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ON PARTIALLY BALANCED TERNARY DESIGNS

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ABSTRACT

The paper consists of three new construction-methods of Partially Balanced Ternary (PBT) design, (i) using the association matrices of Bose and Mesner (1959), (ii) starting from two Partially Balanced Incomplete Block $(PBIB)$ designs on same association scheme having same number of treatments and different block-sizes such that the block-size of one of the designs is twice that of another and (iii) using a Nested Partially Balanced Incomplete Block $(NPBIB)$ design. It also gives some new PBT designs whose existence appears to have been unknown until now.

1. INTRODUCTION

The concept of Balanced $n-$ary designs was initiated, generalizing the concept of Balanced Binary designs, by Tocher (1952). Those Balanced $n-$ary designs are known as Balanced Ternary (BT) design, when $n=3$. The concept of BT design was extended to that of Partially Balanced Ternary (PBT) design by Paik and Federer (1973) and Mehta *et al.* (1975). Paik and Federer (1973) established that Partially Balanced $n-$ary Block designs with binary number association schemes are $n-$ary balanced factorial designs.

Some construction-methods of balanced $n-$ary designs, in particular, balanced ternary designs, are available in Murty and Das (1967), Das and Rao (1968), Saha and Dey (1973), Nigam (1974), Surendran and Sunny (1979). Starting from the incidence matrices of affine $\alpha-$resolvable Balanced Incomplete Block (BIB) designs and group divisible (GD) designs, a class of balanced ternary designs has been obtained by Dey (1970), through $PBIB$ (GD) designs by Saha (1975) and also through a class of $PBIB$ designs by Tyagi and Rizwi (1979).

Many solutions of PBT design, led by a single construction-method under different cases, are scattered in the literatures of Mehta *et al.* (1975), Sinha and Saha (1979) and others. Some new construction-methods of these designs will be proposed later on. Also a few new BT designs, not in the list of known BT designs with $R \le 20$, Saha (1975) are presented later on.

2. DEFINITION

A PBT design with $m-$class associate scheme is a block design having V treatments in B blocks of each size K such that,

a) the $V \times B$ incidence matrix, N, has three entries $0,1,2$,

b) the row sum of N is R, the row sum of squares is Δ,

c) the column sum of N is K and

d) the inner product of any two rows of N is π_i $(i=1,2,\cdots,m)$ if the treatments corresponding to the rows are mutually $i-$th associates.

Thus, it is obvious that $NN' = \Delta B_0 + \sum_{\alpha=1}^{m} \pi_\alpha B_\alpha$, where $B_0 = I, B_1, B_2, \cdots, B_m$ are the association matrices of the association scheme. To the readers, Dey (1986) is referred to for the definitions of Nested Partially Balanced Incomplete Block ($NPBIB$) design and Nested Balanced Incomplete Block ($NBIB$) design and also Raghavarao (1971) for the other definitions.

3. CONSTRUCTION

Consider an $m-$class association scheme along with its incidence matrices, $B_0, B_1, \cdots, B_m$. Taking $N = (B_i + 2B_j)$, $1 \le i \ne j \le m$, as incidence matrix of a design, a series of PBT designs given in the theorem below, can be constructed.

Theorem 3.1: Starting from an $m-$class association scheme with the parameters V, n_α, $p^s_{\alpha\beta}$; $\alpha, \beta, s = 1,2,\cdots,m \ge 3$, a PBT design with the following parameters, can be constructed,

$$V = v = B, \quad R = n_i + 2n_j = K, \quad \Delta = n_i + 4n_j, \quad \pi_s = p^s_{ii} + 4p^s_{ij} + 4p^s_{jj},$$
$$n_\alpha, \quad p^s_{\alpha\beta}.$$

Proof: Clearly,

$$NN' = (B_i + 2B_j)(B_i + 2B_j)' = \sum_{l=0}^{m} B_l \, (p^l_{ii} + 4p^l_{ij} + 4p^l_{jj}).$$

Then $\Delta = p^0_{ii} + 4p^0_{ij} + 4p^0_{jj} = n_i + 4n_j$, since every treatment is $0-$th associate to itself and $\pi_s = p^s_{ii} + 4p^s_{ij} + 4p^s_{jj}$. The other parameters can be obtained with elementary calculations.

An illustration of Theorem 3.1 is given below, with help of an example.

Example 3.1: Consider 3×2 rectangular association scheme, introduced by Vartak (1955):

$$\begin{matrix} \theta_1 & \theta_4 \\ \theta_2 & \theta_5 \\ \theta_3 & \theta_6 \end{matrix}$$

Then

$$B_1 = \begin{bmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0&1&1&0&0&0\\ 1&0&1&0&0&0\\ 1&1&0&0&0&0\\ 0&0&0&0&1&1\\ 0&0&0&1&0&1\\ 0&0&0&1&1&0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0&0&0&0&1&1\\ 0&0&0&1&0&1\\ 0&0&0&1&1&0\\ 0&1&1&0&0&0\\ 1&0&1&0&0&0\\ 1&1&0&0&0&0 \end{bmatrix}$$

Letting $N=(B_1+2B_2)$, a solution of the PBT design with the following parameters, is ready,

$$V=6=B\,,\; R=5=K\,,\; \Delta=9\,,\; \pi_1=0\,,\; \pi_2=4\,,\pi_3=4\,,\; n_1=1\,,\; n_2=2\,,$$

$$n_3=2\,,\; P_1=\begin{bmatrix} 0&0&0\\ 0&0&2\\ 0&2&0 \end{bmatrix}, \quad P_2=\begin{bmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{bmatrix}, \quad P_3=\begin{bmatrix} 0&1&0\\ 1&0&1\\ 0&1&0 \end{bmatrix}.$$

Remark 3.1: With substitution of 2 by p; a positive integer greater than 2 and B_i by B_0, Theorem 3.1 leads to the design given by Sinha and Saha (1979). As B_j can be any one of association matrices $B_1,\cdots,B_{i-1},B_{i+1},\cdots,B_m$, the construction-method of PBT design, defined in Theorem 3.1, produces $m-1$ different series of PBT designs, apart from that of Sinha and Saha (1979).

Theorem 3.2: Given two $PBIB$ designs on same v treatments, having same $m-$class association scheme and with the parameters,

$$D^{(1)}:(v,b^{(1)},r^{(1)},k;\lambda_1^{(1)},\lambda_2^{(1)},\cdots,\lambda_m^{(1)}),$$

$$D^{(2)}:(v,b^{(2)},r^{(2)},k/2;\lambda_1^{(2)},\lambda_2^{(2)},\cdots,\lambda_m^{(2)})$$

the existence of a PBT design, based on the same association scheme, with the parameters,

$$V=v,\ \ B=b^{(1)}+b^{(2)},\ R=r^{(1)}+2r^{(2)},\quad K=k,\ \Delta=r^{(1)}+4r^{(2)},$$

$\pi_i=\lambda_i^{(1)}+4\lambda_i^{(2)};\quad i=1,2,\cdots,m$, is guaranteed.

Proof : Consider $N=[N_1:2N_2]$, as incidence matrix of the PBT design, where N_1 and N_2 are incidence matrices of $D^{(1)}$ and $D^{(2)}$ respectively.

Now,

$$NN'=N_1N_1'+4N_2N_2'=\sum_{i=0}^{m}B_i(\lambda_i^{(1)}+4\lambda_i^{(2)})$$

which completes the proof.

An example of Theorem 3.2 is presented below:

Example 3.2: Consider $2-$class triangular $PBIBD's$, series number $T1$ and $T28$, Clathworthy (1973), with the parameters,

$T28:\quad v=10,\ b=5,\ \ r=2,\ \ k=4,\ \ \lambda_1=1,\ \lambda_2=0,\ n_1=6,n_2=3.$

$T1\ :\quad v=10,\ b=30,\ r=6,\ \ k=2,\ \ \lambda_1=1,\ \lambda_2=0,\ n_1=6,n_2=3.$

Then by the above construction-method of Theorem 3.2, the solution of the triangular $PBIBD$ with the following parameters, is obtained,

$V=10,\ B=35,\ R=14,\ K=4,\ \Delta=26,\ \pi_1=5,\quad \pi_2=0,\ n_1=6,$
$n_2=3.$

If in Theorem 3.2, $D^{(1)}$ is a BIB design, instead of $PBIB$ design, i.e. $\lambda_i^{(1)}=\lambda^{(1)}$, for all i, then the juxtaposition of the incidence matrix of the BIB design with that of the $PBIB$ design as defined in the proof of Theorem 3.2, produces a PBT design as given below:

Corollary 3.1: Given a BIB design and a $PBIB$ design on same v treatments with the parameters,

$$D^{(1)}: (v, b^{(1)}, r^{(1)}, k;\ \lambda^{(1)})$$

$$D^{(2)}: (v, b^{(2)}, r^{(2)}, k/2;\ \lambda_1^{(2)}, \lambda_2^{(2)}, \cdots, \lambda_m^{(2)}),$$

the existence of a *PBT* design, based on the association scheme of the given PBIB design, with the parameters,

$$V = v,\quad B = b^{(1)} + b^{(2)},\ R = r^{(1)} + 2\,r^{(2)},\quad K = k,\ \Delta = r^{(1)} + 4\,r^{(2)},$$

$\pi_i = \lambda^{(1)} + 4\,\lambda_i^{(2)};\quad i = 1, 2, \cdots, m$, is guaranteed.

Similarly, under the condition that $D^{(2)}$ is a *BIB* design with $\lambda_i^{(2)} = \lambda^{(2)}$ for all i, Theorem 3.2 yields a *PBT* design as follows:

Corollary 3.2: Given a *PBIB* design and a *BIB* design on the same v treatments, with the parameters,

$$D^{(1)}: (v, b^{(1)}, r^{(1)}, k;\ \lambda_1^{(1)}, \lambda_2^{(1)}, \cdots, \lambda_m^{(1)})$$

$$D^{(2)}: (v, b^{(2)}, r^{(2)}, k/2;\ \lambda^{(2)})$$

the existence of a *PBT* design, based on the association scheme of the given *PBIB* design, with the parameters,

$$V = v,\quad B = b^{(1)} + b^{(2)},\ R = r^{(1)} + 2\,r^{(2)},\quad K = k,\ \Delta = r^{(1)} + 4\,r^{(2)},$$

$\pi_i = \lambda_i^{(1)} + 4\,\lambda^{(2)}$, is guaranteed.

If both $D^{(1)}$ and $D^{(2)}$ given in Theorem 3.2 are *BIB* designs with the balancing parameter $\lambda^{(1)}$ and $\lambda^{(2)}$ respectively, then the construction of the *BT* design with the balancing parameter $\lambda^{(1)} + 4\,\lambda^{(2)}$, can be done as given below.

Corollary 3.3: Given two *BIB* designs on the same v treatments, with the parameters,

$$D^{(1)}: (v, b^{(1)}, r^{(1)}, k;\ \lambda^{(1)})$$

$$D^{(2)}: (v, b^{(2)}, r^{(2)}, k/2;\ \lambda^{(2)}),$$

a *BT* design with the following parameters always exists,

$$V = v,\quad B = b^{(1)} + b^{(2)},\ R = r^{(1)} + 2\,r^{(2)},\quad K = k,\ \Delta = r^{(1)} + 4\,r^{(2)},$$

$$\pi = \lambda^{(1)} + 4\,\lambda^{(2)}.$$

Remark 3.2: A list of new BT designs, which can be constructed through the construction-method proposed in Corollary 3.3 and are not available in Saha (1975), Table 1, is shown in Table 3.1. '(R)' and '(T)' will denote the series number of BIB design given in Raghavarao (1971), Table 5.10.1 and that given in Takeuchi (1962) respectively. 'E' stands for efficiency factor given by $E = V\pi / RK$.

Table 3.1:

S. No.	$D^{(1)}$	$D^{(2)}$	V	B	R	K	π	E
1	$4(R)$	$3(R)$	5	15	12	4	7	0.73
2	$9(R)$	$6(R)$	6	30	20	4	10	0.75
3	$11(R)$	$12(R)$	7	28	16	4	6	0.66
4	$13(R)$	$10(R)$	7	14	12	6	9	0.88
5	$4(R)$	$20(R)$	9	24	16	6	9	0.84
6	$32(R)$	$29(R)$	11	22	20	10	17	0.95

As a generalization of Theorem 3.2, starting from p $PBIB$ designs, $D^{(i)}$, based on a same association scheme, with the parameters $v, b^{(i)}, r^{(i)}, k^{(i)}, \lambda_j^{(i)}$ such that $i\,k^{(i)} =$ a constant $(< v)$ for all i $(i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, m)$, a PB $p-$ary design on the association scheme, with the parameters given in the following corollary, can be constructed, by considering

$$N = [1N^{(1)} : 2N^{(2)} : \cdots : pN^{(p)}]; \quad p \text{ a natural number,}$$

as its incidence matrix.

Corollary 3.4: The existence of p $PBIB$ designs, based on the same $m-$class association scheme, with the parameters $v, b^{(i)}, r^{(i)}, k^{(i)}, \lambda_j^{(i)}$; such that $i\,k^{(i)} = k$, a constant $(< v)$ for all i $(i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, m)$ implies that of a PB $p-$ary design on the association scheme, with the parameters,

$$V = v, \quad B = \sum_{i=1}^{p} b^{(i)}, \qquad R = \sum_{i=1}^{p} i\, r^{(i)}, \qquad K = k, \quad \Delta = \sum_{i=1}^{p} i^2 [r^{(i)}]^2 ,$$

$$\pi_j = \sum_{i=1}^{p} i^2 \lambda_j^{(i)} .$$

Let there exist a Nested Partially Balanced Incomplete Block ($NPBIB$) design, D, on a common association scheme, with the parameters r, v, b_1, k_1, λ_{1i}, b_2, k_2, λ_{2i}, 2; $(i=1,2,\cdots,m)$. If N_1 and N_2 be the incidence matrices of the two system of blocks in the design, an incidence matrix of a PBT design, based on the association scheme, given in the following theorem, is defined by

$$N=[N_1:2N_2].$$

Theorem 3.3: The existence of a $NPBIB$ design, D, with the parameters r, v, b_1, k_1, λ_{1i}, b_2, k_2, λ_{2i}, 2; $(i=1,2,\cdots,m)$ based on the same association scheme, implies that of a PBT design, based on the association scheme, with the parameters,

$$V=v,\quad B=b_1+b_2,\quad R=r_1+2r_2,\quad K=k_1,\quad \Delta=r_1+4r_2,$$

$$\pi_i=\lambda_{1i}+4\lambda_{2i}.$$

Proof: By the nature of the defined incidence matrix, N, the proof can be completed straight forward.

To illustrate the construction-method given in Theorem 3.3, an example follows.

Example 3.3: Consider the Semi-Regular Group Divisible design, *SR* 35, Clathworthy (1973), which is also an $NPBIB$ design with the parameters,

$r=6$, $v=6$, $b_1=9$, $k_1=4$, $\lambda_{11}=3$, $\lambda_{12}=4$, $b_2=18$, $k_2=2$, $\lambda_{21}=3$, $\lambda_{22}=0$, 2, $m=2$, $n=3$ and having the blocks, given below.

$\{(1,3),(2,4)\}$, $\{(5,1),(6,2)\}$, $\{(3,5),(4,6)\}$

$\{(1,3),(4,6)\}$, $\{(5,1),(2,4)\}$, $\{(3,5),(6,2)\}$

$\{(1,3),(6,2)\}$, $\{(5,1),(4,6)\}$, $\{(3,5),(2,4)\}$.

The association scheme plane is as follows:

G_1	G_2
1	2
3	4
5	6

Then we can construct a PBT design based on Group Divisible association scheme, with the parameters,

$V=6$, $B=27$, $R=18$, $K=4$, $\Delta=30$, $\pi_1=15$, $\pi_2=4$, $m=2$, $n=3$.

Under the condition that λ_{1i}'s and λ_{2i}'s are constants, λ_1 and λ_2 (say), we can get the following result.

Corollary 3.5: The existence of a Nested Balanced Incomplete Block (*NBIB*) design with the parameters, r, v, b_1, k_1, λ_1; b_2, k_2, λ_2, 2 implies that of a BT design with the parameters,

$$V = v,\quad B = b_1 + b_2,\quad R = r_1 + 2r_2,\quad K = k_2,\quad \Delta = r_1 + 4r_2,$$

$$\pi = \lambda_1 + 4\lambda_2.$$

Remark 3.3: A BT design, which is unlisted in Saha (1975), with parameters, $V = 7$, $B = 35$, $R = 18$, $K = 6$, $\pi = 13$, $E = 0.84$, can be constructed starting from the NBIB design [Preece (1967) Table No. 3, Series No. 2], with the parameters $r = 6$, $v = 7$, $b_1 = 7$, $k_1 = 6$, $\lambda_1 = 6$, $b_2 = 14$, $k_2 = 3$, $\lambda_2 = 2$, 2.

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