binom{v}{k}, \binom{v-1}{k-1}, k, \binom{v-2}{k-2}\right].$$

In the literature this design is also known as an unreduced *BIBD* or a trivial *BIB* design. For given v and k, these designs are always available.

Evidently, the parameters of an irreducible *SBIB* (v, k, λ) design are given by

$(v, v-1, v-2)$ i.e. $r = k = v-1$ and $\lambda = v-2$.

2. MAIN RESULTS

We prove below the resistance of variance balanceness property for symmetric irreducible *BIB* designs and asymmetric irreducible *BIB* designs. The main results are as under.

Theorem 2.1: Every symmetric irreducible *BIB* $(v, v-1, v-2)$ design is fully globally resistant of degree $k-2 = v-3$.

Proof: Let the set of v treatments of an irreducible *SBIB* $(v, v-1, v-2)$ design D be denoted by Ω. Consider any subset L of Ω of cardinality p.

Let $x \in L$. Upon deleting a single treatment x from D evidently we get,

$D'_X = (v-1, v-2, v-3)$: an irreducible *SBIBD*

and

$D_{\overline{X}}$ = a complete block of $v-1$ treatments.

Further upon deleting second treatment $y \in L$, we get:

- From D'_X : an irreducible *SBIB* $(v-2, v-3, v-4)$ design and a complete block containing all remaining $v-2$ treatments,
- From $D_{\overline{X}}$: a complete block of $v-2$ treatments.

Thus upon deleting two treatments we get,

i) an irreducible *SBIB* $(v-2, v-3, v-4)$ design and

ii) two complete blocks of size $v-2$.

Continuing deletion of p treatments of L one after another, we get,

i) an irreducible *SBIB* $(v-p, v-p-1, v-p-1)$ design and

ii) p complete blocks of size $v-p$.

Since $v-p-1 \geq 2$ or $v-p-2 \geq 1$ we get,

$p \leq v-3 = k-2$.

Thus every symmetric irreducible *BIB* $(v, v-1, v-2)$ design is fully globally resistant of degree $k-2$. This proves the theorem.

Theorem 5.5 of Hedayat and John (1974) is as under.

"Every *SBIBD* is locally resistant of degree k (but not necessarily for less than k)".

However, for an irreducible *SBIB* designs (subclass of *SBIB* design), deletion of any block leads us to the global resistance of degree k.

The result of Hedayat and John (1974) for class of *SBIB* designs is improved as follows:

Corollary 2.1: Irreducible *SBIB* designs are globally resistant of degree k.

Further, by the result proved above in Theorem 2.1 we get,

Corollary 2.2: Every irreducible *SBIB* design $(v, v-1, v-2)$ is *FGR* $(k-2 = v-3)$.

Now consider a class of asymmetric irreducible *BIB* designs.

Theorem 2.2: Every asymmetric irreducible *BIB* (v, b, r, k, λ) design is fully globally resistant of degree $k-2$.

Proof: Consider an irreducible *BIB* design $D\left[v,\binom{v}{k},\binom{v-1}{k-1},k,\binom{v-2}{k-2}\right]$.

Let Ω denote the set of v treatments of D.

Now, upon deletion of a treatment $x\in\Omega$, we get

$$D'_X \ : \ \left[v-1,\binom{v-1}{k-1},\binom{v-2}{k-2},k-1,\lambda_1\right]$$

which is evidently an irreducible *BIB* design.

$$D_{\overline{X}} \ : \ \left[v-1,\binom{v}{k}-\binom{v-1}{k-1},\binom{v-1}{k-1}-\binom{v-2}{k-2},k,\lambda_2\right].$$

Evidently,

$$\binom{v}{k}-\binom{v-1}{k-1}=\binom{v-1}{k} \text{ and } \binom{v-1}{k-1}-\binom{v-2}{k-2}=\binom{v-2}{k-1}.$$

Thus $D_{\overline{X}}$ is an irreducible *BIBD*.

Further,

$$(v-1)=\binom{v-1}{k} \text{ for } k=v-2.$$

Hence $D_{\overline{X}}$ is a symmetric irreducible *BIBD*.

Thus upon deletion of a treatment from an asymmetric irreducible *BIBD*, we get

i) D'_X as an asymmetric irreducible *BIBD* and

ii) $D_{\overline{X}}$ as a symmetric irreducible *BIBD*.

Upon deletion of one more treatment form D, we get

- From D'_X :

  i) an asymmetric irreducible *BIB* $\left[v-2,\binom{v-2}{k-2},\binom{v-3}{k-3},k-2,\lambda_3\right]$ design and

  ii) symmetric irreducible *BIB* $(v-2,v-3,v-4)$ design.

- From $D_{\overline{X}}$:

  i) symmetric irreducible *BIB* $(v-2,v-3,v-4)$ design and

  ii) a complete block of size $v-2$.

Thus upon deletion of two treatments from D, we get

- an asymmetric irreducible *BIBD* $\left[v-2, \binom{v-2}{k-2}, \binom{v-3}{k-3}, k-2, \lambda_3\right]$
- two symmetric irreducible *BIBD* s $(v-2, v-3, v-4)$ and
- a complete block with $v-2$ treatments.

Continuing deletion of p treatments of L one after another, we get

i) an asymmetric irreducible *BIB* $\left[v-p, \binom{v-p}{k-p}, \binom{v-p-1}{k-p-1}, k-p, \lambda_p\right]$ design.

ii) p symmetric irreducible *BIB* $(v-p, v-p-1, v-p-2)$ designs and

iii) $p-1$ complete blocks of size $v-p$.

Evidently, $k-p \geq 2$ and hence $p \leq k-2$.

Thus every asymmetric irreducible *BIBD* is fully globally resistant of degree $k-2$.

This completes the proof.

Combining results proved above in Theorem 2.1 and Theorem 2.2, we say that

Theorem 2.3: Every irreducible *BIBD* is $FGR(k-2)$.

3. CONCLUSION

By above discussed results, the Theorem 5.5 of Hedayat and John (1974) can be rectified and improved as under:

i) Every *SBIBD* (v, k, λ) except an irreducible *SBIBD* $(v, v-1, v-2)$ is locally resistant of degree k (but not necessarily for less than k).

ii) Every irreducible *SBIBD* $(v, v-1, v-2)$ is $GR(k)$.

iii) Every irreducible *SBIBD* $(v, v-1, v-2)$ is $FGR(k-2=v-3)$.

4. ILLUSTRATIONS

i) Let $D(6,5,4)$ be a symmetric irreducible *BIB* design (Design-1). Upon deletion of any treatment (say A) from D we get a symmetric irreducible *BIB* $(5,4,3)$ and a complete block of size 5 (Design-2). After one by one deletion of any three treatments (say A, B, C) from D we get a symmetric irreducible *BIB* $(3,2,1)$ design and three complete blocks of size 3 (Design-3). Thus given symmetric irreducible *BIB* design is fully globally resistant of degree three.

Design: 1

A	*B*	*C*	*D*	*E*
A	*B*	*C*	*D*	*F*
A	*B*	*C*	*E*	*F*
A	*B*	*D*	*E*	*F*
A	*C*	*D*	*E*	*F*
B	*C*	*D*	*E*	*F*

Design: 2

B	*C*	*D*	*E*
B	*C*	*D*	*F*
B	*C*	*E*	*F*
B	*D*	*E*	*F*
C	*D*	*E*	*F*

B	*C*	*D*	*E*	*F*

Design: 3

D	*E*
D	*F*

E	*F*

D	*E*	*F*
D	*E*	*F*
D	*E*	*F*

ii) Let $D(7,21,15,5,10)$ be an asymmetric irreducible *BIB* design (Design-1). Upon deletion of a treatment form D we get an asymmetric irreducible $BIB(6,15,10,4,6)$ design and a symmetric irreducible $BIB(6,5,4)$ design (Design-2). If we delete one more treatment form D we get an asymmetric irreducible $BIB(5,10,6,3,3)$ design, two symmetric irreducible $BIB(5,4,3)$ designs and a complete block of size 5 (Design-3). After deletion of any three treatments in the same manner from D we get an asymmetric irreducible $BIB(4,6,3,2,1)$ design, three symmetric irreducible $BIB(4,3,2)$ designs and three complete blocks of size 4 each (Design-4).

Design: 1

A	*B*	*C*	*D*	*E*
A	*B*	*C*	*D*	*F*
A	*B*	*C*	*D*	*G*
A	*B*	*C*	*E*	*F*
A	*B*	*C*	*E*	*G*
A	*B*	*C*	*F*	*G*
A	*B*	*D*	*F*	*G*
A	*B*	*E*	*F*	*G*
A	*C*	*D*	*E*	*F*
A	*C*	*D*	*E*	*G*
A	*C*	*D*	*F*	*G*
A	*D*	*E*	*F*	*G*
B	*C*	*D*	*E*	*F*
A	*C*	*E*	*F*	*G*
A	*D*	*E*	*F*	*G*
B	*C*	*D*	*E*	*F*
B	*C*	*D*	*E*	*G*
B	*C*	*D*	*F*	*G*
B	*C*	*E*	*F*	*G*
B	*D*	*E*	*F*	*G*
C	*D*	*E*	*F*	*G*

Design: 2

B	*C*	*D*	*E*
B	*C*	*D*	*F*
B	*C*	*D*	*G*
B	*C*	*E*	*F*
B	*C*	*E*	*G*
B	*C*	*F*	*G*
B	*D*	*E*	*F*
B	*D*	*E*	*G*
B	*D*	*F*	*G*
B	*E*	*F*	*G*
C	*D*	*E*	*F*
C	*D*	*E*	*G*
C	*D*	*F*	*G*
C	*E*	*F*	*G*
D	*E*	*F*	*G*

B	*C*	*D*	*E*	*F*
B	*C*	*D*	*E*	*G*
B	*C*	*E*	*F*	*G*
B	*C*	*E*	*F*	*G*
B	*D*	*E*	*F*	*G*
C	*D*	*E*	*F*	*G*

Design: 3

C	*D*	*E*
C	*D*	*F*
C	*D*	*G*
C	*E*	*F*
C	*E*	*G*
C	*F*	*G*
D	*E*	*F*
D	*E*	*G*
D	*F*	*G*
E	*F*	*G*

C	*D*	*E*	*F*
C	*D*	*E*	*G*
C	*D*	*F*	*G*
C	*E*	*F*	*G*
D	*E*	*F*	*G*

C	*D*	*E*	*F*
C	*D*	*E*	*G*
C	*D*	*F*	*G*
C	*E*	*F*	*G*
D	*E*	*F*	*G*

C	*D*	*E*	*F*	*G*

Design: 4

D	*E*
D	*F*
D	*G*
E	*F*
E	*G*
F	*G*

D	*E*	*F*
D	*E*	*G*
D	*F*	*G*
E	*F*	*G*

D	*E*	*F*
D	*E*	*G*
D	*F*	*G*
E	*F*	*G*

D	*E*	*F*	*G*

D	*E*	*F*
D	*E*	*G*
D	*F*	*G*
E	*F*	*G*

D	*E*	*F*	*G*
D	*E*	*F*	*G*

Thus the given asymmetric irreducible *BIB* design D is fully globally resistant of degree three.

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