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Estimation of Lindley Distribution Using Type-II Hybrid Censored Masked System Lifetime Data

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ABSTRACT

Competing risk situations occur when the systems/subjects under study are subjected to more than one cause of failure. In the present study, we propose Type-II hybrid censored competing risk analysis with masked system failure time data. It is assumed that the lifetime distribution of competing causes of failures follow Lindley distribution. We derive maximum likelihood and Bayes estimates of the model parameters. We also provide asymptotic and two bootstrap (boot-p and boot-t) confidence intervals of the model parameters. Markov Chain Monte Carlo technique such as Gibbs sampler is employed for Bayesian estimation. A simulation study is conducted to check the performances of the considered estimation methods. A masked competing risk real dataset is analyzed for illustrative purpose.

1. Introduction

The model for failure data in a life testing experiment of a multi-component system where system failure occurs as soon as anyone of its component fails is known as the competing risk model. In this regard, the components are connected in a series system. In the competing risk analysis, every unit or item can experience many risk factors until the unit is either failed or censored. Obtaining failure data under competing risk analysis are very abundant in the field of biomedical, engineering and life sciences in terms of various possible risk factors of life testing units observed for an experiment. The following are some real life examples related to competing risk data. For biomedical data, a patient suffers from breast cancer may also suffer from heart disease and other causes. The

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effect of other causes may play an important role in the survival of patient. For engineering data, computer system failure can occur as soon as anyone of the motherboard, disk drives or power supply stop functioning.

In competing risk analysis, we also often encounter the situation when for a given subset of items, the true causes of failure belong to only a subgroup of the causes and cannot be uniquely identified. The failure data obtained from these items generally termed as masked failure data. However, dealing with masked system failure data, a second stage analysis may be conducted to find an exact cause of failure. But second stage analysis takes more time and tends to be more costly. So, one may consider some masked failure data for study.

The analysis of masking with competing risk data have been studied by many authors assuming different lifetime distributions for competing causes of failures. Miyawaka (1984) obtained the maximum likelihood estimators of two-component series system of exponential lifetime distribution by using masked data. Craiu and Reiser (2006) discussed inferences for the dependent competing risks model with masked causes of failure. The study in Sen *et al.* (2010) dealt with the Bayesian approach to competing risk analysis with masked causes of death. Basu (2009) gave inferences of competing risk analysis in the presence of masked failure data. Xu and Tang (2009) dealt with Pareto distribution for performing the Bayesian analysis of masked data. Kumar *et al.* (2014) considered Bayesian estimation of progressive censored masked lifetime data from Rayleigh distribution. The study in Panwar *et al.* (2015) dealt with the parameter estimation of inverse Rayleigh distribution under competing risk model for masked data.

Further, in reliability/survival analysis, obtaining failure data is very expensive and time consuming, and hence censoring is frequently used in life testing experiment. Type-II hybrid censoring scheme introduced by Childs et al. (2003) is widely used in reliability/survival analysis due to its special feature that it assured at least some failures before the termination of the experiment. For more details on Type-II Hybrid censoring scheme, one may refer (Banerjee and Kundu 2008; Balakrishnan and Kundu 2013; Bhattacharya *et al.* 2014; Singh *et al.* 2014; Koley *et al.* 2017; Singh and Goel 2018).

Although several distributions have been considered to draw inferences on competitive risk analysis using masked data to the best of our knowledge, no study has demonstrated competitive risk analysis with masked system failure data under the Type-II hybrid censoring scheme with the Lindley distribution. The Lindley distribution was originally proposed by Lindley (1958, 1965) as a mixture of exponential and gamma distributions, has been used by many authors in various fields of life testing experiments like competing risk analysis, stress-strength reliability analysis, load-sharing scenario, actuarial sciences etc. due to its time dependent failure rate. Ghitany *et al.* (2008) explored the properties of Lindley distribution and showed it as a better lifetime model as compared to the exponential distribution. Mazucheli and Achcar (2011) studied Lindley distribution under competing risk framework. Krishna and Kumar (2011) proposed the estimation of the parameters of Lindley distribution with progressive Type-II censoring scheme. Singh and Gupta (2012) analysed k-component load-sharing parallel system model assuming each component lifetime distribution as Lindley distribution. Gupta and Singh (2013) also derived parameter estimation of Lindley distribution with hybrid censored data. Dube *et al.* (2016) studied the progressive first failure censoring with Lindley distribution. Wang and Li (2019) engaged ML and Bayesian procedures for demonstrating partially observed causes of failure in the presence of generalized progressively hybrid censored data. Recently, Rai *et al.* (2021) analysed the masked data with Lindley failure model.

In the light of above considerations, the present study is focused on the analysis of competing risk model with masked system failure data under Type-II hybrid censoring scheme. It is assumed that the lifetime of each competing cause of failures follows Lindley distribution. The rest of the study is organized as follows. In section 2, we present the notations used for this study and describe the model. Section 3 contains the maximum likelihood estimation of the model parameters. In section 4, we derived asymptotic and two bootstrap confidence intervals of the model parameters. Thereafter, using Gibbs sampler, one of the Markov Chain Monte Carlo (MCMC) technique, Bayesian estimation of the model parameters is performed by assuming gamma and non-informative priors in section 5. For highlighting the theoretical developments, a simulation study is carried out in section 6. For practical illustrations, we present a real data analysis in section 7. Finally, we conclude our findings in section 8.

2. Notations and Model Description

2.1 Notations

Without loss of generality, it is assumed that the system under study experiences only two competing causes of failures. The following notations have been used in this study.

$n =$ The number of systems under consideration on test.

$r_j =$ The number of observed system failures due to cause j, j = 1, 2.

$r_{12} =$ The number of observed system failures due to masked causes of failures.

$T_{ji} =$ Lifetime of the j^{th} component of the i^{th} system, for j=1,2 and $i = 1, 2, ..., n.$

$Z_i = \min\left(T_{1i}, T_{2i}\right).$

$Z_{i:n} = i^{th}$ Ordered statistic of $Z_i, i = 1, 2, ..., n.$

R = Pre-fixed integer.

T = Pre-fixed time point.

$U = \max\left(Z_{R:n}, T\right).$

$r =$ The number of system failures before U .

$r = r_1 + r_2 + r_{12}$.

$\delta_i =$ A set containing the corresponding cause of failure of the i^{th} ordered system.

$\left\{\left(Z_{1:n}, \delta_1\right), ..., \left(Z_{r:n}, \delta_r\right)\right\}$ = The observed data before the termination of test.

2.2 Model Description

Suppose that n systems, each consisting of two components are put on test. In each system, the components are connected in series. Therefore, the system can fail due to any of its components failure. Let T_{ji} ; $j = 1, 2$; $i = 1, 2, ..., n$ be the lifetime of j^{th} component of the i^{th} system, independently follows Lindley distribution with parameter θ_j i.e. $T_{ji} \sim \text{Lindley}(\theta_j)$ for $i = 1, 2, \ldots. n; j = 1, 2$. The probability density function (pdf) of T_{ji} is given by

$$f_j\left(t \middle| \theta_j\right) = \frac{\theta_j^2}{\left(1+\theta_j\right)}(1+t)e^{-\theta_j t} \quad ; j = 1, 2; \ \theta_j, t > 0 \tag{2.2.1}$$

Also, the hazard rate (hrf) and the survival function (sf) of T_{ji} are given by

$$h_j\left(t \middle| \theta_j\right) = \frac{\theta_j^2 (1+t)}{\left(\theta_j + 1 + \theta_j t\right)} \quad \text{And} \quad S_j\left(t \middle| \theta_j\right) = \frac{\left(\theta_j + 1 + \theta_j t\right)}{\left(\theta_j + 1\right)} e^{-\theta_j t} \quad ; \theta_j, t > 0 \tag{2.2.2}$$

Since, the random variables $T_{ji}'s$ are assumed to be independent, and let Z_i denotes the observed lifetime of the i^{th} system where $Z_i = \min\left(T_{1i}, T_{2i}\right)$, then the pdf of Z_i is as follows:

$$f\left(z \middle| \theta\right) = \sum_{j=1}^{2} \frac{\theta_j^2 (1+z)}{\left(\theta_j + 1 + \theta_j z\right)} \prod_{j=1}^{2} \frac{\left(\theta_j + 1 + \theta_j z\right)}{\left(\theta_j + 1\right)} e^{-\theta_j z} \quad ; z > 0 \tag{2.2.3}$$

In the competing risk analysis with masked causes of failures, one can observe the following three different structures

1. A system fails and we observe both, the time to failure and cause of failure.
2. A system fails and we observe only time to failure but not the cause of failure i.e. masking arises in the system.
3. A system survived up to the end of the experiment i.e. censored observation.

Now, suppose $Z_{1:n} < Z_{2:n} < \ldots\ldots < Z_{n:n}$ denotes the ordered values of $Z_1, Z_2, \ldots., Z_n$ and the test is terminated at time point $U = \max\left(Z_{R:n}, T\right)$ where $R < n$ is the pre-fixed integer and T is the pre-fixed time point. Let r be the number of observed system failures up to time point U. Also, assume that δ_i is a set of system's components containing the corresponding cause of system failure. Note that, if δ_i is not singleton, then the corresponding cause of failure is masked. Thus, under Type-II hybrid censoring scheme, we have the following observations at the end of the experiment.

Case I: $\{(Z_{1:n},\delta_1),......,(Z_{R:n},\delta_R)\}$ if $T < Z_{R:n}$

Case II: $\{(Z_{1:n},\delta_1),......,(Z_{d:n},\delta_d)\}$ if $Z_{R:n} < T$ and $Z_{d:n} < T < Z_{d+1:n}$ for $d = R,......n$

Case III: $\{(Z_{1:n},\delta_1),......,(Z_{n:n},\delta_n)\}$ if $Z_{n:n} < T.$

3. Maximum Likelihood Estimation

The Likelihood function of the data point $(Z_{i:n},\delta_i)$ for $i = 1,2,....r$ is given by

$$L(\theta_1,\theta_2|z_i,\delta_i) = \prod_{i=1}^{r}\sum_{j\in\delta_i}\left(f(z_i|\theta_j).\prod_{\substack{k=1\\k\neq j}}^{j}S(z_i|\theta_k)\right)\left(\prod_{\substack{k=1\\k\neq j}}^{j}S(U|\theta_k)\right)^{n-r} \tag{3.1}$$

Let, out of the r observed system failures, r_1 and r_2 respectively denote the number of system failures due to cause 1 and cause 2. Additionally, r_{12} be the number of failed systems due to masked causes of failures. Apparently, $r_1 + r_2 + r_{12} = r$. Now, the likelihood function given in equation (3.1) can be rewritten as

$$L(\theta_1,\theta_2|z_i,\delta_i) = \frac{n!}{r_1!r_2!r_{12}!(n-r)!}\prod_{i=1}^{r_1}\{f(z_i|\theta_1)S(z_i|\theta_2)\}\prod_{i=1}^{r_2}\{f(z_i|\theta_2)S(z_i|\theta_1)\}$$
$$\prod_{i=1}^{r_{12}}\{f(z_i|\theta_1)S(z_i|\theta_2)+f(z_i|\theta_2)S(z_i|\theta_1)\}\left\{\prod_{\substack{k=1\\k\neq j}}^{j}S(U|\theta_k)\right\}^{n-r} \tag{3.2}$$

In view of equations (2.2.1), (2.2.2), and (3.2) we get the following form of the likelihood function

$$L(\theta_1,\theta_2|z_i,\delta_i) = \frac{C.\theta_1^{2r_1}.\theta_2^{2r_2}}{(\theta_1+1)^n.(\theta_2+1)^n}\prod_{i=1}^{r}(1+z_i).e^{-(\theta_1+\theta_2)\left[\sum_{i=1}^{r}z_i+(n-r)U\right]}.\prod_{i=1}^{r_1}\{\theta_2(1+z_i)+1\}.$$
$$\prod_{i=1}^{r_2}\{\theta_1(1+z_i)+1\}.\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\{\theta_2(1+z_i)+1\}+\theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\{\theta_1(1+z_i)+1\}\right].$$
$$\{\theta_1(1+U)+1\}^{n-r}.\{\theta_2(1+U)+1\}^{n-r} \tag{3.3}$$

Where, $r = \begin{cases} R\,;\ \text{for case I} \\ d\,;\ \text{for case II} \\ n;\ \text{for case III} \end{cases}$, $U = \begin{cases} Z_R\,;\ r = R \\ T\ ;\quad r > R \end{cases}$, $C = \dfrac{n!}{r_1!\, r_2!\, r_{12}!\,(n-r)!}$

The log likelihood function is

$$\log L(\theta_1,\theta_2 | z_i,\delta_i) = \log C + 2(r_1 \log\theta_1 + r_2 \log\theta_2) - n\{\log(\theta_1+1) + \log(\theta_2+1)\} + \sum_{i=1}^{r}\log(1+z_i) - (\theta_1+\theta_2)\left\{\sum_{i=1}^{r} z_i + (n-r)U\right\} + \log\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\{\theta_2(1+z_i)+1\} + \theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\{\theta_1(1+z_i)+1\}\right] + \sum_{i=1}^{r_1}\log\{\theta_2(1+z_i)+1\} + \sum_{i=1}^{r_2}\log\{\theta_1(1+z_i)+1\} + (n-r)\left[\log\{\theta_1(1+U)+1\} + \log\{\theta_2(1+U)+1\}\right] \tag{3.4}$$

On differentiating equation (3.4) partially with respect to θ_1, θ_2 and equating to zero, one gets

$$\frac{\partial}{\partial\theta_1}\log L(\theta_1,\theta_2 | z_i,\delta_i) = \frac{2r_1}{\theta_1} - \frac{n}{(\theta_1+1)} - \left(\sum_{i=1}^{r} z_i + (n-r)U\right) + \sum_{i=1}^{r_2}\frac{(1+z_i)}{\theta_1(1+z_i)+1} + \frac{(n-r)(1+U)}{(\theta_1(1+U)+1)} + \frac{2r_{12}.\theta_1^{2r_{12}-1}.\prod_{i=1}^{r_{12}}\{\theta_2(1+z_i)+1\} + \theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}(1+z_i)}{\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\{\theta_2(1+z_i)+1\} + \theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\{\theta_1(1+z_i)+1\}} = 0, \tag{3.5}$$

$$\frac{\partial}{\partial\theta_2}\log L(\theta_1,\theta_2 | z_i,\delta_i) = \frac{2r_2}{\theta_2} - \frac{n}{(\theta_2+1)} - \left(\sum_{i=1}^{r} z_i + (n-r)U\right) + \sum_{i=1}^{r_1}\frac{(1+z_i)}{\theta_2(1+z_i)+1} + \frac{(n-r)(1+U)}{(\theta_2(1+U)+1)} + \frac{\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}(1+z_i) + 2r_{12}.\theta_2^{2r_{12}-1}.\prod_{i=1}^{r_{12}}\{\theta_1(1+z_i)+1\}}{\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\{\theta_2(1+z_i)+1\} + \theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\{\theta_1(1+z_i)+1\}} = 0, \tag{3.6}$$

The maximum likelihood estimators (MLEs) of the model parameters θ_1 and θ_2 can be obtained on solving the non-linear equations (3.5) and (3.6) simultaneously, but these equations have multidimensional complexity and cannot be solve directly. However, one can use any of the optimization functions of R-software like nlm(), mle(), maxLik() to obtain $\hat{\theta}_1$ and $\hat{\theta}_2$.

4. Confidence Intervals

Here, we propose to use two alternative confidence intervals for the parameters θ_1 and θ_2 namely asymptotic confidence intervals and bootstrap confidence intervals.

4.1 Asymptotic Confidence Intervals

Here, we develop asymptotic confidence intervals for the parameters θ_1 and θ_2 by using the asymptotic normality of MLEs for which we first obtain the observed Fisher information matrix $I\left(\hat{\Theta}\right)$ given as

$$I\left(\hat{\Theta}\right)=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}_{\Theta=\hat{\Theta}}$$

Where

$$A_{11}=E\left(-\frac{\partial^2 \log L}{\partial\theta_1^2}\right),\; A_{12}=E\left(-\frac{\partial^2 \log L}{\partial\theta_1\partial\theta_2}\right)=A_{21},\; A_{22}=E\left(-\frac{\partial^2 \log L}{\partial\theta_2^2}\right)$$

The expressions of the second partial derivatives of the log likelihood function with respect to θ_1 and θ_2 are as follows:

$$\frac{\partial^2}{\partial\theta_1^2}\log L\left(\theta_1,\theta_2\left|z_i,\delta_i\right.\right)=\frac{-2r_1}{\theta_1^2}+\frac{n}{\left(\theta_1+1\right)^2}-\sum_{i=1}^{r_2}\frac{\left(1+z_i\right)^2}{\left\{\theta_1\left(1+z_i\right)+1\right\}^2}-\frac{\left(n-r\right)\left(1+U\right)^2}{\left(\theta_1\left(1+U\right)+1\right)^2}+C_1 \qquad (4.1.1)$$

Where,

$$C_1=\frac{\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_2\left(1+z_i\right)+1\right\}+\theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_1\left(1+z_i\right)+1\right\}\right]\left[2r_{12}\left(2r_{12}-1\right)\theta_1^{2r_{12}-2}.\prod_{i=1}^{r_{12}}\left\{\theta_2\left(1+z_i\right)+1\right\}\right]-\left[\theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\left(1+z_i\right)+2r_{12}.\theta_1^{2r_{12}-1}.\prod_{i=1}^{r_{12}}\left\{\theta_2\left(1+z_i\right)+1\right\}\right]^2}{\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_2\left(1+z_i\right)+1\right\}+\theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_1\left(1+z_i\right)+1\right\}\right]^2}$$

$$\frac{\partial^2}{\partial\theta_2^2}\log L\left(\theta_1,\theta_2\left|z_i,\delta_i\right.\right)=\frac{-2r_2}{\theta_2^2}+\frac{n}{\left(\theta_2+1\right)^2}-\sum_{i=1}^{r_1}\frac{\left(1+z_i\right)^2}{\left\{\theta_2\left(1+z_i\right)+1\right\}^2}-\frac{\left(n-r\right)\left(1+U\right)^2}{\left(\theta_2\left(1+U\right)+1\right)^2}+C_2 \qquad (4.1.2)$$

$$C_2=\frac{\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_2\left(1+z_i\right)+1\right\}+\theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_1\left(1+z_i\right)+1\right\}\right]\left[2r_{12}\left(2r_{12}-1\right)\theta_2^{2r_{12}-2}.\prod_{i=1}^{r_{12}}\left\{\theta_1\left(1+z_i\right)+1\right\}\right]-\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\left(1+z_i\right)+2r_{12}.\theta_2^{2r_{12}-1}.\prod_{i=1}^{r_{12}}\left\{\theta_1\left(1+z_i\right)+1\right\}\right]^2}{\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_2\left(1+z_i\right)+1\right\}+\theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_1\left(1+z_i\right)+1\right\}\right]^2}$$

$$\frac{\partial^2}{\partial\theta_1\partial\theta_2}\log L\left(\theta_1,\theta_2\left|z_i,\delta_i\right.\right)=\frac{\begin{array}{l}\left[2r_{12}\prod_{i=1}^{r_{12}}\left(1+z_i\right)\left(\theta_1^{2r_{12}-1}+\theta_2^{2r_{12}-1}\right)\right].\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_2\left(1+z_i\right)+1\right\}+\right.\\ \left.\theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_1\left(1+z_i\right)+1\right\}\right]-\left[2r_{12}\theta_1^{2r_{12}-1}\prod_{i=1}^{r_{12}}\left\{\theta_2\left(1+z_i\right)+1\right\}\right.\\ \left.+\theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\left(1+z_i\right)\right].\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\left(1+z_i\right)+2r_{12}\theta_2^{2r_{12}-1}\prod_{i=1}^{r_{12}}\left\{\theta_1\left(1+z_i\right)+1\right\}\right]\end{array}}{\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_2\left(1+z_i\right)+1\right\}+\theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_1\left(1+z_i\right)+1\right\}\right]^2} \qquad (4.1.3)$$

Now, the $100\times\left(1-\alpha\right)\%$ asymptotic confidence intervals of θ_1 and θ_2 can be constructed as $\hat{\theta}_j\pm Z_{\alpha/2}\sqrt{I_{jj}\left(\hat{\theta}\right)}$; $j=1,2.$ Here, $z_{\alpha/2}$ is the upper $(\alpha/2)^{th}$ percentile of standard normal distribution and I_{jj} is the $(j,j)^{th}$ element of the inverse of Fisher information matrix.

4.2 Bootstrap Confidence Intervals

This subsection is concerned with two parametric bootstrap confidence intervals namely percentile bootstrap (boot-p) and bootstrap-t (boot-t) confidence intervals of the unknown parameters. For the construction of bootstrap confidence intervals, the following steps are made.

Algorithm for boot-p method:

1. Based on the original sample $\underset{\frown}{z}=\left(z_1,z_2,......z_r\right)$ calculate MLEs of the model parameters, say $\hat{\theta}_1$ and $\hat{\theta}_2$.

2. Under the same sampling framework of competing risk analysis with Type-II hybrid censored masked data, simulate sample $\underset{\frown}{z}^* = \left(z_1^*, z_2^*, ...z_r^*\right)$ from the underlined Lindley distribution.
3. Compute the MLEs of θ_1 and θ_2 using $\underset{\frown}{z}^*$ denoted by $\hat{\theta}_1^*$ and $\hat{\theta}_2^*$.
4. Repeat step 2 and step 3, B times to obtain B such Bootstrap estimates.
5. Arrange $\left(\hat{\theta}^*_{1(1)}, \hat{\theta}^*_{1(2)}, ..., \hat{\theta}^*_{1(B)}\right), \left(\hat{\theta}^*_{2(1)}, \hat{\theta}^*_{2(2)}, ..., \hat{\theta}^*_{2(B)}\right)$ in ascending order.
6. A two sided $100\times(1-\alpha)\%$ percentile bootstrap (boot-p) confidence intervals for θ_1 and θ_2 are $\left(\hat{\theta}_1^{*[\alpha B/2]}, \hat{\theta}_1^{*[(1-\alpha/2)B]}\right)$ and $\left(\hat{\theta}_2^{*[\alpha B/2]}, \hat{\theta}_2^{*[(1-\alpha/2)B]}\right)$.
7. Lastly, the percentile bootstrap (boot-p) estimates of θ_1, θ_2 can be respectively obtained as $\hat{\theta}^*_{1b} = \frac{1}{B}\sum_{l=1}^{B}\theta^*_{1(l)}$ and $\hat{\theta}^*_{2b} = \frac{1}{B}\sum_{l=1}^{B}\theta^*_{2(l)}$.

Algorithm for boot-t method:

1. Using the above boot-p sample $\underset{\frown}{z}^*$, determine the statistic $T_1^* = \frac{\left(\hat{\theta}_1^* - \hat{\theta}_1\right)}{\sqrt{V\left(\hat{\theta}_1^*\right)}}$ and $T_2^* = \frac{\left(\hat{\theta}_2^* - \hat{\theta}_2\right)}{\sqrt{V\left(\hat{\theta}_2^*\right)}}$, where $V\left(\hat{\theta}_1^*\right)$ and $V\left(\hat{\theta}_2^*\right)$ are the asymptotic variances of $\hat{\theta}_1^*$ and $\hat{\theta}_2^*$ respectively which can be obtained using Fisher information matrix.
2. Repeat step 1 B times.
3. Arrange $T^*_{1(1)}, T^*_{1(2)}, ...T^*_{1(B)}$ and $T^*_{2(1)}, T^*_{2(2)}, ...T^*_{2(B)}$ in ascending order.
4. The $100\times(1-\alpha)\%$ boot-t confidence intervals for θ_1 and θ_2 are respectively given by $\left(\hat{\theta}_1 - T_1^{*[B(1-\alpha/2)]}\sqrt{V\left(\hat{\theta}_1\right)},\ \hat{\theta}_1 - T_1^{*[B(\alpha/2)]}\sqrt{V\left(\hat{\theta}_1\right)}\right)$ and $\left(\hat{\theta}_2 - T_2^{*[B(1-\alpha/2)]}\sqrt{V\left(\hat{\theta}_2\right)},\ \hat{\theta}_2 - T_2^{*[B(\alpha/2)]}\sqrt{V\left(\hat{\theta}_2\right)}\right)$.

5. Bayesian estimation

For Bayesian estimation of the model parameters θ_1 and θ_2, it is assumed that the model parameters are the random variables, and follow independent gamma prior distributions as:

$$\pi\left(\theta_1\right)=\frac{b_1^{a_1}}{\Gamma a_1}e^{-b_1\theta_1}\theta_1^{a_1-1}\text{ and} \tag{5.1}$$

$$\pi\left(\theta_2\right)=\frac{b_2^{a_2}}{\Gamma a_2}e^{-b_2\theta_2}\theta_2^{a_2-1}\ \theta_i, a_i, b_i \geq 0. \tag{5.2}$$

In view of the above priors and the likelihood function in (3.3), the joint posterior distribution of θ_1 and θ_2 given data is

$$h\left(\theta_1,\theta_2\left|z_i,\delta_i\right.\right)\propto L\left(\theta_1,\theta_2\left|z_i,\delta_i\right.\right).\pi\left(\theta_1\right).\pi\left(\theta_2\right)$$

$$h\left(\theta_1,\theta_2\left|z_i,\delta_i\right.\right)\propto\frac{\theta_1^{2r_1+a_1-1}.\theta_2^{2r_2+a_2-1}.\prod_{i=1}^{r}\left(1+z_i\right)}{\left(\theta_1+1\right)^n.\left(\theta_2+1\right)^n}.e^{-\left(\theta_1+\theta_2\right)\left\{\sum_{i=1}^{r}z_i+\left(n-r\right)U\right\}}$$

$$.\prod_{i=1}^{r_1}\left\{\theta_2\left(1+z_i\right)+1\right\}.\prod_{i=1}^{r_2}\left\{\theta_1\left(1+z_i\right)+1\right\}.\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_2\left(1+z_i\right)+1\right\}+\right. \tag{5.3}$$

$$\left.\theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_1\left(1+z_i\right)+1\right\}\right].\left\{\theta_1\left(1+U\right)+1\right\}^{n-r}\left\{\theta_2\left(1+U\right)+1\right\}^{n-r}.e^{-b_1\theta_1-b_2\theta_2}$$

For obtaining Bayes estimates of θ_1 and θ_2 under squared error loss function $L(\theta^*,\theta)=(\theta^*-\theta)^2$, one needs to obtain marginal posterior expectations which are not possible to determine analytically by using (5.3). Therefore, we use Gibbs sampling approach, one of the Markov Chain Monte Carlo (MCMC) technique to compute approximate Bayes estimates of the model parameters. This approach was given by Geman and Geman (1984). For implementing Gibbs sampler, the full conditional posterior pdfs of the model parameters are given by

$$\Pi\left(\theta_1\left|\theta_2,z_i,\delta_i\right.\right)\propto\frac{\theta_1^{2r_1+a_1-1}}{\left(\theta_1+1\right)^n}.e^{-\theta_1\left(\sum_{i=1}^{r}z_i+\left(n-r\right)U+b_1\right)}.\prod_{i=1}^{r_2}\left\{\theta_1\left(1+z_i\right)+1\right\}.\left\{\theta_1\left(1+U\right)+1\right\}^{n-r}$$

$$\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_2\left(1+z_i\right)+1\right\}+\theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_1\left(1+z_i\right)+1\right\}\right] \tag{5.4}$$

$$\Pi\left(\theta_2 \left|\theta_1, z_i, \delta_i\right.\right) \propto \frac{\theta_2^{2r_2+a_2-1}}{\left(\theta_2+1\right)^n}.e^{-\theta_2\left(\sum_{i=1}^{r} z_i+(n-r)U+b_2\right)}.\prod_{i=1}^{r_1}\left\{\theta_2\left(1+z_i\right)+1\right\}.\left\{\theta_2\left(1+U\right)+1\right\}^{n-r}$$
$$\left[\theta_1^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_2\left(1+z_i\right)+1\right\}+\theta_2^{2r_{12}}\prod_{i=1}^{r_{12}}\left\{\theta_1\left(1+z_i\right)+1\right\}\right] \qquad (5.5)$$

Gibbs algorithm

Using equations (5.4) and (5.5), we generate the Gibbs sequence $\left(\theta_1^0,\theta_2^0\right),\left(\theta_1^1,\theta_2^1\right),........,\left(\theta_1^K,\theta_2^K\right)$ involving the following steps:

1. Start with the initial values of θ_1 and θ_2 say $\left(\theta_1^0,\theta_2^0\right)$.
2. Generate θ_1^1 from $\Pi\left(\theta_1 \left|\theta_2^0, z_i, \delta_i\right.\right)$.
3. Generate θ_2^1 from $\Pi\left(\theta_2 \left|\theta_1^1, z_i, \delta_i\right.\right)$.
4. Repeat the steps 2-3,K times. To nullify the effects of the starting values of the parameters, we record the parametric values $\left(\theta_1^{N+1},\theta_2^{N+1}\right),.......,\left(\theta_1^K,\theta_2^K\right)$ after discarding N burn-in period draws.
5. Bayes estimates of the parameters θ_1, θ_2 and corresponding variances are taken to be the means and variances of generated values θ_1 and θ_2 respectively.
6. Arrange $\left(\theta_1^{N+1},\theta_1^{N+2},.......,\theta_1^K\right)$ and $\left(\theta_2^{N+1},\theta_2^{N+2},.......,\theta_2^K\right)$ in ascending order. Then by using the method given by Chen and Shao (1999),we construct the HPD intervals of θ_1 and θ_2

Similarly, we can obtain the posterior statistics via non-informative priors by setting all gamma priors parameters equal to zero i.e. $a_1 = b_1 = 0$, $a_2 = b_2 = 0$. Also, note that the simulation in the steps 2 and 3 is not easy due to complex posterior densities of θ_1 and θ_2. Therefore, we use Metropolis-Hastings algorithm (Metropolis and Ulam 1949; Hastings 1970), to simulate parametric draws from these densities of θ_1 and θ_2.

6. Simulation Study

Here, we present some illustrations based on simulation study for comparing the performances of different methods of estimation of model parameters under Type-II hybrid censoring schemes for varying sample sizes. In this study, we have used the following algorithms for simulating the data.

Algorithm for Sample Generation

- First, we take model parameters $\theta_1 = 2, \theta_2 = 1$ and generate Lindley lifetimes for the competing causes of failures 1 and 2 using the function LindleyR() of the R software.
- In competing risk analysis, minimum of the failure time of the components is the failure time of the system and the component corresponds to the minimum failure time taken to be the cause of failure. From these observations, we then generate random samples using Type-II hybrid censoring scheme by considering different values of R and T.
- Further, under different levels of masking, we get the final form of the competing risk data with missing cause of failure.
- Thereafter, we repeat the process 1000 times and acquire the average estimates with respective average SE/PSEs and average interval widths of the confidence, bootstrap and HPD credible intervals along with their coverage probabilities (CPs).

  In this study, we use different values of n, R and T as n=30, 50, 100, R=0.80n, 0.60n, T=0.4, 0.7 with different masking levels such as 10%, 20%, 30%.

Algorithm for Classical Estimation

- In maximum likelihood estimation, we utilize the function maxLik() of R software for obtaining the MLE of the model parameters θ_1 and θ_2 with the corresponding standard errors (SE).
- The confidence intervals (CI) and two bootstrap (boot-p and boot-t) confidence intervals with 3000 bootstrap replications are also computed.
- The convergence of the classical parameters has been checked through the various plots of Lindley distribution for the parameters θ_1 and θ_2. We have plot the pdf, cumulative distribution function (cdf), survival function (SF) and hazard rate function (HRF) shown in Fig. 1.

Algorithm for Bayesian Estimation

- In Bayesian setup, we took 20,000 posterior samples of the model parameters θ_1 and θ_2 using M-H algorithm within Gibbs sampler and obtain Bayes estimates of model parameters θ_1 and θ_2 along with their posterior standard errors (PSE) and highest posterior density (HPD) credible intervals.
- The convergence of MCMC chains after discarding suitable burn-in-period has been checked through trace and auto correlation plots.
- Then, we plot the posterior densities of the model parameters θ_1 and θ_2 shown in Fig. 2. In each case, the coverage probabilities (CPs) of the intervals have also been obtained.

  All the simulation results are outlined in Table 1-6. We use R software to develop the programs of the simulation study. From Table 1-6, we observe that-

- All the method of estimation satisfactorily estimates the parameters in terms of average SEs/PSEs and average interval widths associated with their respective CPs of the CI/HPD credible intervals.
- The performance of average SEs/PSEs of estimates improves as any one of the n, R, T increases. Also the estimated errors tend to decrease as we decrease the masking level. The same trend is observed in case of average widths of CI/HPD credible intervals.
- The boot-p estimation and Bayes estimation with non-informative prior's results are similar to the maximum likelihood estimation in terms of respective average SEs and average PSEs.
- Bayes estimation yields more efficient estimates as we look over the maximum likelihood estimation and boot-p estimation in terms of average SEs/PSEs and average interval widths along with their CPs of the CI/HPD credible intervals.
- Bayes estimation with gamma priors gives best results as compared to all other methods of estimation.

7. Real Data Analysis

Here, we demonstrate the real life applicability of the proposed methodology. We consider the real life data which was originally proposed by Nelson (1982) and based on conducting the lifetimes of electric appliances. In this dataset, the lifetimes of electric appliances are considered to be the complete use of cycles of appliance until the appliance got failed. Initially, there were 18 ways in which appliance could fail. This data has also been previously studied by Tomer *et al.* (2014). They analyzed the data after some modifications for presenting the example of masked data. They considered the data to the two causes of failures and introduced 20% masking. The considered data is as follows:

Mode of Failure	**Appliance failure times**
Cause-1	2223, 2400, 4329, 1167, 3112, 2471, 1925, 3214, 1990, 3034, 2551,6976, 3478
Cause-2	958, 35, 170, 2831, 13403, 2702, 6367, 708, 2451, 381, 1062, 1594,2761, 329, 49, 2565
Masked	2568, 3059, 11, 3034, 3504, 7846, 2327

We divide each observation of the dataset by 1000 and checked whether the Lindley distribution fits well the data or not. For this purpose, we calculate Kolmogorov-Smirnov (K-S) statistics along with corresponding p-values. The K-S statistics with the respective p-value for cause-1 and cause-2 respectively come out to be 0.2960 (0.1665) and 0.1985 (0.4924). These values clearly indicate that the Lindley distribution can be considered as the failure time distribution for cause-1 and cause-2. Further, for analyzing this data under Type-II hybrid censoring scheme, we artificially formed 20% censoring scheme by assuming R = 29 and T = 3 from the complete data with masked causes of failures.

Now, assuming Lindley distribution as the failure time distribution of the competing causes of failures, we obtain estimates of the model parameters using maximum likelihood estimation and Bayes method of estimation with gamma and non-informative priors. The maximum likelihood estimates of the model parameters θ_1 and θ_2 turn out to be 0.3107 and 0.4839 with the respective standard errors 0.0662 and 0.0776. Bayes estimates with gamma and non-informative priors of (θ_1, θ_2) respectively are (0.4185, 0.5942) and (0.3202, 0.4611) with posterior standard errors (0.0475, 0.0579) and (0.0763, 0.0831).

The 95% confidence intervals of the parameters θ_1 and θ_2 are (0.1809, 0.4406) and (0.3318, 0.6361). The 95% HPD credible intervals of the parameters θ_1 and

θ_2 with gamma and non-informative priors are (0.3211, 0.4927), (0.4911, 0.6910) and (0.1547, 0.4552), (0.3300, 0.6449) respectively.

8. Concluding Remarks

In this article, we present classical and Bayesian estimation of Type-II hybrid censored competing risk data with masked causes of failures. We assume that the failure time distribution of competing causes of failures is Lindley distribution. In classical setup, we obtain maximum likelihood estimates of the model parameters along with their estimated standard errors. We also provide the asymptotic and two bootstrap confidence intervals of the model parameters. Thereafter, we obtain Bayes estimates along with HPD credible intervals by assuming gamma and non-informative priors of the model parameters. Since Bayes estimates are not found in the closed form expressions, therefore, we use MCMC technique such as Gibbs sampler for obtaining Bayes estimates of the model parameters. The efficiency of the above proposed methods of estimation are examined based on various sample sizes with different combinations of Type-II hybrid censoring parameters R and T under different masking levels. For practical implementation of the above methodology, a masked competing risk real data analysis is provided.

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Table 1: Various average estimates with respective average SE/PSE for $\theta_1 = 2, \theta_2 = 1$ at different masking level for n=30 under Type-II hybrid censoring.

Average Estimates [Average SE/PSE]	**Masking Level**	**R=24, T=0.7**		**R=18, T=0.4**	
		θ_1	θ_2	θ_1	θ_2
MLEs [SE]	10%	1.9353 [0.3789]	1.1264 [0.2946]	2.0196 [0.4439]	1.1335 [0.3378]
	20%	2.0316 [0.4002]	1.0351 [0.2959]	2.0462 [0.4629]	1.0936 [0.3508]
	30%	2.1330 [0.3917]	0.8927 [0.2872]	2.1252 [0.4538]	0.9509 [0.3408]
Boot-p [SE]	10%	1.9910 [0.3869]	1.1449 [0.2984]	2.0248 [0.4446]	1.1920 [0.3458]
	20%	2.0987 [0.4115]	1.0727 [0.3027]	2.0479 [0.4584]	1.0861 [0.3458]
	30%	2.1860 [0.4032]	0.9606 [0.2905]	2.1294 [0.4537]	1.0022 [0.3476]
Bayes with Gamma Priors [PSE]	10%	1.9892 [0.1319]	1.0120 [0.0940]	1.9971 [0.1333]	1.0104 [0.0956]
	20%	2.0009 [0.1321]	1.0004 [0.0936]	2.0009 [0.1341]	1.0049 [0.0952]
	30%	2.0145 [0.1331]	0.9838 [0.0928]	2.0124 [0.1353]	0.9901 [0.0942]
Bayes with Non-Informative Priors [PSE]	10%	2.0122 [0.3900]	1.1111 [0.2966]	1.8921 [0.4239]	1.1796 [0.3383]
	20%	2.0552 [0.4015]	1.0464 [0.3003]	1.8822 [0.4332]	1.1368 [0.3474]
	30%	2.0697 [0.4147]	0.9797 [0.2994]	2.1085 [0.4814]	0.9927 [0.3508]

Table 2: Various average CI/HPD intervals widths along with their CPs for $\theta_1 = 2, \theta_2 = 1$ at different masking level for n=30 under Type-II hybrid censoring.

Average CI/HPD intervals	**Masking Level**	**R=24, T=0.7**		**R=18, T=0.4**	
		θ_1	θ_2	θ_1	θ_2
CI	10%	1.4856 [0.96]	1.1550 [0.98]	1.7401 [0.95]	1.3245 [0.97]
	20%	1.5688 [0.95]	1.1601 [0.95]	1.8147 [0.97]	1.3751 [0.96]
	30%	1.5357 [0.96]	1.1783 [0.85]	1.7788 [0.91]	1.3186 [0.86]
Boot-p CI	10%	1.5708 [0.99]	1.2214 [1.00]	1.7981 [0.98]	1.4035 [1.00]
	20%	1.7425 [1.00]	1.3112 [1.00]	1.8845 [0.99]	1.4810 [1.00]
	30%	1.9003 [0.99]	1.3101 [1.00]	2.1810 [1.00]	1.7255 [1.00]
Boot-t CI	10%	1.5764 [0.85]	1.3020 [0.87]	1.9224 [0.90]	1.4934 [0.75]
	20%	1.7445 [0.87]	1.3302 [0.85]	1.9687 [0.90]	1.6162 [0.74]
	30%	2.0268 [0.87]	1.4668 [0.87]	2.5715 [0.80]	1.5968 [0.76]
HPD with Gamma priors	10%	0.5057 [1.00]	0.3589 [1.00]	0.5100 [1.00]	0.3643 [1.00]
	20%	0.5064 [1.00]	0.3572 [1.00]	0.5130 [1.00]	0.3635 [1.00]
	30%	0.5106 [1.00]	0.3550 [1.00]	0.5180 [1.00]	0.3610 [1.00]
HPD with Non-Informative priors	10%	1.4940 [0.93]	1.1246 [0.93]	1.6224 [0.89]	1.2794 [0.95]
	20%	1.5453 [0.95]	1.1304 [0.94]	1.6609 [0.86]	1.3056 [0.92]
	30%	1.5941 [0.92]	1.1302 [0.88]	1.8470 [0.94]	1.3054 [0.91]

Table 3: Various average estimates with respective average SE/PSE for $\theta_1 = 2, \theta_2 = 1$ at different masking level for n=50 under Type-II hybrid censoring.

Average Estimates [Average SE/PSE]	**Masking Level**	**R=40, T=0.7**		**R=30, T=0.4**	
		θ_1	θ_2	θ_1	θ_2
MLEs [SE]	10%	1.9612 [0.2976]	1.1157 [0.2290]	1.9347 [0.3332]	1.0926 [0.2569]
	20%	2.0561 [0.2963]	0.9507 [0.2038]	2.0442 [0.3453]	0.9965 [0.2480]
	30%	2.1345 [0.3028]	0.8965 [0.1987]	2.1670 [0.3505]	0.8960 [0.2597]
Boot-p [SE]	10%	1.9812 [0.2968]	1.0873 [0.2242]	1.9721 [0.3384]	1.1081 [0.2601]
	20%	2.0343 [0.2947]	0.9836 [0.2073]	2.0873 [0.3524]	1.0277 [0.2538]
	30%	2.1461 [0.3045]	0.9149 [0.2009]	2.0915 [0.3580]	0.9086 [0.2567]
Bayes with Gamma Priors [PSE]	10%	1.9906 [0.1266]	1.0180 [0.0909]	1.9887 [0.1305]	1.0106 [0.0929]
	20%	2.0097 [0.1275]	0.9873 [0.0892]	2.0016 [0.1298]	0.9948 [0.0917]
	30%	2.0202 [0.1280]	0.9787 [0.0895]	2.0207 [0.1311]	0.9841 [0.0917]
Bayes with Non-Informative Priors [PSE]	10%	1.9087 [0.2937]	1.1305 [0.2310]	1.9747 [0.3423]	1.1099 [0.2639]
	20%	2.0341 [0.3031]	0.9849 [0.2274]	2.0643 [0.3619]	1.0446 [0.2690]
	30%	2.0605 [0.3207]	0.9452 [0.2301]	2.1768 [0.3762]	0.9578 [0.2677]

Table 4: Various average CI/HPD intervals widthsalong with their CPsfor $\theta_1 = 2, \theta_2 = 1$ at different masking level for n=50 under Type-II hybrid censoring.

Average CI/HPD intervals	Masking Level	R=40, T=0.7		R=30, T=0.4	
		θ_1	θ_2	θ_1	θ_2
CI	10%	1.1669 [0.94]	0.7976 [0.97]	1.3064 [0.92]	1.0073 [0.95]
	20%	1.1616 [0.94]	0.7992 [0.87]	1.3536 [0.93]	0.9723 [0.89]
	30%	1.1872 [0.96]	0.7792 [0.88]	1.3742 [0.98]	0.9805 [0.87]
Boot-p CI	10%	1.1890 [0.97]	0.8191 [1.00]	1.3685 [0.96]	1.0741 [1.00]
	20%	1.2142 [0.99]	0.8724 [1.00]	1.5066 [1.00]	1.1041 [1.00]
	30%	1.2385 [0.98]	0.8765 [0.97]	1.5189 [0.99]	1.1091 [1.00]
Boot-t CI	10%	1.1837 [0.85]	0.8667 [0.78]	1.4327 [0.81]	1.1466 [0.73]
	20%	1.2633 [0.90]	0.8562 [0.80]	1.5761 [0.90]	1.1990 [0.78]
	30%	1.2851 [0.90]	0.8725 [0.78]	1.7119 [0.91]	1.1116 [0.75]
HPD with Gamma priors	10%	0.4853 [1.00]	0.3481 [1.00]	0.5021 [1.00]	0.3551 [1.00]
	20%	0.4894 [1.00]	0.3400 [1.00]	0.4957 [1.00]	0.3495 [1.00]
	30%	0.4888 [1.00]	0.3414 [1.00]	0.5013 [1.00]	0.3522 [1.00]
HPD with Non-Informative priors	10%	1.1310 [0.93]	0.8115 [0.91]	1.3212 [0.93]	1.0055 [0.94]
	20%	1.1739 [0.94]	0.8277 [0.95]	1.3971 [0.92]	1.0163 [0.94]
	30%	1.2380 [0.96]	0.8682 [0.88]	1.4565 [0.95]	1.0042 [0.87]

Table 5: Various average estimates with respective average SE/PSE for $\theta_1 = 2, \theta_2 = 1$ at different masking level for n=100 under Type-II hybrid censoring.

Average Estimates [Average SE/PSE]	**Masking Level**	**R=80, T=0.7**		**R=60, T=0.4**	
		θ_1	θ_2	θ_1	θ_2
MLEs [SE]	10%	1.9894 [0.2092]	1.0901 [0.1463]	1.9949 [0.2391]	1.0736 [0.1682]
	20%	2.0743 [0.2120]	0.9916 [0.1477]	2.0614 [0.2400]	0.9750 [0.1674]
	30%	2.0693 [0.2090]	0.9122 [0.1494]	2.1761 [0.2492]	0.9011 [0.1682]
Boot-p [SE]	10%	1.9404 [0.2021]	1.0449 [0.1497]	1.9863 [0.2378]	1.0791 [0.1681]
	20%	2.0391 [0.2080]	0.9765 [0.1451]	2.0401 [0.2374]	0.9846 [0.1671]
	30%	2.1474 [0.2157]	0.9080 [0.1419]	2.1031 [0.2404]	0.8951 [0.1697]
Bayes with Gamma Priors [PSE]	10%	1.9934 [0.1162]	1.0259 [0.0842]	1.9969 [0.1217]	1.0169 [0.0870]
	20%	2.0200 [0.1174]	0.9954 [0.0824]	2.0120 [0.1211]	0.9935 [0.0856]
	30%	2.0173 [0.1162]	0.9710 [0.0812]	2.0430 [0.1221]	0.9746 [0.0848]
Bayes with Non-Informative Priors [PSE]	10%	1.9600 [0.2079]	1.0755 [0.1557]	1.9443 [0.2380]	1.1127 [0.1842]
	20%	2.0387 [0.2117]	0.9653 [0.1485]	2.0642 [0.2484]	1.0073 [0.1803]
	30%	2.0833 [0.2149]	0.9055 [0.1461]	2.0996 [0.2574]	0.9229 [0.1801]

Table 6: Various average CI/HPD intervals widthsalong with their CPsfor $\theta_1 = 2, \theta_2 = 1$ at different masking level for n=100 under Type-II hybrid censoring.

Average CI/HPD intervals	Masking Level	R=80, T=0.7		R=60, T=0.4	
		θ_1	θ_2	θ_1	θ_2
CI	10%	0.8204 [0.97]	0.5128 [0.93]	0.9374 [0.97]	0.6486 [0.97]
	20%	0.8311 [0.93]	0.5793 [0.93]	0.9408 [0.97]	0.6562 [0.87]
	30%	0.8393 [0.91]	0.5704 [0.86]	0.9769 [0.90]	0.6397 [0.87]
Boot-p CI	10%	0.7873 [0.97]	0.5549 [1.00]	0.9193 [0.98]	0.6124 [1.00]
	20%	0.8088 [0.99]	0.5606 [0.99]	0.9065 [0.97]	0.6483 [1.00]
	30%	0.8193 [0.95]	0.5692 [0.98]	0.9344 [0.98]	0.6438 [1.00]
Boot-t CI	10%	0.7922 [0.81]	0.5639 [0.79]	0.9270 [0.85]	0.6136 [0.72]
	20%	0.8200 [0.79]	0.5704 [0.81]	0.9376 [0.77]	0.6596 [0.85]
	30%	0.8255 [0.83]	0.5773 [0.72]	0.9598 [0.85]	0.6494 [0.70]
HPD with Gamma priors	10%	0.4434 [1.00]	0.3014 [0.99]	0.4653 [1.00]	0.3213 [1.00]
	20%	0.4491 [1.00]	0.3137 [0.99]	0.4620 [1.00]	0.3264 [1.00]
	30%	0.4454 [1.00]	0.3084 [0.99]	0.4672 [1.00]	0.3231 [1.00]
HPD with Non-Informative priors	10%	0.7998 [0.96]	0.5568 [0.95]	0.9183 [0.96]	0.6587 [0.93]
	20%	0.8120 [0.94]	0.5664 [0.91]	0.9600 [0.96]	0.6891 [0.94]
	30%	0.8269 [0.95]	0.5538 [0.81]	0.9920 [0.95]	0.6793 [0.85]

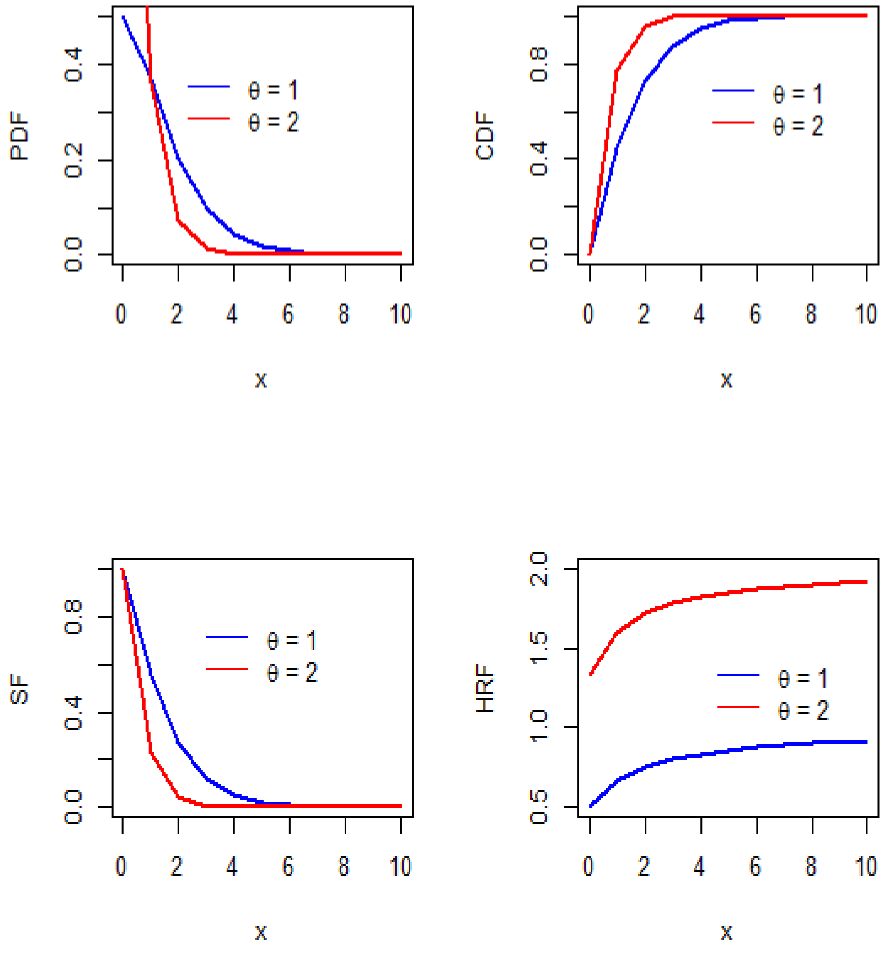

Fig. 1 Plots for pdf, cdf, survival function, and hazard function of Lindley Distribution.

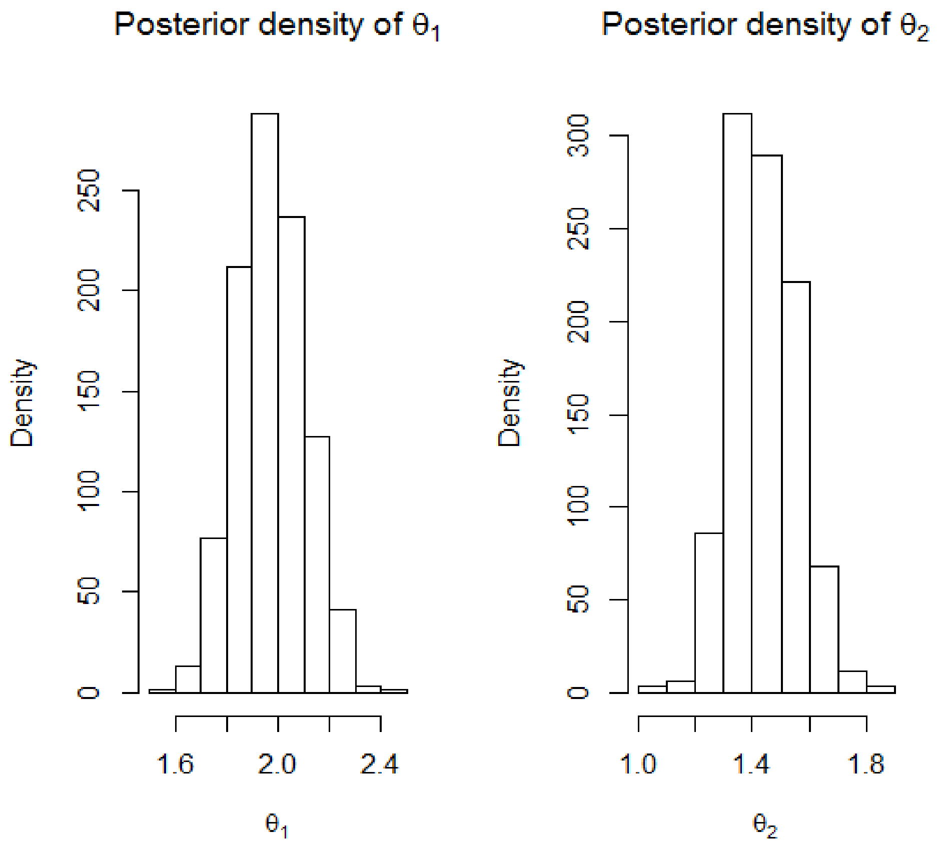

Fig. 2 Estimated Posterior density plots for the parameters θ_1 and θ_2 .