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CHARACTERIZATIONS OF CERTAIN CONTINUOUS UNIVARIATE DISTRIBUTIONS BASED ON TRUNCATED MOMENT OF THE FIRST ORDER STATISTIC

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ABSTRACT

Some well-known univariate continuous distributions are characterized based on a truncated moment of the 1st order statistic.

1. INTRODUCTION

The problem of characterizing a distribution is an important problem which has attracted the attention of many researchers in recent years. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions one of which is in terms of the truncated moments. The characterization results presented here will be in terms of a truncated moment of the 1st order statistic.

Let $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ be the order statistics of a random sample of size n from a continuous cumulative distribution function (cdf) $F(x)$. In our previous work Ahsanullah and Hamedani (2007), we characterized power function and beta of first kind distributions based on a truncated moment of $X_{1,n}$ or of $X_{n,n}$ extending some known characterizations of the power function and uniform distributions [see Ahsanullah (1973, 89), Wesolowski and Ahsanullah (2004), Hamedani and Volkmer (2005) and Oncel *et al.*(2005)]. In the present work we will establish characterizations of certain continuous univariate distributions based on a truncated moment of $X_{1,n}$.

2. CHARACTERIZATIONS

We will take up the distributions in alphabetical order rather than that of their practical or theoretical importance. All the characterizations presented here, except that of Generalized Beta 1 are based on the following general theorem.

Theorem 2.1: Let $X:\Omega \to (a,\infty)$, $a \ge 0$, be a continuous random variable with *cdf* F such that $\lim_{x\to\infty} x^{\delta}(1-F(x))^n = 0$, for some $\delta > 0$. Let $g(x,\delta,n)$ be a real-valued function which is differentiable with respect to x and $\int_a^\infty \frac{\delta x^{\delta-1}}{g(x,\delta,n)}dx = \infty$. Then

$$E[X_{1,n}^{\delta} \mid X_{1,n} > t] = t^{\delta} + g(t,\delta,n), \quad t > a \tag{2.1}$$

implies that

$$F(t) = 1 - \left(\frac{g(a,\delta,n)}{g(t,\delta,n)}\right)^{1/n} e^{-\int_a^t \frac{\delta x^{\delta-1}}{n\,g(x,\delta,n)}dx}, \quad t \ge a. \tag{2.2}$$

Proof: Condition (2.1) and the assumption $\lim_{x\to\infty} x^{\delta}(1-F(x))^n = 0$ imply

$$\int_t^{\infty} \delta x^{\delta-1}(1-F(x))^n dx = g(t,\delta,n)\,(1-F(t))^n \tag{2.3}$$

Differentiating (2.3) with respect to t, we obtain

$$-\delta t^{\delta-1}(1-F(t))^n = \left[\frac{\partial}{\partial t} g(t,\delta,n)(1-F(t)) - ng(t,\delta,n)f(t)\right](1-F(t))^{n-1},$$

from which

$$\frac{f(t)}{1-F(t)} = \frac{\frac{\partial}{\partial t} g(t,\delta,n)}{n\,g(t,\delta,n)} + \frac{\delta t^{\delta-1}}{n\,g(t,\delta,n)} \tag{2.4}$$

Integrating (2.4) with respect to t from a to x, results in

$$F(x) = 1 - \left(\frac{g(a,\delta,n)}{g(x,\delta,n)}\right)^{1/n} e^{-\int_a^x \frac{\delta t^{\delta-1}}{n\,g(t,\delta,n)}dt}, \quad x \ge a.$$

Remark 2.1: In Theorem 2.1 the interval (a, ∞) may be taken to be closed.

i) Burr Type XII Distribution

The probability density function (pdf) for this distribution is given by

$$f(x;\beta) = \beta(1+x)^{-(\beta+1)}, \quad x > 0, \tag{2.5}$$

where $\beta > 0$ is a parameter.

Proposition 2.1: Let $X : \Omega \to \Re^{+}$ be a continuous random variable with *cdf* F such that $\lim_{x\to\infty} x(1-F(x))^n = 0$. Then X has *pdf* (2.5) for $n\beta > 1$ if and only if

$$E[X_{1,n} \mid X_{1,n} > t] = \frac{n\beta t + 1}{n\beta - 1} \quad \text{for } t > 0 \tag{2.6}$$

Proof: If X has *pdf* (2.5) with $n\beta > 1$, then it can easily be shown that (2.6) holds.

Now, assume (2.6) holds, then

$$g\,(t,\delta,n)=\frac{1+t}{n\beta-1}\,,\qquad \frac{\partial}{\partial t}g\,(t,\delta,n)=\frac{1}{n\beta-1}\,,\text{ and by Theorem 2.1}$$

$$F\,(x)=1-\left(\frac{1}{1+x}\right)^{1/n}e^{-\int_0^x \frac{1}{n\left(\frac{1+t}{n\beta-1}\right)}dt}=1-(1+x)^{-\beta}\,,\qquad x\geq 0$$

and

$$f\,(x)=\beta\,(1+x)^{-(\beta+1)}\,,\quad x>0\,.$$

ii) Generalized Beta 1 Distribution

The *pdf* for this family is given by

$$f\,(x;\alpha,\beta,\gamma)=\alpha\,\gamma\,\beta^{-\alpha}x^{\alpha-1}\left(1-\left(\frac{x}{\beta}\right)^{\alpha}\right)^{\gamma-1},\qquad 0\leq x\leq\beta\,, \tag{2.7}$$

where $\alpha>0$, $\beta>0$, and $\gamma>0$ are parameters.

Proposition 2.2: Let $X:\Omega\to[0,\beta]$ be a continuous random variable with *cdf* F. The random variable X has *pdf* (2.7) if and only if

$$E[X_{1,n}^{\alpha}\mid X_{1,n}>t]=\frac{n\gamma t^{\alpha}+\beta^{\alpha}}{n\gamma+1}\,,\qquad 0<t<\beta\,. \tag{2.8}$$

Proof: If X has *pdf* (2.7), then (2.8) holds.

Now, if (2.8) holds, then

$$\int_t^{\beta}\alpha\,x^{\alpha-1}(1-F(x))^n\,dx=\frac{\beta^{\alpha}}{n\gamma+1}\left(1-\left(\frac{t}{\beta}\right)^{\alpha}\right)(1-F(t))^n \tag{2.9}$$

Differentiating (2.9) with respect to t, we obtain

$$-\alpha\,t^{\alpha-1}(1-F(t))^n=\frac{\beta^{\alpha}}{n\gamma+1}\left\{-\frac{\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1}(1-F(t))\right.$$

$$\left.-n\left(1-\left(\frac{t}{\beta}\right)^{\alpha}\right)f\,(t)\right\}(1-F\,(t))^{n-1}$$

and after some simplifications

$$\frac{f(t)}{1-F(t)} = \alpha\gamma\,\beta^{-\alpha} t^{\alpha-1}\left(1-\left(\frac{t}{\beta}\right)^{\alpha}\right)^{-1}.$$

For $0 \le x \le \beta$, we have

$$\int_0^x \frac{f(t)}{1-F(t)}dt = \int_0^x \alpha\gamma\,\beta^{-\alpha} t^{\alpha-1}\left(1-\left(\frac{t}{\beta}\right)^{\alpha}\right)^{-1} dt$$

Or $\;-\ln(1-F(x)) = -\ln\left(1-\left(\frac{x}{\beta}\right)^{\alpha}\right)^{\gamma}$

Or $\;F(x) = 1-\left(1-\left(\frac{x}{\beta}\right)^{a}\right)^{\gamma}.$

iii) Generalized Beta 2 Distribution

The *pdf* for this family is given by

$$f(x;\alpha,\beta,\gamma) = \alpha\gamma\,\beta^{-\alpha} x^{\alpha-1}\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{-(\gamma+1)}, \quad x > 0, \tag{2.10}$$

where $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ are parameters.

Proposition 2.3: Let $X : \Omega \to \Re^{+}$ be a continuous random variable with *cdf* F such that $\lim_{x\to\infty} x^{\alpha}(1-F(x))^{n} = 0$. The random variable X has *pdf* (2.10) for $n\gamma > 1$ if and only if

$$E[X_{1,n}^{\alpha} \mid X_{1,n} > t] = \frac{n\gamma\, t^{\alpha} + \beta^{\alpha}}{n\gamma - 1}, \qquad t > 0. \tag{2.11}$$

Proof: If X has *pdf* (2.10), then (2.11) holds.

Now, if (2.11) holds, then

$$g(t,\alpha,n) = \frac{\beta^{\alpha} + t^{\alpha}}{n\gamma - 1}, \quad \frac{\partial}{\partial t} g(t,\alpha,n) = \frac{\alpha t^{\alpha-1}}{n\gamma - 1},$$ and by Theorem 2.1

$$F(x) = 1-\left(\frac{\beta^{\alpha}}{\beta^{\alpha}+x^{\alpha}}\right)^{1/n} e^{-\int_0^x \frac{\alpha t^{\alpha-1}}{n\left(\frac{\beta^{\alpha}+t^{\alpha}}{n\gamma-1}\right)}dt} = 1-\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{-\gamma}, \; x \ge 0.$$

iv) Generalized Pareto Distribution

For this family the *pdf* is given by

$$f(x;\alpha,\beta)=\frac{\beta+1}{\alpha}\left(1+\frac{\beta}{\alpha}x\right)^{-\left(\frac{1}{\beta}+2\right)}, \qquad x\geq 0, \tag{2.12}$$

where $\alpha>0$ and $\beta>0$ are parameters.

Proposition 2.4: Let $X:\Omega\to[0,\infty)$ be a continuous random variable with *cdf* F such that $\lim_{x\to\infty} x(1-F(x))^n=0$. The random variable X has *pdf* (2.12) if and only if

$$E[X_{1,n}\mid X_{1,n}>t]=\frac{n(1+\beta)t+\alpha}{n(1+\beta)-\beta}, \quad t>0. \tag{2.13}$$

Proof: If X has *pdf* (2.12), then (2.13) holds.

Now, if (2.13) holds, then

$$g(t,\delta,n)=\frac{\alpha+\beta t}{n(1+\beta)-\beta}, \quad \frac{\partial}{\partial t}g(t,\delta,n)=\frac{\beta}{n(1+\beta)-\beta}, \text{ and by Theorem 2.1}$$

$$F(x)=1-\left(\frac{\alpha}{\alpha+\beta x}\right)^{1/n} e^{-\int_0^x \frac{1}{n\left(\frac{\alpha+\beta t}{n(1+\beta)-\beta}\right)}dt}$$

$$=1-\left(1+\frac{\beta}{\alpha}x\right)^{-\left(\frac{1}{\beta}+1\right)}, x\geq 0.$$

v) Pareto of First Kind Distribution

This family has *pdf* of the form

$$f(x,\alpha,\beta)=\alpha\,\beta^{\alpha}\,x^{-(\alpha+1)}, \quad x\geq\beta, \tag{2.14}$$

where $\alpha>0$ and $\beta>0$ are parameters.

Proposition 2.5: Let $X:\Omega\to[\beta,\infty)$ be a continuous random variable with *cdf* F such that $\lim_{x\to\infty} x(1-F(x))^n=0$. The random variable X has *pdf* (2.14) for $n\alpha>1$ if and only if

$$E[X_{1,n}\mid X_{1,n}>t]=\frac{n\alpha\, t}{n\alpha-1}, \qquad t>\beta. \tag{2.15}$$

Proof: If X has *pdf* (2.14), then (2.15) holds.

Now, if (2.15) holds, then

$g(t,\delta,n)=\dfrac{t}{n\alpha-1}$, $\dfrac{\partial}{\partial t}g(t,\delta,n)=\dfrac{1}{n\alpha-1}$, and by Theorem 2.1

$$F(x)=1-\left(\frac{\beta}{x}\right)^{1/n} e^{-\int_\beta^x \frac{1}{n\left(\frac{t}{n\alpha-1}\right)}dt}=1-\left(\frac{x}{\beta}\right)^{-\alpha}, \quad x\ge\beta .$$

Remarks 2.2: (a) Pareto of the second kind has *pdf*

$$f(x;\alpha,\beta)=\alpha\beta^{\alpha}(x+\beta)^{-(\alpha+1)}, \quad x\ge 0, \tag{216}$$

where $\alpha>0$ and $\beta>0$ are parameters.

Note that if X has *pdf* (2.16), then $Y=X+\beta$ has *pdf* (2.14). So, similar characterization can easily be established for a random variable with *pdf* (2.16).

(b) If X has *pdf* (2.14), then $Y=X^{-1}$ has *pdf*

$$f(y;\alpha,\beta)=\alpha\beta^{\alpha}y^{\alpha-1}, \qquad 0<y<\beta^{-1}, \tag{2.17}$$

which is *pdf* of power function (or Pearson Type I) distribution. The latter distribution was discussed in Ahsanullah and Hamedani (2007).

vi) Weibull Distribution

The *pdf* for this family is given by

$$f(x,\alpha,\beta)=\alpha\beta\, x^{\beta-1}e^{-\alpha x^{\beta}}, \quad x>0, \tag{2.18}$$

where $\alpha>0$ and $\beta>0$ are parameters.

Proposition 2.6: Let $X:\Omega\to\Re^{+}$ be a continuous random variable with *cdf* F such that $\lim_{x\to\infty} x^{\beta}(1-F(x))^{n}=0$. Then X has *pdf* (2.18) if and only if

$$E[X_{1,n}^{\beta}\mid X_{1,n}>t]=\frac{n\alpha t^{\beta}+1}{n\alpha}, \quad t>0. \tag{2.19}$$

Proof: If X has *pdf* (2.18), then (2.19) holds.

Now, if (2.19) holds, then

$g(t,\beta,n)=\dfrac{1}{n\alpha}$, $\dfrac{\partial}{\partial t}g(t,\beta,n)=0$, and by Theorem 2.1

$$F(x)=1-e^{-\int_0^x \frac{\beta t^{\beta-1}}{n\left(\frac{1}{n\alpha}\right)}dt}=1-e^{-\alpha x^{\beta}}, \quad x\ge 0 .$$

Remarks 2.3: (a) The following distributions are special cases of Weibull distribution:

Burr Type X; Chi-Square; Extreme Value (Gumbel) Type 2; Gamma; Rayleigh.

(b) If X has Extreme Value (Gumbel) Type 2 distribution, then $Y = -X$ has similar characterization.

(c) Similar characterizations can be established for truncated exponential, truncated Gamma, and truncated Pareto distributions.

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